Recap:

- Introduced 4-acceleration,

$$\vec{a} = \frac{d\vec{u}}{d\tau}$$

- From $\vec{u}\cdot\vec{u} = -c^2$, simple to see that $\vec{u}\cdot\vec{a} = 0$. 4-velocity and 4-acceleration are "spacetime orthogonal"

- Defined constant acceleration to be

$$\vec{a}\cdot\vec{a} = \text{constant}$$

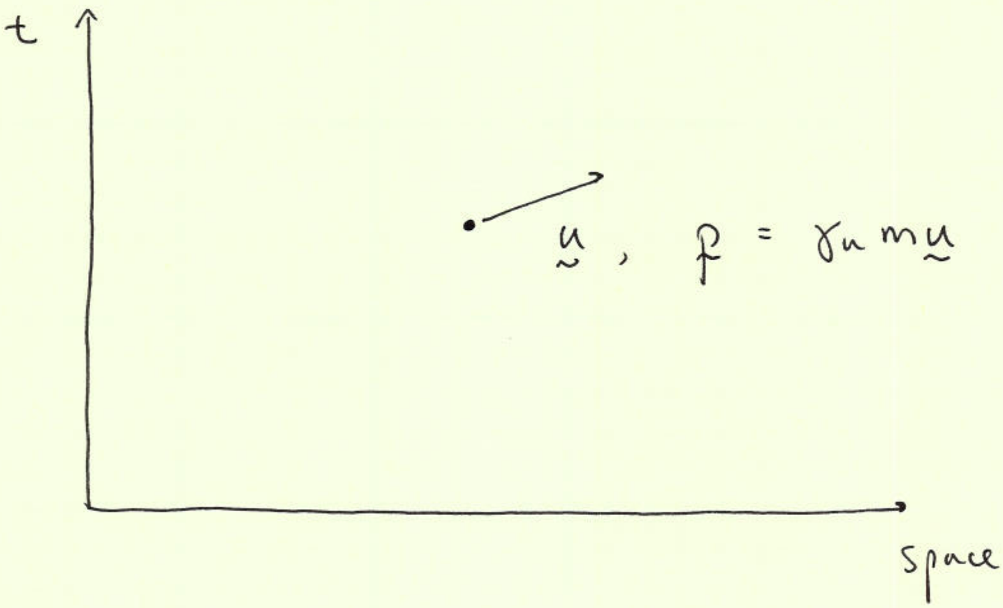
→ Means that in momentarily comoving reference frame, where

$$\vec{a} \underset{\text{MCRF}}{\doteq} (0,\ a^x,\ a^y,\ a^z)$$

ordinary Newtonian acceleration

the observer feels a constant acceleration.

Forces:




Frame S pictured here. Frame S' moves with
$\underset{\sim}{v} = v \underset{\sim}{e}_x$ (not pictured.).

2 notions of force are useful:

4-force: $\vec{F} = \frac{d\vec{p}}{d\tau} = m\vec{a}$

Useful on theoretical grounds, since it's a 4-vector - it's a geometric object in spacetime, we know how to transform its components:

$$F^{\mu'} = \sum_{\alpha=0}^{3} \Lambda^{\mu'}{}_{\alpha} F^{\alpha}$$

However, $\vec{F}$ is not quite what we would <u>measure</u>. If we do an experiment that measures forces, we measure the rate of change of momentum per unit <u>our</u> <u>own</u> <u>time</u>, not the forced object's proper time.

In other words, we measure the 3-force:

$$\underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt}$$

3-force in frame S

$\underset{\sim}{p} = \gamma(u) m \underset{\sim}{u}$ = 3-momentum in S

$t$ = time measured in frame S

Last lecture, showed how 3-vector in frame S transforms to 3-vector in S':

$$(\underset{\sim}{F})^{x'} = \frac{(\underset{\sim}{F})^x - (v/c^2)\,\underset{\sim}{F}\cdot\underset{\sim}{u}}{1 - v(\underset{\sim}{u})^x/c^2}$$

$$(\underset{\sim}{F})^{y'} = \frac{(\underset{\sim}{F})^y}{\gamma(v)\left[1 - v(\underset{\sim}{u})^x/c^2\right]}$$

$$(\underset{\sim}{F})^{z'} = \frac{(\underset{\sim}{F})^z}{\gamma(v)\left[1 - v(\underset{\sim}{u})^x/c^2\right]}$$

(Note similarity in structure to the velocity addition formula.)

Both are important notions of force: one more naturally fits into "spacetime"-oriented phrasing, the other is more naturally suited to the "space" + "time" used by a particular observer.

How do we relate them? Again, similar to the 4-velocity. Consider spatial components first:

$$F^i = \frac{dp^i}{d\tau}$$

In frame S, $d\tau = dt/\gamma(u)$, so

$$F^i = \gamma(u)\frac{dp^i}{dt} = \gamma(u)\left(\underset{\sim}{F}\right)^i$$

Next, consider timelike component:

$$F^0 = \frac{dp^0}{d\tau} = \gamma(u)\frac{d}{dt}\left(\frac{E}{c}\right)$$

$$= \frac{\gamma(u)}{c}\frac{dE}{dt}$$

But, we showed last time that $dE/dt = \underset{\sim}{F}\cdot\underset{\sim}{u}$.

Putting this all together, we have

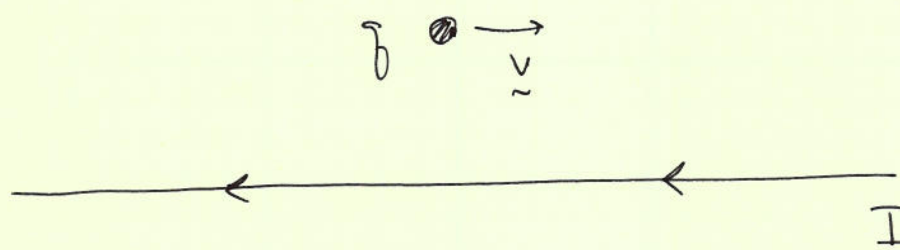
$$\boxed{\begin{aligned} F^0 &= \gamma(u)\,\underset{\sim}{F}\cdot\underset{\sim}{u} \\ F^i &= \gamma(u)\left(\underset{\sim}{F}\right)^i \end{aligned}}$$

This lets you easily translate back & forth between "spacetime" and "space" and "time" view of forces.

Key example of a force that we really care about:
Electromagnetic force.

Electromagnetic considerations are what kicked off relativity, so it clearly must work! But, there are some subtleties.

Consider a charge moving near a wire. Wire is neutral, but carries a current $I$:




Magnetic field generated by the wire: $\underset{\sim}{B} = \dfrac{\mu_0 I}{2 r \pi} \, (\hat{in})$

Electric field generated by the wire: $\underset{\sim}{E} = 0$ since it's neutral.

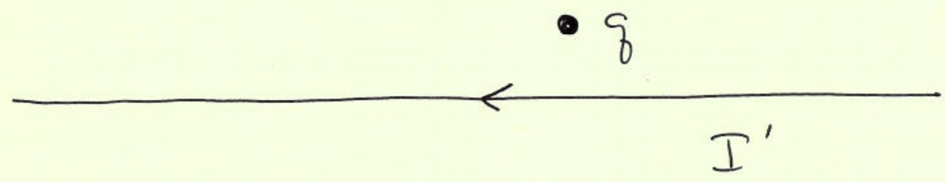
(The unit vector $(\hat{in})$ points into the page/board at the charge $q$'s location.)

This charge feels a 3-force

$$\underset{\sim}{F} = q \, \underset{\sim}{v} \times \underset{\sim}{B} = \frac{\mu_0 q v I}{2 r \pi} \, (\hat{up})$$

$\rightarrow$ Charge is repelled from the wire.

Now, goto a Lorentz frame in which $q$ is at rest:




What's the magnetic force on $q$ in this frame?

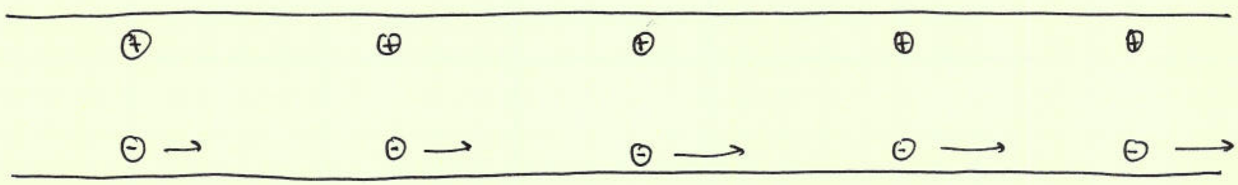
ZERO: It's not moving, so there CANNOT be a magnetic force!

There must be some force that acts on the charge, or else the results are inconsistent. Only one possibility: IN THIS NEW FRAME, THERE IS AN ELECTRIC FIELD THAT ACTS TO REPEL THE CHARGE.

Where does this electric field come from? Only possible source is the wire! Although neutral in the original "lab" frame, it cannot be neutral in the charge's rest frame.

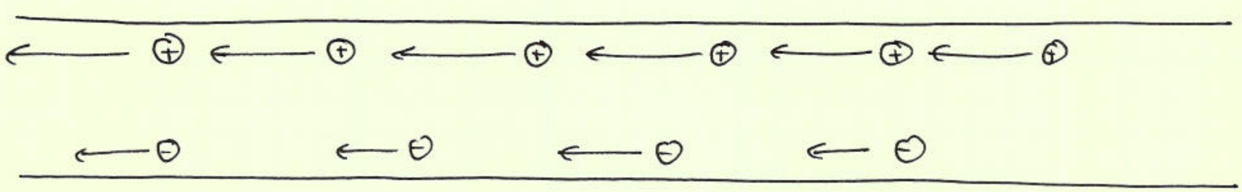
How is this possible? Consider view of wire in the two frames:

Lab frame:



Negative charges IN MOTION precisely balance positive charges.

View from charge's rest frame:



In this frame

- Positive charges are in motion: Lorentz contraction boosts their density.

- Negative charges are in different motion than in lab frame.

→ Precise balance must be disrupted!

Another way to think of it: Lorentz transform mixes up electic and magnetic field components.

Not too surprising - Lorentz transform mixed up space and time, momentum and energy - why shouldn't it mix up $\underset{\sim}{E}$ and $\underset{\sim}{B}$?

Suggests it will be useful to think of a way to represent $\underset{\sim}{E}$ and $\underset{\sim}{B}$ using a spacetime geometric object.

Can we do this with a 4-vector?

No: $\underset{\sim}{E}$ has 3 components,
$\underset{\sim}{B}$ has 3 components.

A 4-vector has 4 components - not enough to "hold" $\underset{\sim}{E}$ & $\underset{\sim}{B}$ together. (Remember, goal is to regard them as aspects of a single underlying object.)

So, we want a single mathematical structure that can hold all 6 components of $\underset{\sim}{E}$ & $\underset{\sim}{B}$.

Considerations toward this:

1. Spacetime has 4 dimensions. The objects which describe the electromagnetic field must reflect this: It should have components along these 4 dimensions.
2. 4-vectors don't work, but ...
3. 4 x 4 matrices might work!

General 4x4 matrix:

$$\mathbb{A} \doteq \begin{bmatrix} A^{tt} & A^{tx} & A^{ty} & A^{tz} \\ A^{xt} & A^{xx} & A^{xy} & A^{xz} \\ A^{yt} & A^{yx} & A^{yy} & A^{yz} \\ A^{zt} & A^{zx} & A^{zy} & A^{zz} \end{bmatrix}$$

This has 16 components ... too big.

What if we constrain it? If the matrix is symmetric, $A^{\alpha\beta} = A^{\beta\alpha}$, then the only independent components are

- 4 components on the diagonal, and
- 6 components above or below the diagonal.

→ 10 components. Still too big.

How about *anti* symmetric? $A^{\alpha\beta} = -A^{\beta\alpha}$.

For off diagonal, this is just like the symmetric case – except with a minus sign – so 6 components there.

→ The diagonal components must all be zero:

$$A^{\alpha\alpha} = -A^{\alpha\alpha} \rightarrow A^{\alpha\alpha} = 0.$$

→ An antisymmetric $4 \times 4$ matrix has *exactly* the right number of components to "hold" both the electric and magnetic fields.

Such a matrix actually represents the components of a *tensor*: A geometric object that "points" in multiple spacetime directions:

$$\mathbb{A} = A^{\mu\nu}\, \vec{e}_\mu \otimes \vec{e}_\nu$$

↑

"outer product": Means that we keep both of these unit vectors in our description of the tensor.

Key thing for us is that a tensor is itself a frame-invariant geometric object, just like a 4-vector.

As such, it is simple to see how its components transform between refrence frames:

Frame $S$ : $\quad \mathbb{A} = A^{\mu\nu}\, \vec{e}_\mu \otimes \vec{e}_\nu$

Frame $S'$ : $\quad \mathbb{A} = A^{\alpha'\beta'}\, \vec{e}_{\alpha'} \otimes \vec{e}_{\beta'}$

Given that

$$\vec{e}_{\alpha'} = \sum_{\mu=0}^{3} \Lambda^{\mu}{}_{\alpha'}\, \vec{e}_\mu$$

It is straightforward to show that

$$A^{\alpha'\beta'} = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \Lambda^{\alpha'}{}_{\mu} \Lambda^{\beta'}{}_{\nu} A^{\mu\nu}$$

Using the requirement that the geometric object $\mathbb{A}$ be the same in both representations.