Recap:

- Requiring that the <u>geometry</u> of 4-vectors in spacetime be invariant to their representation yields rules for how their components and their unit vectors transform:

$$\vec{V} = \sum_{\alpha=0}^{3} V^{\alpha} \vec{e}_{\alpha} = \sum_{\mu'=0}^{3} V^{\mu'} \vec{e}_{\mu'}$$

$$V^{\mu'} = \sum_{\alpha=0}^{3} \Lambda^{\mu'}{}_{\alpha} V^{\alpha}$$

$$\vec{e}_{\mu'} = \sum_{\alpha=0}^{3} \Lambda^{\alpha}{}_{\mu'} \vec{e}_{\alpha}$$

$\Lambda^{\mu'}{}_{\alpha}$ = Element of Lorentz transformation matrix that goes from unprimed frame to primed frame. Row μ', column α.

$\Lambda^{\mu}{}_{\alpha'}$ $\equiv$ Element of matrix for the Lorentz transformation that inverts the previous one.

General rule: We go <u>from</u> the coordinates of the bottom index <u>to</u> the coordinates of the top index.

Mathematical expression of inverseness:

$$\sum_{\mu'=0}^{3} \Lambda^{\alpha}{}_{\mu'} \Lambda^{\mu'}{}_{\beta} = \delta^{\alpha}{}_{\beta}$$

$$\sum_{\alpha=0}^{3} \Lambda^{\mu'}{}_{\alpha} \Lambda^{\alpha}{}_{\nu'} = \delta^{\mu'}{}_{\nu'}$$

Velocity addition rules in special relativity

Observer in frame T sees something move with velocity $\underset{\sim}{u}_T$ in that frame.

Frame T moves with $\underset{\sim}{v} = v\,\underset{\sim}{e}_x$ relative to frame S. What is the velocity $\underset{\sim}{u}_S$?

$$u_S^x = \frac{u_T^x + v}{1 + u_T^x v / c^2}$$

$$u_S^y = \frac{u_T^y}{\gamma\left(1 + u_T^x v / c^2\right)}$$

$$u_S^z = \frac{u_T^z}{\gamma\left(1 + u_T^x v / c^2\right)}$$

$$\gamma = \left(1 - v^2 / c^2\right)^{-1/2}$$

If velocity gets messed up, momentum will get messed up too. Begin our analysis by carefully laying out how things work in Newtonian mechanics.

Imagine we have N_i bodies that come together in some fashion (i for "initial"), and we end up with N_f bodies at the end. Conservation of momentum tells us

$$\sum_{i=1}^{N_i} m_i^{\text{initial}} \underset{\sim}{u}_i^{\text{initial}} = \sum_{i=1}^{N_f} m_i^{\text{final}} \underset{\sim}{u}_i^{\text{final}}$$

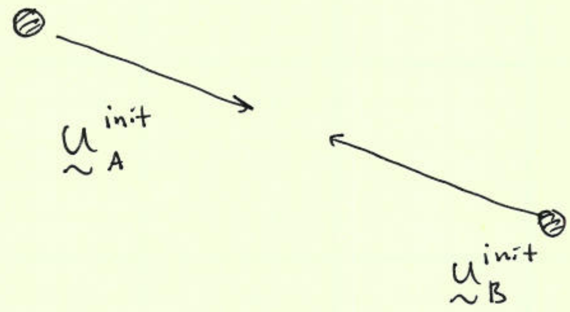
As long as $\sum_{i=1}^{N_i} m_i^{\text{initial}} = \sum_{i=1}^{N_f} m_i^{\text{final}}$, this relation holds in all Galilean inertial reference frames.

Does this work when we go from Galilean frames to Lorentz frames? Yes... but we have to make some modifications.

To see why, consider an elastic collision between 2 identical particles.

Before collision:

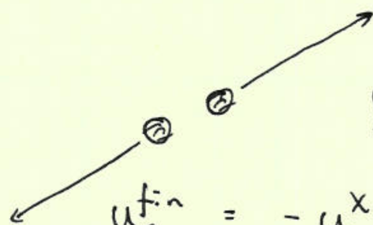
$$m_a = m_b \equiv m$$



$$\underset{\sim}{u}_A^{init} = u^x \underset{\sim}{e}_x - u^y \underset{\sim}{e}_y$$

$$\underset{\sim}{u}_B^{init} = -u^x \underset{\sim}{e}_x + u^y \underset{\sim}{e}_y$$

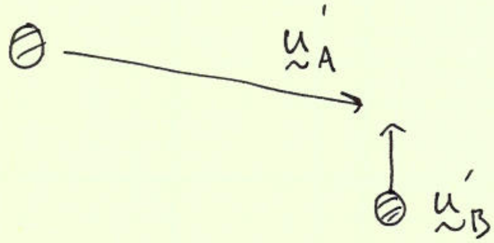
After:



$$\underset{\sim}{u}_A^{fin} = u^x \underset{\sim}{e}_x + u^y \underset{\sim}{e}_y$$

$$\underset{\sim}{u}_B^{fin} = -u^x \underset{\sim}{e}_x - u^y \underset{\sim}{e}_y$$

Now, examine this situation from the frame that cancels out the horizontal motion of B:



An observer at rest in the first frame we used will see that this frame moves with $\underset{\sim}{v} = -u^x \underset{\sim}{e}_x$.

What are the velocity vectors in this frame? Use our new velocity addition formula to find out:

$$u_A^{x'} = \frac{u^x + u^x}{1 + (u^x)^2/c^2}$$

$$u_B^{x'} = \frac{u^x - u^x}{1 - (u^x)^2/c^2} = 0$$

Horizontal momentum no longer balances... but we don't expect it to after changing frames.

$$u_A^{y'} = \frac{-u^y}{\gamma\,(1 + (u^x)^2/c^2)} = -\frac{u^y\sqrt{1 - (u^x)^2/c^2}}{1 + (u^x)^2/c^2}$$

$$u_B^{y'} = \frac{u^y}{\gamma\,(1 - (u^x)^2/c^2)} = \frac{+u^y}{\sqrt{1 - (u^x)^2/c^2}}$$

→ Notice that vertical components no longer balance: Changing to a frame that moves along the x-direction has broken conservation of momentum along the <u>vertical</u> direction.

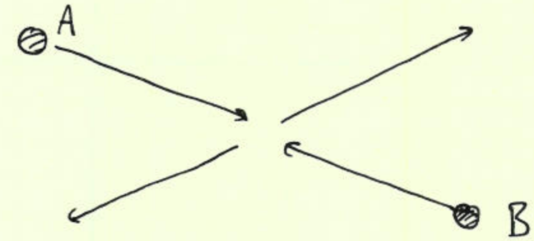
→ This is disturbing!

Hypothesis: We can define a modified version of momentum which allows us to recover a useful notion of momentum conservation.

Let us postulate that $\underset{\sim}{p} = \alpha(|\underset{\sim}{u}|)\, m \underset{\sim}{u}$

Scalar correction to "normal" momentum law.

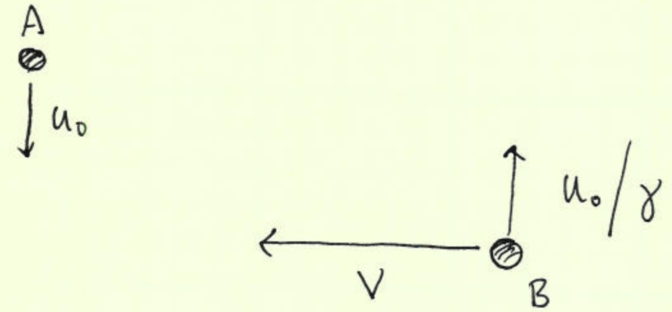
Re-examine the collision:



In particular, we'll now analyze this in "Frame A," the Lorentz frame in which body A has no velocity in the x-direction. We will also consider "Frame B," the Lorentz frame in which body B has no velocity along x.

Physics: The bodies have the same "mass." (I.e., when at rest, they each have mass m.) In Frame A, the magnitude of particle A's velocity is u_0; in Frame B, the magnitude of B's velocity is u_0.

Examine collision in Frame A. Before collision:

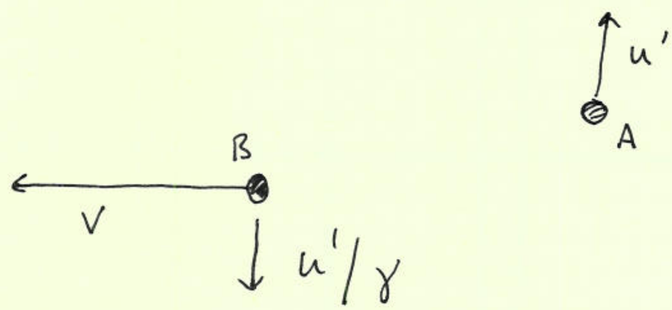


$$\gamma = \left(1 - v^2/c^2\right)^{-1/2}$$

V is the result for B's horizontal component after we transform into Frame A.

Why u_0/γ for the vertical component of B's velocity? In frame B, the vertical component of B is u_0. Use velocity addition formula to change to Frame A - get u_0/γ.

After collision in Frame A:



Logic here: The horizontal speed cannot be affected, so it remains V. The vertical speed might be affected, and the vertical direction must reverse.

Impose conservation of horizontal momentum:

$$\alpha\left(\sqrt{V^2 + (u_0/\gamma)^2}\right) mV = \alpha\left(\sqrt{V^2 + (u'/\gamma)^2}\right) mV$$

→ Only true if $u_0 = u'$: y-components only switch direction, no change in magnitude.

Now, require y-component to be conserved:

$$-\alpha(u_0)\, m u_0 + \alpha\left(\sqrt{v^2 + u_0^2/\gamma^2}\right) m u_0/\gamma$$

$$= \alpha(u_0)\, m u_0 - \alpha\left(\sqrt{v^2 + u_0^2/\gamma^2}\right) m u_0/\gamma$$

$$\longrightarrow \quad \alpha\left(\sqrt{v^2 + u_0^2/\gamma^2}\right) = \gamma\, \alpha(u_0)$$

Let's assume $\alpha(0) = 1$. This means that our formula reduces to "normal" momentum for small velocities. Then,

$$\alpha(v) = \gamma$$

and we obtain

$$\boxed{\underset{\sim}{p} = \gamma(v)\, m \underset{\sim}{v}}$$

This notion of momentum is conserved during interactions in special relativity.

Note: Some texts define $m(\underset{\sim}{v}) = \gamma m_0$ (where m_0 is the "rest mass") as the "relativistic mass." This term is now deprecated; modern terminology leaves m as an invariant, and builds the relativistic stuff in elsewhere.

Energy: In Newtonian physics, the change in kinetic energy is the work done on a body:

$$K_b - K_a = \int_a^b \frac{d\underset{\sim}{p}}{dt} \cdot d\underset{\sim}{r}$$

$$= \int_a^b \frac{d}{dt}\left(m\underset{\sim}{u}\right) \cdot \underset{\sim}{u}\, dt$$

$$= \int_a^b m\, \underset{\sim}{u} \cdot d\underset{\sim}{u}$$

$$= \frac{1}{2} m \left(u_b^2 - u_a^2\right)$$

Now, consider how it will look in relativity:

$$K_b - K_a = \int_a^b \frac{d}{dt}\left[\gamma(u)\, m\underset{\sim}{u}\right] \cdot \underset{\sim}{u}\, dt$$

$$= \int_a^b \underset{\sim}{u} \cdot d\left[\frac{m\underset{\sim}{u}}{\sqrt{1 - u^2/c^2}}\right]$$

$$\underset{\sim}{u} \cdot d\left[\frac{\underset{\sim}{u}}{\sqrt{1-u^2/c^2}}\right] = d\left[\frac{u^2}{\sqrt{1 - u^2/c^2}}\right] - \frac{\underset{\sim}{u} \cdot d\underset{\sim}{u}}{\sqrt{1 - u^2/c^2}}$$

So,

$$K_b - K_a = \left.\frac{mu^2}{\sqrt{1 - u^2/c^2}}\right|_a^b - \int_a^b \frac{m\underset{\sim}{u} \cdot d\underset{\sim}{u}}{\sqrt{1 - u^2/c^2}}$$

$$= \left.\frac{mu^2}{\sqrt{1 - u^2/c^2}}\right|_a^b + \left. mc^2\sqrt{1 - \frac{u^2}{c^2}}\,\right|_a^b$$

Define: $\underset{\sim}{u}(b) \equiv \underset{\sim}{u}$, $\underset{\sim}{u}(a) = 0 \rightarrow K_a = 0$

$$\rightarrow \quad K_b \equiv K = \frac{mu^2}{\sqrt{1-u^2/c^2}} + mc^2\sqrt{1-\frac{u^2}{c^2}} - mc^2$$

-or-

$$\boxed{\begin{aligned} K &= \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2 \\ &= \left(\gamma(u) - 1\right) mc^2 \end{aligned}}$$

Interpretation: Define

$$E = \gamma mc^2$$

as the total energy of a body, and let

$$E_0 = mc^2$$

be its "rest energy." The quantity K is then the "energy of motion" - ie, the kinetic energy - associated with the body.