Recap:

- Momentum, $\underline{p} = \gamma(u) m \underline{u}$, and energy, $E = \gamma(u) mc^2$, are most usefully combined into a single 4-vector:

$$\vec{p} = \sum_{\mu=0}^{3} p^\mu \vec{e}_\mu$$

$$p^t = E/c \ , \quad p^i = (\underline{p})^i$$

This quantity transforms between Lorentz frames like any 4-vector: $p^{\alpha'} = \sum_{\mu=0}^{3} \Lambda^{\alpha'}{}_{\mu} p^\mu$

  - Note that energy & momentum get mixed up just as space & time do in the Lorentz transformation of space & time!

- The 4-momentum is conserved in any interaction: combine conservation of energy and momentum into one law.
- $\vec{p} \cdot \vec{p} = -m^2 c^2$. Scalar product of two 4-vectors is a Lorentz invariant.

- 4-velocity: $\vec{u} = \frac{\vec{p}}{m} = \frac{d\vec{x}}{d\tau}$. Tells us about movement through spacetime per unit <u>proper</u> time - time as measured by the moving observer.

Twin "paradox"

Twin A stays on Earth. Twin B travels on a rocket ship to Alpha Centauri 4 light years away, at relativistic speed.

Twin B turns around and comes back. When they compare notes, who is older?

Essence of the ~~pa~~ paradox: According to special relativity no inertial observer is preferred:

- A can say B is in motion; therefore, B's clock runs slow, and B is younger.
- B can say A is in motion, and therefore A is younger.

Who is right?

Resolution: B is not an inertial observer! Twin B accelerates, and this breaks the symmetry. One of the twins will be unambiguously older.

Which one? Need to learn about acceleration.

We want to go beyond things moving at constant velocity. How do we do this?

To begin, consider the quantity

$$\frac{d\vec{u}}{d\tau} = \frac{d^2\vec{x}}{d\tau^2}$$

This is clearly an acceleration. Let's examine some of its properties.

We know that $\vec{u}\cdot\vec{u} = -c^2$.

(Two ways to show this: 1. $\vec{u} = \vec{p}/m$; $\vec{p}\cdot\vec{p} = -m^2c^2$.
2. In rest frame $u^t = c$, $u^i = 0$. Then, use fact that $\vec{u}\cdot\vec{u}$ is a Lorentz invariant.)

Take derivative of $\vec{u}\cdot\vec{u} = -c^2$:

$$\frac{d}{d\tau}\left(\vec{u}\cdot\vec{u}\right) = \frac{d}{d\tau}\left(-c^2\right)$$

$$\rightarrow \quad \vec{u}\cdot\frac{d\vec{u}}{d\tau} = 0$$

So, this notion of acceleration must always be "orthogonal" to 4-velocity in spacetime.

To wrap our heads around the physics of acceleration, introduce a particular reference frame:

the MCRF = "Momentarily Comoving Reference Frame"

This is a Lorentz frame that, at least for one moment, has the same velocity as the accelerating observer. In other words, this observer is at rest in the MCRF for a moment.

In the MCRF, $u^t_{MCRF} = c$, $u^i_{MCRF} = 0$.

$$d\tau = dt_{MCRF}$$

$$\rightarrow \quad \frac{d\vec{u}}{d\tau} \underset{MCRF}{\doteq} \left(0, \frac{du^x}{dt}, \frac{du^y}{dt}, \frac{du^z}{dt}\right)$$

"The geometric object on the left has the components on the right in the specified frame."

→ In the MCRF, $d\vec{u}/d\tau$ is clearly just the normal acceleration we know from Newtonian classical mechanics!

Let's call it $\vec{a} \equiv \frac{d\vec{u}}{d\tau}$: The 4-acceleration.

Example: A uniformly accelerated observer.

An observer starts at rest with respect to us, but experiences uniform acceleration of magnitude g. ("Uniform" means the observer feels the same acceleration at all times ... hence, it has the same value in the MCRF.)

Compute the 4-velocity at later times.

How to proceed? Let's say the acceleration is in the x direction. We know

$$\vec{u}(\tau=0) = c\,\vec{e}_t$$

$$\vec{a}(\tau=0) = g\,\vec{e}_x$$

We also know

$$\vec{u}\cdot\vec{u} = -c^2 \quad \text{at all times}$$

$$\vec{u}\cdot\vec{a} = 0 \quad \text{at all times}$$

$$\vec{a}\cdot\vec{a} = g^2 \quad \text{at all times.}$$

How do we know this last one? We know that $\vec{a}$ has one non-zero component in the MCRF. In the MCRF, $\vec{a}\cdot\vec{a} = g^2$ ... and the scalar product is a Lorentz invariant.

Write out those dot products in terms of components in the original rest frame:

$$-(u^t)^2 + (u^x)^2 = c^2$$

$$-u^t a^t + u^x a^x = 0$$

$$-(a^t)^2 + (a^x)^2 = g^2$$

Form of $\vec{u}\cdot\vec{u}$ and $\vec{a}\cdot\vec{a}$ suggests that the solution will be in terms of hyperbolic functions: Let's try

$$u^t = c\cosh[A\tau], \quad u^x = c\sinh[A\tau]$$

We must choose A so that $\vec{a}\cdot\vec{a}$ is correct:

$$a^t = \frac{du^t}{d\tau} = cA\sinh[A\tau]$$

$$a^x = \frac{du^x}{d\tau} = cA\cosh[A\tau]$$

$$-(a^t)^2 + (a^x)^2 = c^2A^2\left[-\sinh^2[A\tau] + \cosh^2[A\tau]\right]$$

$$= g^2$$

$$\longrightarrow \quad A = \frac{g}{c}$$

So, here's our solution: $\vec{u} = c\cosh[g\tau/c]\,\vec{e}_t + c\sinh[g\tau/c]\,\vec{e}_x$

$$\vec{a} = g\sinh[g\tau/c]\,\vec{e}_t + g\cosh[g\tau/c]\,\vec{e}_x$$

(Easy to verify $\vec{u}\cdot\vec{a} = 0$.)

Let's use this to explore what happens when someone is uniformly accelerated. Two questions:

1. After travelling for time T as measured by the accelerating observer, how far have we travelled?

2. After travelling for time T as measured by the accelerating observer, how much time has elapsed "back home"?

Both questions are easily found by integrating the 4-velocity.

$$\Delta x = \int_0^T u^x \, d\tau$$

$$= c \int_0^T \sinh\left(\frac{g\tau}{c}\right) d\tau$$

$$= \frac{c^2}{g}\left[\cosh\left(\frac{gT}{c}\right) - 1\right]$$

$$\frac{c^2}{g} = 0.97 \text{ lightyears}$$

$\frac{g}{c} \cdot (\text{one year}) = 1.03$ (i.e, $c/g = 0.97$ year)

$\rightarrow \Delta x \,(\text{one year}) = 0.56$ ly

$\Delta x \,(\text{two years}) = 2.9$ ly

$\Delta x \,(\text{ten years}) = 14{,}400$ ly !!

How much time elapses while doing this?

$$\Delta t = \frac{1}{c}\int_0^T u^t \, d\tau$$
$$= \int_0^T \cosh\left(\frac{g\tau}{c}\right) d\tau$$
$$= \frac{c}{g} \sinh\left(\frac{gT}{c}\right)$$

$$\frac{c}{g} = 0.97 \text{ year}$$

$\rightarrow$ Δt (one year) = 1.2 years

Δt (two years) = 3.7 years

Δt (ten years) = 14,400 years

As seen by observers "back home," the accelerated observer is getting closer and closer to the speed of light, and so is experiencing enormous amounts of time dilation. Their 10 year interval is our 14,400 year interval – the moving clock is running wicked slowly.

Forces

We want a way of describing forces that is consistent with our intuition regarding Newton's 2ND law, $\underset{\sim}{F} = d\underset{\sim}{p}/dt$. The problem we face is that $\underset{\sim}{p}$ and t vary from frame to frame.

Two general conceptual frameworks are used:

1. Define a "4-force",

$$\vec{F} = \frac{d\vec{p}}{d\tau}$$

In terms of this, $\vec{a} = \vec{F}/m$. Our "constant acceleration" example was in fact a "constant 4-force" calculation!

2. Use the usual 3-force, $\underset{\sim}{F} = d\underset{\sim}{p}/dt$, but carefully transform pieces between different frames.

Although option 1 is preferred in principle, since it is based on properly constructed spacetime invariants, option 2 is often used in practice, since we tend to measure 3-forces in a particular frame.

The important thing is then to understand how the 3-force transforms between reference frames.

Frame S: Particle has 3-momentum $\underset{\sim}{p} = \gamma(u) m \underset{\sim}{u}$, 3-vel $\underset{\sim}{u}$, energy $E = \gamma(u) mc^2$.

Frame S': Moves with velocity $\underset{\sim}{v} = v \underset{\sim}{e}_x$ according to

If force $\underset{\sim}{F}$ acts in frame S, what force $\underset{\sim}{F}'$ acts in frame S'?

Work this out using Lorentz transformations:

$$F^{x'} = \frac{dp^{x'}}{dt'} = \frac{\gamma(dp^x - v\,dE/c^2)}{\gamma(dt - v\,dx/c^2)}$$

$$= \frac{F^x - (v/c^2)\,dE/dt}{1 - vu^x/c^2}$$

We can simplify this a bit more:

$$E^2 = p^2c^2 + m^2c^4$$

$$E\frac{dE}{dt} = \underset{\sim}{p} \cdot \frac{d\underset{\sim}{p}}{dt}\,c^2$$

$$\gamma mc^2 \frac{dE}{dt} = \gamma m\underset{\sim}{u} \cdot \frac{d\underset{\sim}{p}}{dt}\,c^2 \rightarrow \quad \frac{dE}{dt} = \underset{\sim}{F} \cdot \underset{\sim}{u}$$

So, $$\boxed{F^{x'} = \frac{F^x - (v/c^2)\,\underset{\sim}{F}\cdot\underset{\sim}{u}}{1 - vu^x/c^2}}$$

$$\boxed{\begin{aligned} F^{y'} &= \frac{dp^{y'}}{dt'} = \frac{dp^y}{\gamma(dt - v\,dx/c^2)} = \frac{F^y}{\gamma(1 - vu^x/c^2)} \\ F^{z'} &= \frac{F^z}{\gamma(1 - vu^x/c^2)} \end{aligned}}$$