Recap: RW metric + EFE → Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G\rho}{3} - \frac{K}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

Recipe to build a universe:

1. Pick spatial curvature: $k \in [-1, 0, 1] \rightarrow$ [open, flat, closed].
   Recall $K = k/R_0^2$, where R_0 is fiducial scale.

2. Pick a mixture of "species" that contribute to density:

$$\rho = \sum_i \rho_i$$

   2a. Know the ~~analogy~~ equation of state for each species:

$$P_i = w_i \rho_i$$

$$\nabla_\mu T^\mu{}_0 = 0 \rightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

$$\rightarrow \rho_i = \rho_{i0}\, a^{-n_i} \qquad n_i = 3(1+w_i)$$

3. Attack!

Key element of interest to us: How does varying these terms change the evolution of the scale factor? Would like to find an observation surrogate for $a(t)$. Can then map out growth history, use it to understand the underlying physics.

Important aside: Killing tensors.

generalization of Killing vectors: $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$

$\rightarrow \quad \nabla_{(\alpha} \xi_{\beta)} = 0$

Rank n Killing tensor satisfies $\nabla_{(\alpha} K_{\beta\gamma\delta\cdots)} = 0$

$\hookrightarrow$ n indices.

From this, it is simple to show that $K = K_{\alpha\beta\gamma\delta\cdots} u^\alpha u^\beta u^\gamma u^\delta \cdots$

is conserved on a geodesic whose tangent vector is $\vec{u}$:

$$u^\alpha \nabla_\alpha K = 0 \qquad \text{if} \qquad u^\beta \nabla_\beta u^\alpha = 0.$$

Reason we introduce this: FRW spacetimes admit the Killing tensor

$$K_{\mu\nu} = a^2(t)\left[g_{\mu\nu} + u_\mu u_\nu\right]$$

where u_μ is the 4-vel of a comoving observer.

Consider this for null geodesics: $\underset{\text{momentum}}{V^\mu} V_\mu = 0$, build

$$K = K_{\mu\nu} V^\mu V^\nu = a^2(t)\left[\cancel{g_{\mu\nu} V^\mu V^\nu}^{\,0} + (u_\mu V^\mu)(u_\nu V^\nu)\right]$$

$$\rightarrow \quad K = a^2(t)\, E^2 = \text{constant}$$

since $E = -u_\mu V^\mu$ is the energy the comoving observer assigns to V^μ.

Hence, energy measured by comoving observers varies as a^{-1}!

Use $E = \hbar\omega$:

$$\frac{\omega_{obs}}{\omega_{emitted}} = \frac{a_{emitted}}{a_{observed}} = \frac{\lambda_{emitted}}{\lambda_{observed}} \qquad a_{obs} = 1$$

Scale factor encoded in spectra!

Definition of redshift: $z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$

$$\rightarrow \quad a_{em} = \frac{1}{1+z}$$

Find well understood spectra... read out z ... know a!
Direct probe of scale factor.

Only part of the task! Need to connect scale to time if our goal is to build $a(t)$. Since we use radiation, what we want is to understand <u>distance</u> <u>measures</u>.

Simplest measure: proper distance from us to source:

$$ds^2 = -dt^2 + a^2(t) R_0^2 \left[d\chi^2 + S_k^2(\chi)\, d\Omega^2 \right]$$

Proper distance between us ($\chi = 0$) and some source at χ is just

$$D_p = a(t) R_0 \chi$$

Carroll: "Instantaneous physical distance" - emphasizes that we + source are taken to be on some moment of simultaneity.

Note: Hubble law built in! $dD_p/dt = \dot{a} R_0 \chi$

$$\rightarrow \quad v_{apparent} = \frac{\dot{a} D_p}{a} = H_0 D_p .$$

Not very useful! We and source are <u>NOT</u> at same moment.

If space were Euclidean, have at least 3 ways to determine distance:

1. Compare intrinsic luminosity of source to its apparent brightness:

$$F = \frac{L}{4\pi D_L^2}$$

measured flux → "Luminosity distance"

2. Compare physical size to angular size:

$$\Delta(\text{angle}) = \Delta L / D_A$$

→ "Angular diameter distance"

3. Compare transverse speed to apparent angular speed:

$$\dot{\Theta} = V_\perp / D_M$$

→ "Proper motion distance"

Excellent summary, with emphasis on quantities useful for observations:
David Hogg, "Distance measures in cosmology", astro-ph/9905116.

All of these distance measures develop from Friedmann #1, written as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \sum_i \rho_i$$

$i \in$ All contributions, including curvature.

Assume power law: $\rho_i = \rho_{i0}\, a^{-n_i} = \rho_{i0}\,(1+z)^{n_i}$

Then, $H(z) = H_0\, E(z)$

where $E(z) = \left[\sum_i \Omega_{i0}\,(1+z)^{n_i}\right]^{1/2}$

Common form: $E(z) = \sqrt{\Omega_\Lambda + \Omega_m (1+z)^3}$

Carroll does D_L; Pset does D_M; we do D_A.

Consider source that subtends angle $\Delta\phi$. Thanks to spherical symmetry, we orient coords so that it lives in $\theta = \pi/2$ plane. Proper size of the source is

$$\Delta L = a(t_{em}) R_0 S_k(\chi_{em}) \Delta\phi$$

from $ds^2 = -dt^2 + R_0^2 a^2(t)\left[d\chi^2 + S_k(\chi) d\Omega^2\right]$

$$\rightarrow \quad D_A \equiv \frac{\Delta L}{\Delta\phi} = a(t_{em}) R_0 S_k(\chi_{em}) = \frac{R_0 S_k(\chi_{em})}{(1+z)}$$

χ is not observable! Get rid of it using null path:

$$0 = -dt^2 + a^2(t) R_0^2 d\chi^2$$

$$\rightarrow \quad \chi(t) = \frac{1}{R_0}\int_{t_{em}}^{t_{obs}} \frac{dt}{a(t)}$$

$$= \frac{1}{R_0}\int_{a_{em}}^{a_{obs}} \frac{da}{a\dot{a}}$$

$$= \frac{1}{R_0}\int_{a_{em}}^{a_{obs}} \frac{da}{a^2 H(a)}$$

$$= \frac{1}{R_0}\frac{1}{H_0}\int_0^z \frac{dz'}{E(z')} \qquad \text{using } H(z) = H_0 E(z).$$

Finally, need to eliminate R_0 - arbitrary and unobservable.

$$\Omega_{c0} = \frac{-k}{R_0^2 H_0^2} \rightarrow R_0 = \frac{H_0^{-1}}{\sqrt{|\Omega_{c0}|}} \quad \text{for } k = \pm 1.$$

$\sin x,\ x,\ \sinh x$

Using this and the definition of $S_k(x)$, we finally get

$$D_A = (1+z)\, H_0^{-1} \times \begin{cases} |\Omega_{c0}|^{-1/2} \sin\left[|\Omega_{c0}|^{1/2} \int_0^z \frac{dz'}{E(z')} \right] & (k=+1) \\ \int_0^z dz' / E(z') & (k=0) \\ |\Omega_{c0}|^{-1/2} \sinh\left[|\Omega_{c0}|^{1/2} \int_0^z \frac{dz'}{E(z')} \right] & (k=-1) \end{cases}$$

Analyze all 3 distances, find:

$$D_L = (1+z)\, D_M = (1+z)^2 D_A$$

Our universe: Current observations point to cosmographic parameters

$\cancel{H_0 = 73 \pm 3 \text{ (km/sec) Mpc}^{-1}}$ ~~71.9 ± 2.6~~

0.0441 ± 0.003

$\cancel{\Omega_M = 0.238 \pm 0.019, \quad \Omega_b = 0.042 \pm 0.015}$ $(w=0)$

$\cancel{\Omega_\Lambda = 0.72 \pm 0.04 \quad (w=-1)}$ ↳ OUT OF DATE! See linked paper for up to date figures.

$\cancel{\Omega_c = -0.01 \pm 0.013}$

Measurements come from a variety of sources:

1. Cosmic microwave background: Size of "spots" is a standard ruler; distribution of their sizes lets us fix Ω_c.
2. Type Ia SNe: standard candle. Believed to be sufficiently well-modeled that we can determine D_L... Hence Ω_M + Ω_c.
3. Galaxy clustering: Allows us to determine how much mass is out there, fixes Ω_M.
4. Chemical abundances: Rate and relative abundance at which elements freeze out from early, high density state. Depends on density of "baryonic" matter: protons + neutrons. Nucleosynthesis theory + observations → Ω_b. CMB confirms.

5. Nearby standard candles: Probe local Hubble law → H_0.

NOTE: All entangled! Need to do a joint analysis of all data sets to understand everything.

Two mysteries.

1. Why is the universe so nearly flat?

   Recall Friedmann 1:

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G \rho}{3} - \frac{K}{a^2} \quad \text{or}$$

$$1 = \Omega + \Omega_c \qquad \Omega = \frac{8\pi G \rho}{3H^2}$$

$$\Omega_c = \frac{-K}{H^2 a^2}$$

Expectation is that early on universe was a mixture of matter ($\rho \propto a^{-3}$) and radiation ($\rho \propto a^{-4}$).

Notice: $\frac{\Omega_c}{\Omega} \propto a$ for matter

$\propto a^2$ for radiation

If universe is just slightly non-spatially flat, then the deviation from flatness should **GROW** as the universe expands!

2. Why is the CMB so homogeneous?

Our "standard" model suggests the universe was a hot plasma at very early times. Cooled as it expanded ... eventually became sufficiently cool that plasma "re" combined into neutral hydrogen. Thompson scattering of photons ended ... become free streaming.

Run "standard" model backwards, find that points on opposite sides of the sky were not in causal contact when transition occurred. HOWEVER, they are at the same temperature — appear to be in thermal equilibrium!

Solution to both problems, proposed by Guth: have a period of exponential <u>inflation</u> driven expansion at some point. Recall cosmological constant solution: $a \propto \exp\left[+\sqrt{\frac{\Lambda}{3}}\, t\right]$, $\rho_\Lambda = \text{const.}$

$$H^2 = \frac{8\pi G \rho_\Lambda}{3} - \frac{K}{a^2} \rightarrow \frac{\Omega_c}{\Omega_\Lambda} \propto a^{-2} \propto e^{-2\sqrt{\frac{\Lambda}{3}}\, t}$$

Term rapidly becomes unimportant as a inflates.

If Ω ~~is not quite 1~~ differs from 1, exponential expansion rapidly drives it close to 1.

Can likewise show that points which were in causal contact fall out after expansion ... hence, look disconnected now, but were connected long ago.

How do we make this happen?

Recall cosmological constant is equivalent to a vacuum energy density. Trick is to put the universe into a "false" vacuum":

Introduce a scalar field ϕ, satisfies evolution equation:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \; ; \; F1 \rightarrow \; H^2 = \frac{8\pi G}{3}\left(V(\phi) + \frac{\dot{\phi}^2}{2}\right)$$

Evolution viewed as ϕ "rolling" down potential, but with drag term $3H$.

"Slow roll" regime: field changes "slowly":

$$\dot{\phi}^2 \ll V(\phi), \quad |\ddot{\phi}| \ll 3H\dot{\phi}$$

Then, $V(\phi)$ acts like ρ_Λ ... universe adiabatically traces the form of the inflationary potential.

(Trick: How do we end it??)

Big game: test this.