Another route to the field equations - somewhat more systematic, helps to clarify some of the choices we make in our derivation: Einstein field equations via the Einstein-Hilbert action.

Schematically, the action is given by integrating a Lagrangian density over all of spacetime:

$$S = \int d^4x \, \mathcal{L}$$

↳ Depends on fields you are studying.

Action must be a Lorentz scalar, so this is sometimes written

$$S = \int d^4x \sqrt{-g} \, \hat{\mathcal{L}}$$

↳ Lorentz scalar.

Extremization amounts to the requirement that the action be stationary with respect to variations in the fields:

$$\delta S = \int d^4x \left[\frac{\delta \mathcal{L}}{\delta(\text{fields})} \right] \delta(\text{fields})$$

$= 0$ to be stationary

$$\rightarrow \frac{\delta \mathcal{L}}{\delta[\text{fields}]} = 0$$

↳ Leads to Euler Lagrange equations for fields.

Simple example: A Lagrangian that depends on fields Φ^a and their derivatives $\partial_\mu \Phi^a$. (Superscript just labels different fields - not intended to be an index.)

$$\rightarrow \mathcal{L} = \mathcal{L}(\Phi^a, \partial_\mu \Phi^a)$$

Note: treating field and derivative as separate degrees of freedom.

Focus on flat spacetime: Put $\Phi^a \rightarrow \Phi^a + \delta\Phi^a$

$$\partial_\mu \Phi^a \rightarrow \partial_\mu \Phi^a + \partial_\mu(\delta\Phi^a).$$

So,
$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^a} \delta\Phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \partial_\mu(\delta\Phi^a) \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) \right] \delta\Phi^a$$

(Int. by parts)

Stationarity of action leads to Euler-Lagrange field equations:

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) = 0$$

Example:
$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\cancel{\tfrac{1}{2}}\,\eta^{\mu\nu}\partial_\nu \phi$$

$$\rightarrow \Box \phi - m^2 \phi = 0$$

$$\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu$$

Klein-Gordon equation of a massive scalar field.

We would like to develop a Lagrangian appropriate for general relativity

Principles we apply:

1. Action must be a scalar, so the Lagrangian must be a scalar (modulo $\sqrt{-g}$, depending on how we write it).

2. Lagrangian should be built out of curvature tensors — cannot be eliminated at some point by changing coordinates.

The simplest action of this type is

$$S = \frac{1}{16\pi G} \int dV^4 \, R$$

$$= \frac{1}{16\pi G} \int d^4x \, \sqrt{-g} \, R$$

$R = g^{\alpha\beta} R_{\alpha\beta}$ is the curvature scalar.

"Field" is metric. Let's vary and see what happens:

$$\delta S = \frac{1}{16\pi G} \int d^4x \, \frac{\delta}{\delta g^{\alpha\beta}} \left[\sqrt{-g} \, R \right] \delta g^{\alpha\beta}$$

$$\downarrow$$

$$\sqrt{-g} \, g^{\alpha\beta} R_{\alpha\beta}$$

Pieces we need for this:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\; g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

$$\delta R_{\alpha\beta} = \nabla_\mu\left(\delta\Gamma^\mu{}_{\alpha\beta}\right) - \nabla_\alpha\left(\delta\Gamma^\mu{}_{\mu\beta}\right)$$

$$\delta\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}\Big[\nabla_\gamma\left(g_{\alpha\delta}\, g_{\beta\lambda}\, g^{\mu\gamma}\,\delta g^{\delta\lambda}\right) - \nabla_\alpha\left(g_{\beta\gamma}\,\delta g^{\mu\gamma}\right) - \nabla_\beta\left(g_{\alpha\gamma}\,\delta g^{\mu\gamma}\right)\Big]$$

$$g^{\alpha\beta}\,\delta R_{\alpha\beta} = \nabla_\alpha\nabla_\beta\left(g^{\alpha\beta}\, g_{\mu\nu}\,\delta g^{\mu\nu} - \delta g^{\alpha\beta}\right) \equiv \nabla_\alpha V^\alpha$$

$$\rightarrow\; \delta\left(\sqrt{-g}\; g^{\alpha\beta}R_{\alpha\beta}\right) = \sqrt{-g}\Big[\Big(R_{\alpha\beta} - \frac{1}{2}\left(g^{\mu\nu}R_{\mu\nu}\right)g_{\alpha\beta}\Big)\delta g^{\alpha\beta} + \nabla_\alpha V^\alpha\Big]$$

$$\rightarrow\; \delta S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,\Big[G_{\alpha\beta}\,\delta g^{\alpha\beta} + \nabla_\alpha V^\alpha\Big]$$

$$\delta\left(\sqrt{-g}\; g^{\alpha\beta}R_{\alpha\beta}\right) = \delta\sqrt{-g}\; g^{\alpha\beta}R_{\alpha\beta} + \sqrt{-g}\,R_{\alpha\beta}\,\delta g^{\alpha\beta} + \sqrt{-g}\; g^{\alpha\beta}\,\delta R_{\alpha\beta}$$

show this 1ST

Almost good! However, we've got an extra term: $\nabla_\alpha v^\alpha$.

Carroll notes that a term of this kind can be eliminated by invoking divergence theorem, treating it as a discardable boundary term.

Very glib! (As Carroll himself notes.) Two rigorous approaches exist:

1. Palatini variation: Consider the metric + the connection to be separate degrees of freedom.

$$S = \frac{1}{16\pi G} \int d^4x \, \sqrt{-g} \, g^{\alpha\beta} R_{\alpha\beta}(\Gamma)$$

Vary with respect to g and Γ.

Result: metric compatibility is forced on us (ie, $\Gamma \equiv$ Christoffel, such that $\nabla_\gamma g_{\alpha\beta} = 0$) AND get no $\nabla_\alpha v^\alpha$ term.

2. Define boundary more carefully, treat the integral on the boundary with some sophistication. Requires understanding how one treats the curvature of a "slice" of spacetime - notion of "extrinsic curvature".

Find we must adjust the Lagrangian slightly. "Boundary terms" end up canceling $\nabla_\alpha v^\alpha$ bit.

See Appendix E of Wald.

Punchline: Extremization of "Einstein - Hilbert" action leads to

$$\frac{\sqrt{-g}}{16\pi G} \, G_{\alpha\beta} = 0$$

→ vacuum Einstein equation!

More generally, action should be EH plus "matter":

$$S = \int d^4x \, \sqrt{-g} \left[\frac{R}{16\pi G} + \mathcal{L}_M \right]$$

$$\rightarrow \delta S = \int d^4x \left[\frac{\partial(\sqrt{-g}\, R)}{\partial g^{\alpha\beta}} \frac{1}{16\pi G} + \frac{\partial(\sqrt{-g}\, \mathcal{L}_M)}{\partial g^{\alpha\beta}} \right] \delta g^{\alpha\beta}$$

$$\rightarrow \frac{\sqrt{-g}}{16\pi G} G_{\alpha\beta} + \frac{\partial(\sqrt{-g}\, \mathcal{L}_M)}{\partial g^{\alpha\beta}} = 0$$

Define: $$T_{\alpha\beta} = \frac{-2}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, \mathcal{L}_M)}{\partial g^{\alpha\beta}}$$

→ get Einstein equation!

Can we reconcile this definition of $T_{\mu\nu}$ with what is done in field theory?

Field theory: typically assume a flat background, derive a tensor $S^{\mu\nu}$ from applying Noether's theorem to a field Lagrangian & spacetime translations: e.g.,

$$S^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial^\mu \Phi)} \partial^\nu \Phi - \eta^{\mu\nu} \mathcal{L}$$

For flat spacetime & scalar fields, this agrees with our procedure for $T^{\mu\nu}$ (up to a constant).

<u>However</u>, $S^{\mu\nu}$ is not automatically symmetric, is not always gauge invariant (!), and makes no sense in curved spacetime.

Hence, our approach is preferred by the pros.

Example: $\mathcal{L}_M = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

$$S_{EM} = -\frac{1}{4} \int d^4x \underbrace{\sqrt{-g}\; g^{\alpha\mu} g^{\beta\nu}}_{\text{vary these}} \overbrace{F_{\alpha\beta} F_{\mu\nu}}^{\text{DON'T vary these!}}$$

$$\delta\left[\sqrt{-g}\; g^{\alpha\mu} g^{\beta\nu}\right] = \sqrt{-g}\; g^{\alpha\mu}\, \delta g^{\beta\nu} + \sqrt{-g}\; g^{\beta\nu}\, \delta g^{\alpha\mu}$$

$$- \frac{1}{2} \sqrt{-g}\; g^{\alpha\mu} g^{\beta\nu} g_{\sigma\rho}\, \delta g^{\sigma\rho}$$

$$\rightarrow \delta S = -\frac{1}{4} \int d^4x \sqrt{-g} \Big[F_{\alpha\beta} F^{\alpha}{}_{\nu}\, \delta g^{\beta\nu} \qquad \alpha \to \mu,\ \beta \to \alpha,\ \nu \to \beta$$

$$+ F_{\alpha\beta} F_{\mu}{}^{\beta}\, \delta g^{\alpha\mu} \qquad \mu \to \beta,\ \beta \to \mu$$

$$- \frac{1}{2} F^{\mu\nu} F_{\mu\nu}\, g_{\sigma\rho}\, \delta g^{\sigma\rho} \Big] \qquad \sigma \to \alpha,\ \rho \to \beta$$

$$= -\frac{1}{4} \int d^4x \sqrt{-g} \left[F_{\mu\alpha} F^{\mu}{}_{\beta} + F_{\alpha\mu} F_{\beta}{}^{\mu} - \frac{1}{2} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right] \times \delta g^{\alpha\beta}$$

$$\rightarrow \delta S = -\frac{1}{2} \int d^4x \sqrt{-g}\, \underbrace{\left[F_{\mu\alpha} F^{\mu}{}_{\beta} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right]}_{T_{\alpha\beta}.} \delta g^{\alpha\beta}$$

Benefit of this approach: Allows us to modify the theory in a systematic way.

Example: suppose you want "corrections" to kick in at "small" curvature:

$$S_G = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - \frac{\alpha}{R} \right)$$

Kicks in when $R \lesssim \sqrt{\alpha}$.

Note we've introduced a scale!

Vary $S_G + S_M$:

$$G_{\alpha\beta} + \frac{\alpha}{R^2}\left[R_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} R \right]$$
$$+ \alpha \left[g_{\alpha\beta} \nabla_\mu \nabla^\mu - \nabla_{(\alpha} \nabla_{\beta)} \right] R^{-2} = 8\pi G T_{\alpha\beta}$$

Carroll etal, PRD 70, 043528 (2004)

Can also imagine high order corrections:

$$S_g = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R + \beta R^2 \right]$$

Expect 2^{ND} term to kick in when $R \sim 1/\beta$.

(Some theoretical prejudice that $\beta \sim l_p^2 \sim \hbar^{-1}$.)

Appears in theories in which GR "emerges" effectively after course graining over microscopic degrees of freedom.

Another interesting possibility: a scalar field that couples to curvature:

$$S_g = \int d^4x \sqrt{-g}\, f(\phi)\, R$$

$$S_\phi = \int d^4x \sqrt{-g} \left[g(\phi) \nabla^\mu \nabla_\mu \phi - V(\phi) \right]$$

Scalar-tensor theories: see Sec 4.8 of Carroll.