Recap:

- Found that velocity addition formula broke momentum conservation. Fix: Tweak the definition: $\underset{\sim}{p} = \gamma(u)\, m \underset{\sim}{u}$ is conserved during interactions among bodies.

- This notion of momentum makes kinetic energy more complicated. Integrating the work done on a body, we find that a body with speed u has kinetic energy

$$K = (\gamma(u) - 1)\, mc^2$$

- Interpreting the <u>total</u> energy as $E = \gamma(u)\, mc^2$, this implies a <u>rest</u> energy $E_0 = mc^2$.

Suppose we have a body with energy E and momentum $\underset{\sim}{p}$ as measured in our lab.

What are the energy E' and momentum $\underset{\sim}{p}'$ as measured by an observer whose velocity relative to the lab is $\underset{\sim}{v} = v\,\underset{\sim}{e}_x$?

Recipe to follow:

1. Compute the velocity of the body in the lab from $\underset{\sim}{p}$ and E.

2. Using the velocity addition formulas, compute the velocity in the new ("moving") frame.

3. From the velocity, assemble E' & $\underset{\sim}{p}'$.

Result:

$$E' = \gamma\,(E - v p^x)$$
$$p^{x'} = \gamma\,(p^x - vE/c^2)$$
$$p^{y'} = p^y \, , \quad p^{z'} = p^z$$

-or-

$$\begin{bmatrix} E'/c \\ p^{x'} \\ p^{y'} \\ p^{z'} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E/c \\ p^x \\ p^y \\ p^z \end{bmatrix}$$

→ Energy & momentum form a set of quantities that transform under a Lorentz transformation!

Recall $\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ is a "Lorentz invariant": All Lorentz frames agree on its value. Can we do something similar with energy and momentum?

Inspired by the spacetime invariant, let's think of energy as the "time-like" piece of momentum. Using it, we'll construct something similar:

$$-\frac{E^2}{c^2} + |\underline{p}|^2 = ?$$

Plug in
$$E^2 = \gamma^2 m^2 c^4 = \frac{m^2 c^4}{1 - u^2/c^2}$$
$$|\underline{p}|^2 = \gamma^2 m^2 u^2 = \frac{m^2 u^2}{1 - u^2/c^2}$$

So,
$$-\frac{E^2}{c^2} + |\underline{p}|^2 = \frac{-m^2 c^2 + m^2 u^2}{1 - u^2/c^2}$$
$$= -m^2 c^2 \left[\frac{1 - u^2/c^2}{1 - u^2/c^2}\right]$$

$$\rightarrow \boxed{\begin{aligned} & E^2 - |\underline{p}|^2 c^2 = m^2 c^4 \\ \text{or} \quad & E^2 = |\underline{p}|^2 c^2 + m^2 c^4 \end{aligned}}$$

Notice: If $m = 0$, we have $|\underline{p}| = E/c$

→ Momentum carried by massless bodies. This corresponds perfectly to energy and momentum carried by electromagnetic radiation (c.f. the Poynting vector.)

Not difficult to show that this relationship is an invariant: $E^2 = |\vec{p}|^2 c^2 + m^2 c^4$ in all Lorentz frames.

We can make this invariance a little clearer by recognizing that E/c, p^x, p^y, p^z are, by their Lorentz transformation rule, components of a 4-vector.

We've briefly met this notion already in the context of the displacement 4-vector:

$$\Delta\vec{x} = \sum_{\mu=0}^{3} \Delta x^\mu \vec{e}_\mu$$

$$\begin{aligned}
\Delta x^0 &= \Delta x^t = c\,\Delta t \\
\Delta x^1 &= \Delta x^x = \Delta x \\
\Delta x^2 &= \Delta x^y = \Delta y \\
\Delta x^3 &= \Delta x^z = \Delta z
\end{aligned}$$

Already saw that under our rules for transforming components and unit vectors, this geometric object is invariant as we change between Lorentz frames:

$$\Delta\vec{x} = \sum_{\mu=0}^{3} \Delta x^\mu \vec{e}_\mu = \sum_{\mu'=0}^{3} \Delta x^{\mu'} \vec{e}_{\mu'}$$

We similarly define the "4-momentum" to be the geometric object in spacetime whose components transform between Lorentz frames according to the rules we have found:

$$\vec{p} = \sum_{\mu=0}^{3} p^\mu \vec{e}_\mu$$

$$p^0 = E/c$$

$$p^x = p^x, \quad p^y = p^y, \quad p^z = p^z$$

Very useful definition and theorem:

Given two 4-vectors, $\vec{a} = \sum_{\mu=0}^{3} a^\mu \vec{e}_\mu$ and $\vec{b} = \sum_{\mu=0}^{3} b^\mu \vec{e}_\mu$, we define

$$\vec{a}\cdot\vec{b} = -a^t b^t + a^x b^x + a^y b^y + a^z b^z$$

Notice that Δs^2, the invariant interval, is equal to $\Delta\vec{x}\cdot\Delta\vec{x}$. Likewise, we can see that $\vec{p}\cdot\vec{p} = -m^2c^2$.

Theorem: For any two 4-vectors, $\vec{a}$ & $\vec{b}$, the scalar product $\vec{a}\cdot\vec{b}$ is a Lorentz invariant.

Forgot to mention on the previous page: The 4-momentum gives us a good tool to combine conservation of momentum and energy together. Our combined rule reads as follows:

Given N_i bodies that interact and produce N_f bodies, 4-momentum is conserved:

$$\boxed{\begin{gathered}\sum_{i=1}^{N_i} \vec{p}_i^{\,init} = \sum_{i=1}^{N_f} \vec{p}_i^{\,final} \\ \vec{p}_i^{\,init} = \text{initial 4-momentum of particle } i \\ \vec{p}_i^{\,final} = \text{final 4-momentum of particle } i\end{gathered}}$$

Corollary: Given a 4-vector $\vec{a}$, the scalar product of $\vec{a}$ with itself is a value that all observers agree upon. If

$$\vec{a}\cdot\vec{a} < 0$$

we say $\vec{a}$ is "timelike": There exists a Lorentz frame in which $\vec{a}$ points purely in the timelike direction. If

$$\vec{a}\cdot\vec{a} > 0$$

we say $\vec{a}$ is "spacelike": There exists a Lorentz frame in which $\vec{a}$ points purely in a spatial direction. If

$$\vec{a}\cdot\vec{a} = 0$$

$\vec{a}$ is "lightlike" or "null": In all Lorentz frames, $\vec{a}$ points along light cones.

Notice: $\vec{p}\cdot\vec{p} = -m^2c^2$: $\vec{p}$ is timelike if $m > 0$.

Another definition: $\vec{u} = \frac{1}{m}\vec{p}$

We call $\vec{u}$ the "4-velocity." What does it mean? Look at its components:

$$u^t = p^t/m = E/mc = \gamma mc^2/mc$$
$$= \gamma c$$

$$u^x = p^x/m = \gamma\,(\underset{\sim}{u})^x$$
$$u^y = p^y/m = \gamma\,(\underset{\sim}{u})^y$$
$$u^z = p^z/m = \gamma\,(\underset{\sim}{u})^z$$

[Notation: $(\underset{\sim}{u})^i \equiv$ the i-component of 3-vector $\underset{\sim}{u}$.]

The spatial components look just like "normal" velocity, but with an annoying factor of γ. To interpret this, consider someone passing by you with 3-velocity $\underset{\sim}{u}$. That person's clocks run slow according to you:

$$dt = \gamma\, d\tau$$

Interval between clock ticks you measure.

Interval between clock ticks they measure

(In other words, the moving clock runs slow: In their time interval $d\tau$, we see $\gamma\, d\tau$ go past.)

Definition: $d\tau$ is an interval of "proper" time, time as measured by the person we say is moving.

"Proper" comes from French "propre," and refers to the fact that it is the moving observer's <u>own</u> time. (German sometimes calls it "eigenzeit.")

The key reason that this is useful is that $d\tau$, properly interpreted, is a Lorentz invariant:

<u>All</u> observers agree that the person in motion measures a time interval $d\tau$. Indeed, $d\tau$ is the time interval as measured in the frame in which the observer is at rest!

With this in mind, re-examine components of the 4-velocity:

$$u^x = \gamma \frac{dx}{dt} = \frac{dx}{d\tau}$$

$$u^y = \gamma \frac{dy}{dt} = \frac{dy}{d\tau}$$

$$u^z = \gamma \frac{dz}{dt} = \frac{dz}{d\tau}$$

Last component: $u^t = c\,\frac{dt}{d\tau}$. Modulo the factor of c, this tells us how the two observers interpret intervals of time.

Compare with the displacement 4-vector: $\boxed{\vec{u} = d\vec{x}/d\tau}$

$\rightarrow$ $\vec{u}$ is spacetime displacement per unit <u>proper</u> time.

4-velocity versus 3-velocity

We now have two important ways to characterize a moving body's motion:

3-velocity: $\underset{\sim}{u} = \dfrac{d\underset{\sim}{x}}{dt}$ → Motion through SPACE - as seen by some observer - per unit TIME - as seen by some observer.

→ Clearly strongly depends on the frame in which it is measured!

4-velocity: $\vec{u} = \dfrac{d\vec{x}}{d\tau}$ → Motion through SPACETIME - as seen by all observers - per unit proper time - as agreed to by all observers.

→ Frame-independent, geometric object.

Components that a given observer assigns to 4-velocity are assembled from the 3-velocity:

$$u^0 = \gamma c$$

$$u^1 = \gamma (\underset{\sim}{u})^x \qquad \gamma = \left(1 - |\underset{\sim}{u}|^2 / c^2\right)^{-1/2}$$

$$u^2 = \gamma (\underset{\sim}{u})^y$$

$$u^3 = \gamma (\underset{\sim}{u})^z$$

Key difference is how we describe them in different Lorentz frames:

4-vel: Lorentz transformation.

$$u^{\alpha'} = \sum_{\mu=0}^{3} \Lambda^{\alpha'}{}_{\mu}\, u^{\mu}$$

$\Lambda^{\nu'}{}_{\mu} \equiv$ elements of transformation matrix relating two Lorentz frames.

$$\vec{u} = \sum_{\mu=0}^{3} u^{\mu} \vec{e}_{\mu} = \sum_{\alpha'=0}^{3} u^{\alpha'} \vec{e}_{\alpha'}$$

→ same object, but different components.

3-vel: Velocity addition formula.

$$(\underset{\sim}{u})^{x'} = \frac{(\underset{\sim}{u})^{x} + v}{1 + (\underset{\sim}{u})^{x} v / c^2}$$

$$(\underset{\sim}{u})^{y'} = \frac{\underset{\sim}{u}^{y}}{\gamma\left(1 + (\underset{\sim}{u})^{x} v / c^2\right)}$$

$$(\underset{\sim}{u})^{z'} = \frac{\underset{\sim}{u}^{z}}{\gamma\left(1 + (\underset{\sim}{u})^{z} v / c^2\right)}$$

→ Actually a different 3-vector!