Recap: Tensors: $\binom{0}{N}$ tensor is a Geometric ~~mathematical~~ object which linearly maps N vectors into Lorentz invariant scalars.

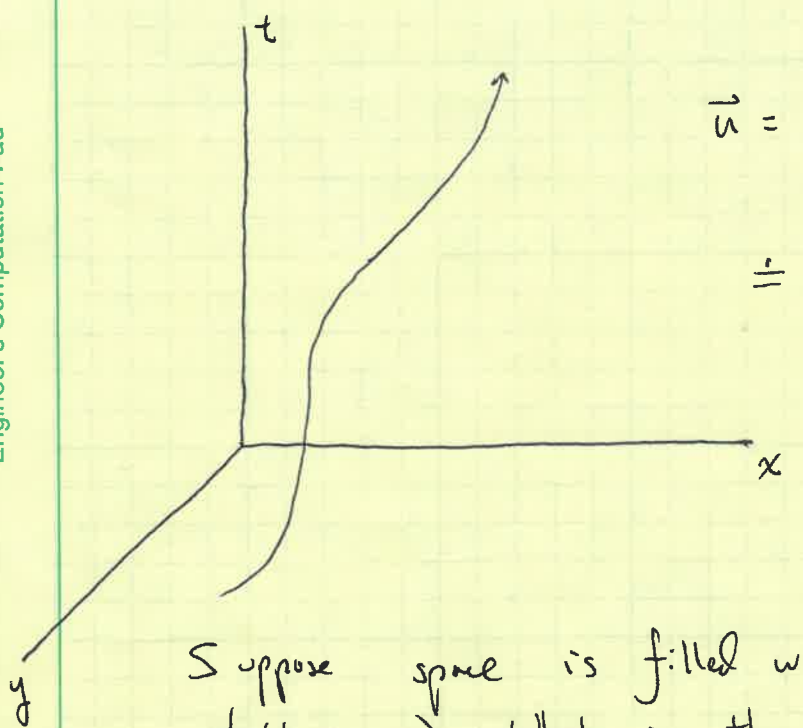
1-forms: Special name for $\binom{0}{1}$ tensors: map a single vector into scalars: $\tilde{p}(\vec{A}) = p_\alpha A^\alpha$

1-forms live in a vector space; basis for this given by basis 1-forms $\tilde{\omega}^\alpha$

Can write $\tilde{p} = p_\alpha \tilde{\omega}^\alpha$

Define basis 1-forms via $\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha{}_\beta$.

Example of a 1-form: consider some trajectory,



$$\vec{u} = \frac{d\tilde{x}}{d\tau}\Big|_{\text{along traj}}$$

$$\doteq \left(\frac{dt}{d\tau}, \frac{d\underset{\sim}{x}}{d\tau} \right)$$

τ: parameter along traj.

Proper time – time as measured along the trajectory.

Suppose space is filled with some scalar field $\phi(t,x,y,z)$. What is the rate of change of ϕ along this curve?

3-space Euclidean intuition: $$\frac{d\phi}{dt} = \frac{dx}{dt}\frac{\partial\phi}{\partial x} + \frac{dy}{dt}\frac{\partial\phi}{\partial y} + \frac{dz}{dt}\frac{\partial\phi}{\partial z}$$
$$= \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\phi$$

generalize to spacetime:

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{dx}{d\tau}\frac{\partial\phi}{\partial x} + \cdots$$
$$= u^t \frac{\partial\phi}{\partial t} + u^x \frac{\partial\phi}{\partial x} + \cdots$$

$\longrightarrow$ $$\frac{d\phi}{d\tau} = u^\alpha \frac{\partial\phi}{\partial x^\alpha} \equiv u^\alpha \partial_\alpha \phi$$
$$\equiv u^\alpha \nabla_\alpha \phi$$

components of a 1-form.

caution on this notation!

Gradient is a 1-form: $\tilde{\nabla}\phi = \tilde{d}\phi \doteq \{\partial_\alpha \phi\}$

Note: notation for directional derivative: $\boxed{\frac{d\phi}{d\tau} \equiv u^\alpha \partial_\alpha \phi \equiv \nabla_{\vec{u}} \phi}$

Notion of gradient as a 1-form gives us a nice alternate way of writing basis 1-forms: Recall basis defined via

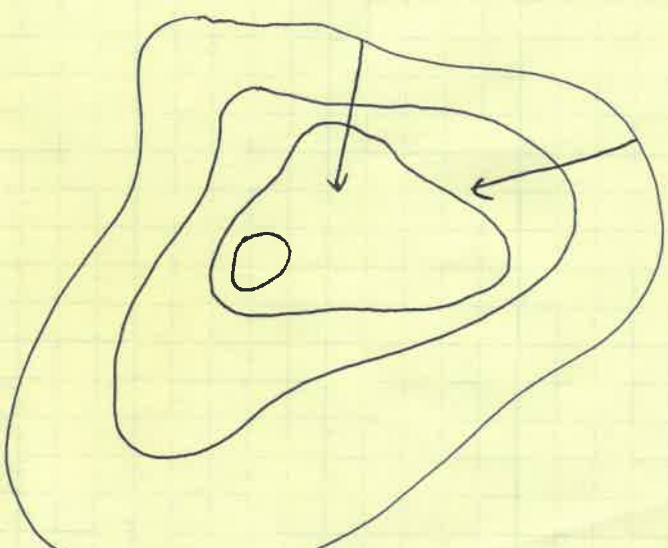
$$\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha{}_\beta$$

We also have $\partial_\beta x^\alpha = \delta^\alpha{}_\beta$

$$\equiv \tilde{d}x^\alpha(\vec{e}_\beta)$$

Thus, $\boxed{\tilde{\omega}^\alpha \equiv \tilde{d}x^\alpha}$ Basis 1-forms are gradient of coordinates.

Picture of a 1-form: Draw level surfaces of a function. The closer the surfaces, the "larger" the 1-form associated with its gradient



$h(x, y)$ = height

$\Delta\vec{x}$ = displacement vector

$\tilde{d}h$ = 1-form of height

$$\tilde{d}h(\Delta\vec{x}) = \Delta x^\alpha \partial_\alpha h$$

$$\equiv \Delta h$$

→ Proportional to number of contours pierced by arrow.

Nice aspect of this picture is that basis 1-forms $\tilde{d}x^\alpha$ are level surfaces of constant coordinate.

Will be very useful when we discuss fluxes!

Metric with both "slots" filled yields a number:

$$\vec{A}\cdot\vec{B} = \bar{\bar{\eta}}(\vec{A}, \vec{B})$$

What about with only one slot filled?

$$\bar{\bar{\eta}}(\vec{A}, -) = \text{object thats takes 1 vector \& produces a number}$$
$$= \text{1-form.}$$

Definition: $\tilde{A}(-) \equiv \bar{\bar{\eta}}(\vec{A}, -)$

Components:
$$A_\alpha \equiv \tilde{A}(\vec{e}_\alpha)$$
$$= \bar{\bar{\eta}}(\vec{A}, \vec{e}_\alpha)$$
$$= \bar{\bar{\eta}}(A^\beta \vec{e}_\beta, \vec{e}_\alpha)$$
$$\rightarrow \boxed{A_\alpha = \eta_{\alpha\beta} A^\beta}$$

Metric converts vector into 1-form by "lowering" index.

Invertible procedure: Define $\eta^{\alpha\beta}$ by $\delta^\alpha{}_\gamma = \eta^{\alpha\beta}\eta_{\beta\gamma}$. (Note, $\eta^{\alpha\beta}$ has the same matrix representation as $\eta_{\beta\gamma}$!)

Then, $A^\alpha = \eta^{\alpha\beta} A_\beta$

1-form $\tilde{A}$ is <u>dual</u> to vector $\vec{A}$.

Notice:
$$\vec{A}\cdot\vec{B} = \bar{\bar{\eta}}(\vec{A}, \vec{B}) = \tilde{A}(\vec{B}) = \tilde{B}(\vec{A})$$
$$= \eta_{\alpha\beta} A^\alpha B^\beta = A_\alpha B^\alpha = A^\alpha B_\alpha$$
$$= \eta^{\alpha\beta} A_\alpha B_\beta$$

All the same, all invariant. Distinction among objects getting silly!

Dualism tells us that vectors are themselves tensors: They map 1-forms to Lorentz scalars:

$$\vec{A}(\tilde{p}) = A^{\alpha} p_{\alpha} = A_{\alpha} p^{\alpha} = \tilde{A}(\vec{p}) = \tilde{p}(\vec{A})$$
$$= \vec{p}(\tilde{A})$$

Operationally, the distinction between "operator" and "operatee" is becoming irrelevant! Index notation highlights fundamental equality of the two species of objects.
$\rightarrow \langle \tilde{p}, \vec{A} \rangle = \langle \tilde{A}, \vec{p} \rangle$

Vector: $\binom{1}{0}$ tensor, maps $\underline{\text{one}}$ 1-form onto scalars.

$\binom{M}{0}$ tensor, maps M 1-forms onto scalars.

Further:

A tensor of type $\binom{M}{N}$ is a linear mapping of M 1-forms and N vectors to Lorentz scalars.

Such an object is represented in some frame by an object with M "upstairs" indices and N "downstairs" indices.

Silly distinction, though, since metric lets us raise + lower indices.

$$\binom{M}{N} \xrightarrow{\text{lower}} \binom{M-1}{N+1} \qquad R_{\alpha\beta\gamma\delta} = \eta_{\alpha\mu} R^{\mu}{}_{\beta\gamma\delta}$$

$$\xrightarrow{\text{raise}} \binom{M+1}{N-1} \qquad S^{\alpha}{}_{\beta\gamma} = \eta^{\alpha\mu} S_{\mu\beta\gamma}$$

Do we need a basis for tensors?

$$\bar{\bar{\eta}} = \eta_{\alpha\beta}\, \bar{\bar{\omega}}^{\alpha\beta} = \eta^{\alpha\beta}\, \bar{\bar{e}}_{\alpha\beta}$$

Imagine so. Let's examine these objects:

We know

$$\eta_{\mu\nu} = \bar{\bar{\eta}}(\vec{e}_\mu, \vec{e}_\nu)$$
$$= \eta_{\alpha\beta}\, \bar{\bar{\omega}}^{\alpha\beta}(\vec{e}_\mu, \vec{e}_\nu)$$

Requires

$$\bar{\bar{\omega}}^{\alpha\beta}(\vec{e}_\mu, \vec{e}_\nu) = \delta^\alpha{}_\mu\, \delta^\beta{}_\nu$$
$$= \tilde{\omega}^\alpha(\vec{e}_\mu)\, \tilde{\omega}^\beta(\vec{e}_\nu)$$

$\bar{\bar{\omega}}^{\alpha\beta}$ is just the "outer product" or "tensor product" of 2 basis 1-forms!

$$\bar{\bar{\omega}}^{\alpha\beta} = \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta$$

Likewise,

$$\bar{\bar{e}}_{\alpha\beta} = \vec{e}_\alpha \otimes \vec{e}_\beta$$

generalizes:

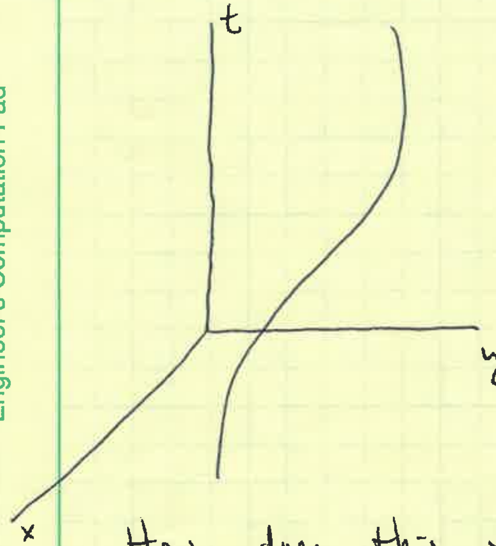
$$\bar{\bar{R}}(\_,\_,\_,\_) = R^\alpha{}_{\beta\gamma\delta}\, \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \otimes \tilde{\omega}^\delta$$

A lot of baggage! Typically stick with components: It's understood basis tensors are coming along for the ride.

Note: Using basis tensors, transformations are obvious.

$$T^{\bar{\alpha}\bar{\beta}}{}_{\bar{\gamma}\bar{\delta}}{}^{\bar{\epsilon}} = T^{\alpha\beta}{}_{\gamma\delta}{}^{\epsilon}\, \Lambda^{\bar{\alpha}}{}_\alpha\, \Lambda^{\bar{\beta}}{}_\beta\, \Lambda^\gamma{}_{\bar{\gamma}}\, \Lambda^\delta{}_{\bar{\delta}}\, \Lambda^{\bar{\epsilon}}{}_\epsilon$$

One place where it is useful to remember existence of basis tensors: Derivatives of tensors.



Trajectory parameterized by τ, defines $\vec{u} \equiv d\vec{x}/d\tau$.

Now, imagine a tensor field fills all of space:

$$\overset{=}{T} = T^{\alpha}{}_{\beta}\, \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta}$$

How does this tensor vary along the trajectory?

Start with the old fashioned definition of a derivative:

$$\frac{d\overset{=}{T}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\overset{=}{T}(\tau + \Delta\tau) - \overset{=}{T}(\tau)}{\Delta\tau}$$

$$= \frac{dT^{\alpha}{}_{\beta}}{d\tau}\, \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta}$$

basis is constant ... now! It won't be later, and that will make things a bit messy.

$$\frac{dT^{\alpha}{}_{\beta}}{d\tau} = u^{\gamma} \partial_{\gamma} T^{\alpha}{}_{\beta} = \nabla_{\vec{u}} T^{\alpha}{}_{\beta}$$

But this is also a tensor - the gradient! (components of)

$$\nabla \overset{=}{T} = \partial_{\gamma} T^{\alpha}{}_{\beta}\, \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta} \otimes \tilde{\omega}^{\gamma}$$

Then, $\dfrac{d\overset{=}{T}}{d\tau} = \vec{u} \cdot \nabla \overset{=}{T}$ ← HORRIBLE notation!

$$= \nabla_{\vec{u}} \overset{=}{T}$$

indicates contracting on gradient index.

Physics again! Quantities we've introduced so far good for kinematics of particles:

$$\vec{u} \doteq (\gamma, \gamma \underset{\sim}{v}) \qquad \vec{u}\cdot\vec{u} = -1$$

(Note: not good for a photon!)

$$\vec{p} = m\vec{u}$$

$$\doteq (E, \underset{\sim}{p}) \qquad \vec{p}\cdot\vec{p} = -m^2$$

$$= \hbar\omega(1, \hat{\underset{\sim}{k}})$$

for a photon. $\vec{u}$ not defined!

↳ $\hat{\underset{\sim}{k}}$ = dir of propagation.

How do we describe more interesting matter?

Simplest continuum form of matter: dust.

Define: A collection of non-interacting particles (no pressure!) at rest in some inertial frame.

Start by examining dust in that frame. Simplest characterization: How many particles do we have per unit volume?

$n_0 \equiv$ number density in the rest frame of dust (Density of an "element")

Now, move into a different reference frame. Total number of particles in "element" must be invariant - Lorentz scalar! - but volume contracts.

$n \equiv$ # density in new frame

$$= \gamma\, n_0 = \frac{n_0}{\sqrt{1-v^2}} \qquad v = \underset{\sim}{v}\cdot\underset{\sim}{v}$$

In this IRF, dust is moving: can define a flux — number of particles crossing unit area in unit time.

$$\underset{\sim}{n} = n\underset{\sim}{v} = \gamma n_0 \underset{\sim}{v}$$

These quantities are screaming to be combined into a 4-vector!

$$\vec{N} \doteq (n, n\underset{\sim}{v}) \doteq n_0(\gamma, \gamma\underset{\sim}{v})$$

$$= n_0\vec{u} \rightarrow \text{"Number flux 4-vector"}$$

Notice: $n_0 = \sqrt{-\vec{N}\cdot\vec{N}}$

→ rest density maps to "magnitude" of $\vec{N}$.

Systematic way to pick out flux across a surface:

Recall $\tilde{d}x^\alpha \equiv$ 1-form describing surfaces at unit ticks of coordinate x^α. Then,

$$\langle \tilde{d}x^\alpha, \vec{N} \rangle \equiv \text{flux in the } x^\alpha \text{ direction}$$

Note: $\langle \tilde{d}t, \vec{N} \rangle \equiv N^0 = n.$

Density is just "flux" in the time direction!

$\#/\text{area}\cdot\text{time} \equiv \#/\text{volume}$ when $c = 1$.

More generally: define some surface as the solution to

$$\psi(t, x, y, z) = \text{const}$$

$$\tilde{d}\psi = \text{normal 1-form to that surface}$$

$$\tilde{n} \equiv \tilde{d}\psi / \sqrt{|\tilde{d}\psi \cdot \tilde{d}\psi|} \equiv \underline{\text{unit}} \text{ normal 1-form.}$$

$$\rightarrow \langle \tilde{n}, \vec{N} \rangle \equiv \tilde{n}_\alpha N^\alpha = \text{flux in } \bar{n} \text{ direction.}$$

Conservation: The flux at must come at the expense of density already there: $\frac{\partial n}{\partial t} = - \underset{\sim}{\nabla} \cdot \underset{\sim}{n}$

$$\longrightarrow \boxed{\partial_\alpha N^\alpha = 0}$$

Integral form of conservation law: Begin by considering some particular inertial reference frame. Intuitively clear that

$$\frac{\partial}{\partial t} \int_{V^3} N^0 \, dV = - \int_{\partial V^3} \underset{\sim}{N} \cdot d\underset{\sim}{a}$$

$V^3 \equiv$ some 3-volume

$\partial V^3 \equiv$ boundary of that 3-volume

In words: the rate of change of the number of particles in a 3-volume equals the integral of the flux through the boundary of that 3-volume.

Can we put this in geometric, <u>covariant</u> language?