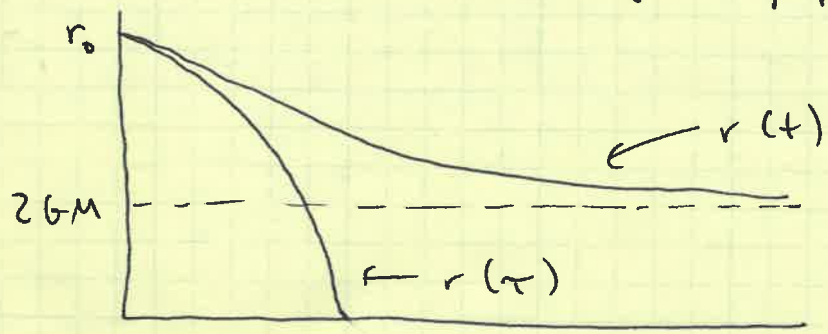
Recap: Examining a spacetime that is Schwarzschild everywhere: $ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$

- Vacuum
- Has mass M (defined by orbits)
- $r = 0$ tidal singularity
- $r = 2M$ screwy coordinate: Takes infinite coord time to reach it, but finite proper time:



What's going on?

Imagine as the body falls in that emits a radio pulse with frequency ω. Far away ($r \gg 2GM$), we have

$$p_\alpha^{radio} = \hbar\omega(-1, 1, 0, 0)$$

Recall: 1. Energy measured by observer with 4-vel $\vec{u}$ is

$$E = -\vec{p} \cdot \vec{u}$$

2. In a time-independent spacetime, p_0 = constant

For a static observer in schwarzschild,

$$u^\alpha = \left[\left(1 - \frac{2GM}{r}\right)^{-1/2}, 0, 0, 0\right]$$

Compare: Energy emitted at r vs energy observed at $r \gg 2GM$:

$$\frac{E_{obs}}{E_{emit}} = \frac{-p_\alpha u^\alpha (r\to\infty)}{-p_\alpha u^\alpha (r)} = \sqrt{1 - \frac{2GM}{r}}$$

$\rightarrow$ Pulse redshifts away! $r \rightarrow 2GM$ is a surface of infinite redshift.

Correspondingly, can show if pulses are spaced by ΔT in the ~~[illegible]~~ falling frame, measured interval far away is

$$\Delta T_\infty = \Delta T \left(1 - \frac{2GM}{r}\right)^{-1/2}$$

$\rightarrow \infty$ as body approaches $2GM$.

Recall when define time in flat region via "Einstein synchronization procedure", and coordinate t hooks this time up to strong field.

Goes to hell! We never see the body cross the event horizon because the photons we would use to view it never reach us. t sucks for strong field dynamics.

Need better coordinates to build a good view of the strong field spacetime! Much of the pathology encoded in null geodesics, so focus on them.

Radial null geodesics:

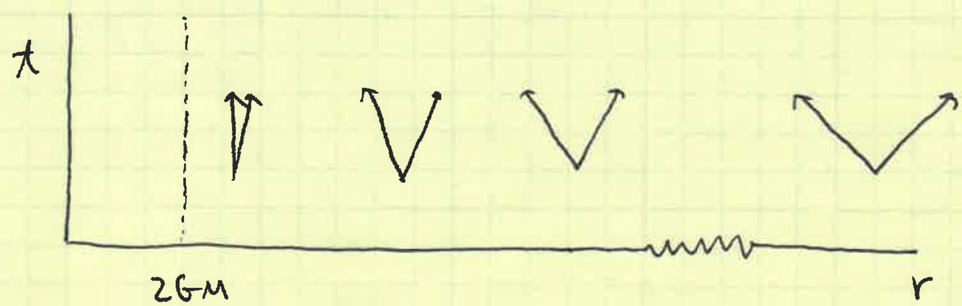
$$0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

$$\rightarrow \quad \frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1}$$

dt/dr defines the opening angle of "light cones".

Large r: $45°$ to vertical. intuitive behavior.

$r \to 2GM$: Angle collapses to $0°$:



1st step to better coordinates: Can move pathology from time to radius by choosing "tortoise coordinate":

$$dt = \pm\, dr_* \qquad \text{where}$$

$$r_* = r + 2GM \ln\left[\frac{r}{2GM} - 1\right]$$

In this coord system: light cones are $45°$ everywhere... but $r = 2GM$ maps to $r_* = -\infty$. Infinitely stretched the strong field region.

We use r_* as an intermediary for designing coords adapted to radiation:

$$v = t + r_* \equiv \text{constant for ingoing radial null rays}$$

$$u = t - r_* \equiv \text{constant for \underline{outgoing} radial null rays.}$$

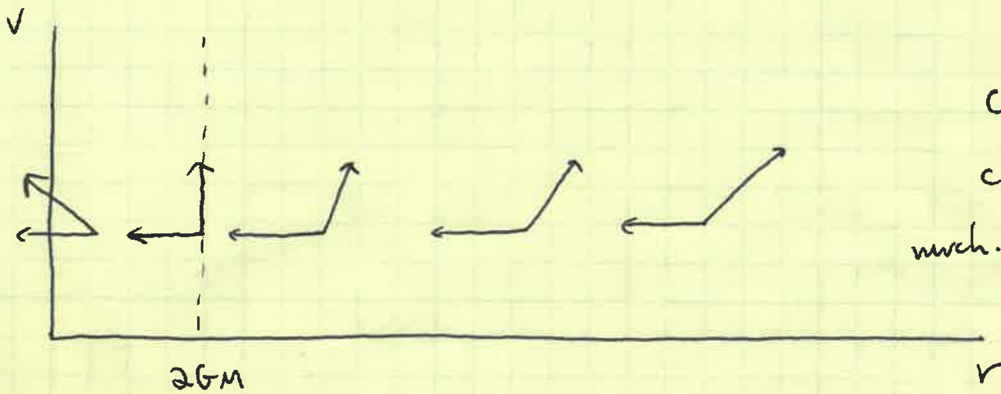
Can choose either v or u to replace Schwarzschild t.
Choose v – yields Schw. metric in "ingoing Eddington-Finkelstein" coordinates:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega^2$$

Radial null curves are given by

$$\frac{dv}{dr} = 0 \qquad \text{(ingoing)}$$

$$= 2\left(1 - \frac{2GM}{r}\right)^{-1} \qquad \text{(outgoing)}$$



cones don't close up as much... tip over one side!

(Note: Fig 5.10 of Carroll not quite drawn right.)

All timelike trajectories must lie between sides of the null cone – extreme limits of timelike behavior!

→ At $r \le 2GM$, NO timelike trajectory "gets out":
All future directed trajectories must go to smaller radii!

NOTHING that crosses $r = 2GM$ is ever coming back.
This is a HORIZON: no events inside the horizon can have a causal influence on events outside.
Hence, "Event horizon". Spacetimes with mass and event horizons are called Black holes.

Final useful but rather obtuse transformation: Put

$$v' = \exp\left[v/4GM \right]$$

$$u' = -\exp\left[-u/4GM \right]$$

Then define $T = \frac{1}{2}(v'+u')$, $R = \frac{1}{2}(v'-u')$

Simple to show

$$\left.\begin{aligned} T &= \pm\sqrt{\frac{r}{2GM}-1}\; e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \\ R &= \pm\sqrt{\frac{r}{2GM}-1}\; e^{r/4GM} \cosh\left(\frac{t}{4GM}\right) \end{aligned}\right\} r \geq 2GM$$

$$\left.\begin{aligned} T &= \pm\sqrt{1-\frac{r}{2GM}}\; e^{r/4GM} \cosh\left(\frac{t}{4GM}\right) \\ R &= \pm\sqrt{1-\frac{r}{2GM}}\; e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \end{aligned}\right\} r \leq 2GM$$

Can invert to original Schwarzschild coordinates via

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$$

$$\begin{aligned} T/R &= \tanh(t/4GM) \qquad r \geq 2GM \\ &= \coth(t/4GM) \qquad r \leq 2GM \end{aligned}$$

(T, R, θ, ϕ) are "Kruskal - Szekeres" coordinates.

Why?? First, get metric that is well-behaved everywhere

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}\left(-dT^2 + dR^2\right) + r^2 d\Omega^2$$

Notice radial null geodesics make 45° trajectories in these coordinates! $ds^2 = 0 \rightarrow dT = \pm dR$.

Second, these coordinates highlight the causal structure of the spacetime: Which events can influence which other events.

Notice: surfaces of constant r form hyperbolae in K-S:

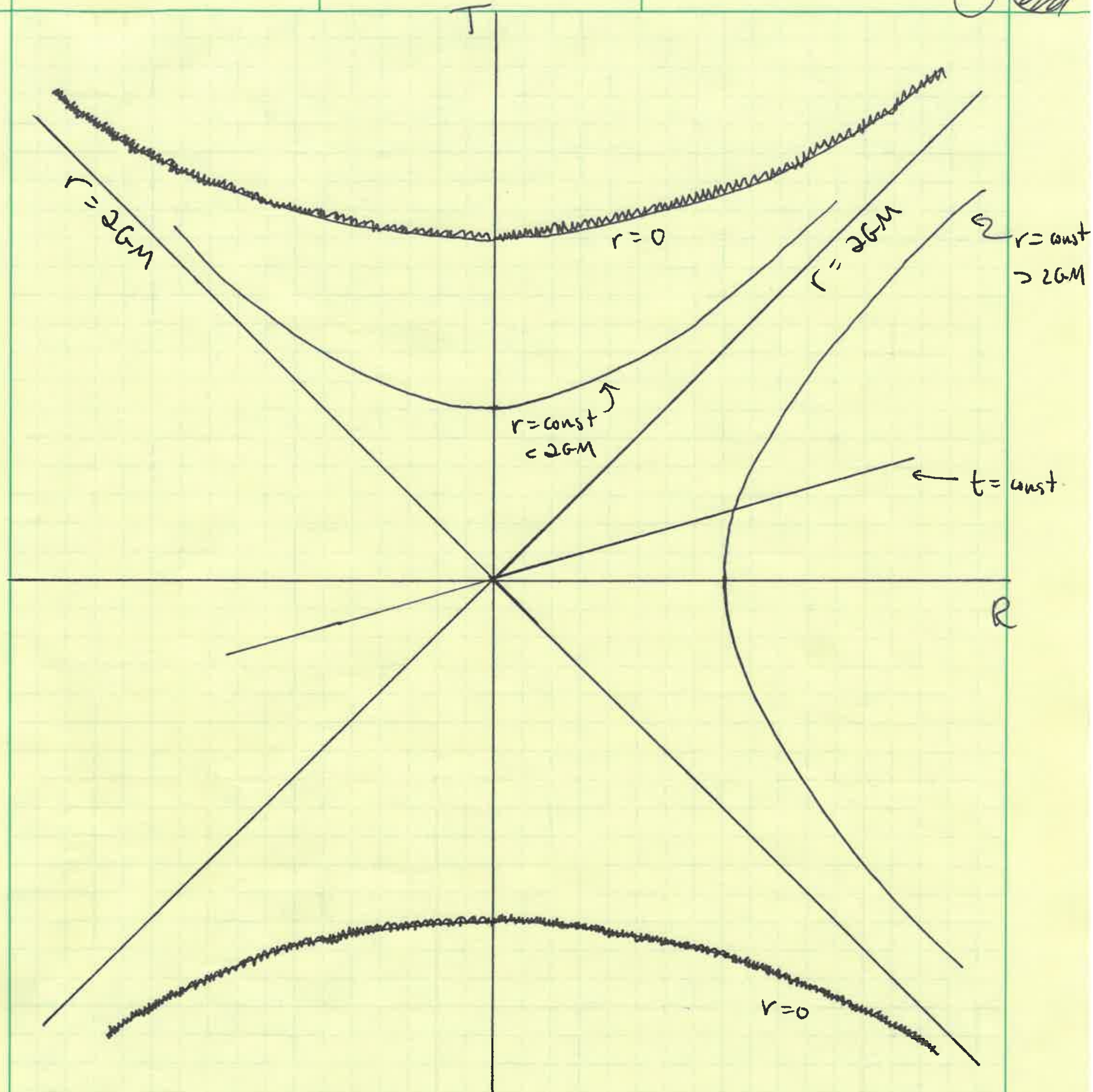
$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} = \text{const}$$

Event horizon: $r = 2GM \rightarrow T = \pm R$

Surfaces of constant t form lines:

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right) = \text{const.}$$

Note: $T = \pm R \rightarrow t \rightarrow \pm\infty$!

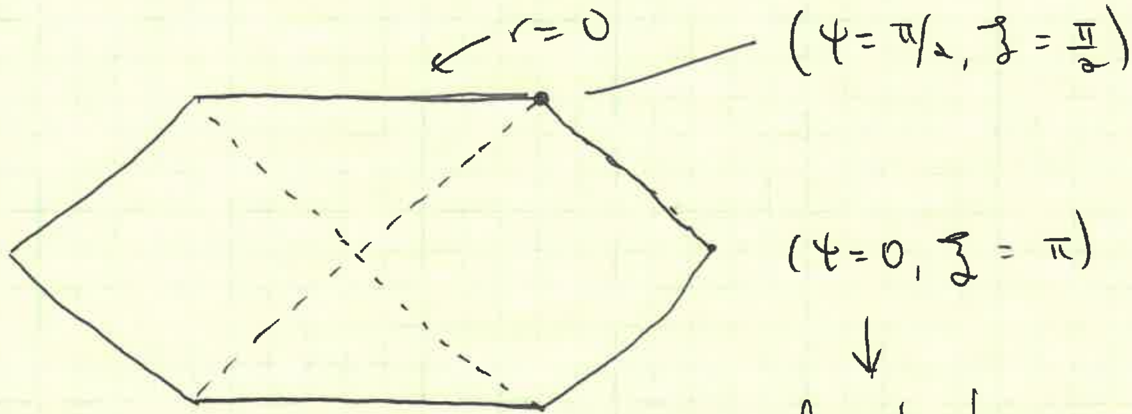


Since speed of light trajectories are 45° lines on this diagram, clear that no physical trajectory can move from r < 2GM to r > 2GM. Interior is "causally disconnected"

One final coordinate transformation:

$$v+u = \tan\left[(\psi+\xi)/2\right]$$

$$v-u = \tan\left[(\psi-\xi)/2\right]$$



$\downarrow$

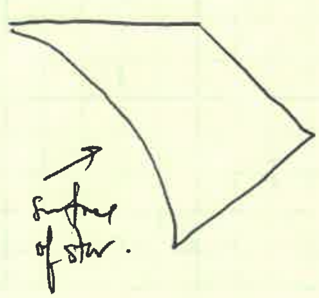
limit of points for $r\to\infty$, $t\to$ finite.

"Spacelike" Infinity

$\psi=\pi/2$, $\xi=\pi/2$: limit of points for $r\to$ finite, $t\to$ infinity "Time like" Infinity

Between them: NULL infinity.

Penrose diagram. With collapse, we get



Summary:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right) dr^2 + r^2 d\Omega^2$$

is a black hole.

Vacuum everywhere ... except for singular field equations at $r=0$.

Bad coordinates at $r = 2GM$: "Surface of infinite redshift"

"Event horizon": Once in, you never get out. All physical trajectories hit the singularity at $r=0$.

Other black holes:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) dt^2 +$$
$$\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Comes from $T^{\mu}{}_{\cancel{\mu}\nu} = \mathrm{diag}\left[-\frac{Q^2}{8\pi r^4}, -\frac{Q^2}{8\pi r^4}, \frac{Q^2}{8\pi r^4}, \frac{Q^2}{8\pi r^4}\right]$

→ Black hole metric with a Coulomb electric field!

Horizon is at root of function:

$$r_{horiz} = G\left[M + \sqrt{M^2 - Q^2}\right]$$

Notice: if $|Q| > M$, no horizon! Still a singularity at $r=0$ - "naked".

How do we know that this where the event horizon is located? In general, we don't! Most generally, need to know the entire future null history of the spacetime. Then, event horizon is the null surface which divides regions that allows light rays to reach infinity from those which do not.

If the spacetime is stationary, then we get to chose a clever radial coordinate that makes things easier. Choose r such that it asymptotes to spherical r coordinate at large radius, and such that $r =$ constant surfaces represent timelike worldtubes.

These worldtubes will be intersected by an outgoing light ray! If the spacetime has an event horizon, a well chosen r coordinate will become null at some radius r_H. The diagnostic of this:

$$(\omega^r)_\mu = \partial_\mu r$$

has a norm that goes to zero:

$$g^{\mu\nu} \partial_\mu r \, \partial_\nu r = 0$$

$$\rightarrow \quad g^{rr}(r_H) = 0 .$$

If coordinates are such that $g^{rr}(r_H) = 0$ for all time and all angles, then $r_H =$ event horizon.

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2$$
$$+ \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$
$$- \frac{4GMar\sin^2\theta}{\rho^2}\,dt\,d\phi$$

where $a = \frac{|\vec{S}|}{M}$, $\vec{S}$ = black hole spin

$$\Delta = r^2 - 2GMr + a^2$$
$$\rho^2 = r^2 + a^2\cos^2\theta$$

"Kerr black hole in Boyer-Lindquist coordinates"

Vacuum solution! Simplest derivation:
Chandrasekhar, "Mathematical Theory of Black Holes", p 273-292.

Horizon at root $\Delta = 0$: $r_H = \left[GM + \sqrt{G^2M^2 - a^2}\right]$
(Note: Require $|a| \leq GM$, or naked singularities!)

Coordinates designed to reduce to Schw. as $a \to 0$.

Noteworthy features:

1. NOT spherically symmetric! Cannot find a radial coordinate such that $g_{\phi\phi} = \sin^2\theta\, g_{\theta\theta}$.

2. Connection between t + ϕ:
$$g_{t\phi} = -\frac{2GMar\sin^2\theta}{\rho^2}$$

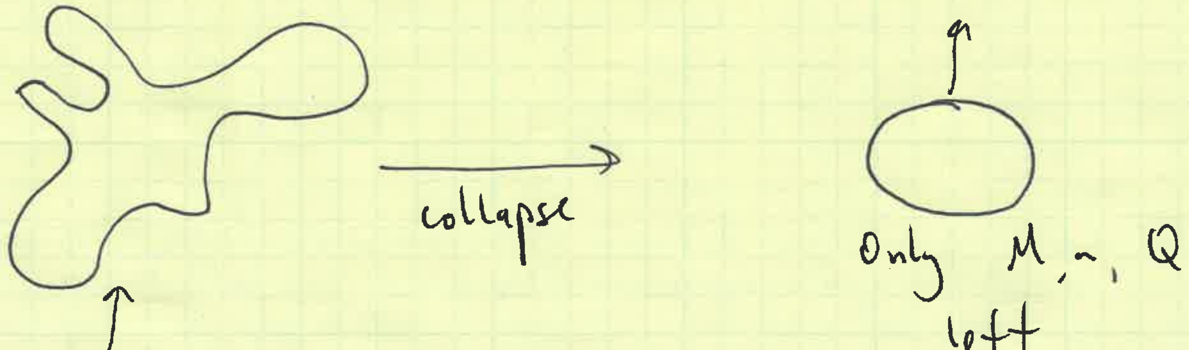
Causes "frame dragging" - a geodesic will tend to wrap around in ϕ, parallel to the hole's spin.

Charged, rotating solution also exists: "Kerr - Newman"

Remarkable Theorem: The ONLY stationary spacetimes in 3+1 dimensions with event horizons are the Kerr - Newman black holes, parametrized by mass, spin, and charge.

"stationary" means time independent. In astrophysical contexts, net charge is rapidly neutralized by environmental plasma → Kerr is most relevant.

Known as "No-hair" theorem. Enforcement is interesting: Consider collapse of some complicated object to a black hole



mess of mass,
charge, spin, fields

During collapse, strongly radiates: E&M, GWs ...
backreaction of radiation removes all structure except M, a, Q.

"Price's theorem": Everything that can be radiated IS radiated.