Motion in black hole spacetimes

Naive approach: Grind out all connection coefficients, study geodesic equation.

See Carroll Eq (5.53) for result!

Not wrong, but not useful. Better to examine symmetry and see what symmetries allow us to simplify things.

Schwarzschild:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- Spherical: I can always rotate coordinates such that they lie in the "equatorial" plane, $\theta = \pi/2$. Also guarantees that orbit remains confined to its initial plane: No way to make a torque that changes the plane's orientation.
- $\partial_t g_{\mu\nu} = 0$. Means p_t = constant → means there is a time like Killing vector → means orbits have a notion of conserved energy, $E = -p_t$.
- $\partial_\phi g_{\mu\nu} = 0$. Means p_ϕ = constant → axial Killing vector → conserved angular momentum, $L_z = p_\phi$.

Holds for Reissner-Nordstrom as well! 2ND points hold for Kerr.

4-momentum of a body moving in Schwarzschild:

$$p^\mu \doteq m\left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$$

rest mass of body

choice of orbital plane.

Look at components that are constant:

$$p_\mu = g_{\mu\nu} p^\nu$$

$$\rightarrow \quad p_t = -m\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \equiv -E$$

Defines conserved energy of the orbit.

$$\rightarrow \quad p_\phi = m r^2 \sin^2\theta \, \frac{d\phi}{d\tau} \equiv L_z$$

choose $\theta = \pi/2$

$$= m r^2 \frac{d\phi}{d\tau} \equiv L.$$

Defines conserved angular momentum of orbit.

Now, use $g_{\mu\nu} p^\mu p^\nu = -m^2$:

$$-m^2\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + m^2\left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + m^2 r^2\left(\frac{d\phi}{d\tau}\right)^2 = -m^2$$

Replace: $\frac{dt}{d\tau} = + \frac{E}{m}\left(1 - \frac{2GM}{r}\right)^{-1} \equiv \hat{E}\left(1 - \frac{2GM}{r}\right)^{-1}$

$$\frac{d\phi}{d\tau} = \frac{L}{m r^2} \equiv \frac{\hat{L}}{r^2}$$

+ multiply everything by $1 - 2GM/r$:

$$\rightarrow \quad -m^2 \hat{E}^2 + m^2 \left(\frac{dr}{d\tau}\right)^2 + m^2 \left(1 - \frac{2GM}{r}\right) \frac{\hat{L}^2}{r^2} = -m^2 \left(1 - \frac{2GM}{r}\right)$$

Rearrange:

$$\boxed{\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \hat{E}^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right) \\ &\equiv \hat{E}^2 - V_{eff}(\hat{L}, r) \end{aligned}}$$

Studying trajectories of bodies near Schw. black holes boils down to a simple recipe:

1. Pick energy per unit mass $\hat{E}$ and angular momentum per unit mass $\hat{L}$.

2. Pick initial position (r, ϕ).

3. Integrate $\left(\frac{dr}{d\tau}\right)^2 = \hat{E}^2 - V_{eff}(\hat{L}, r)$

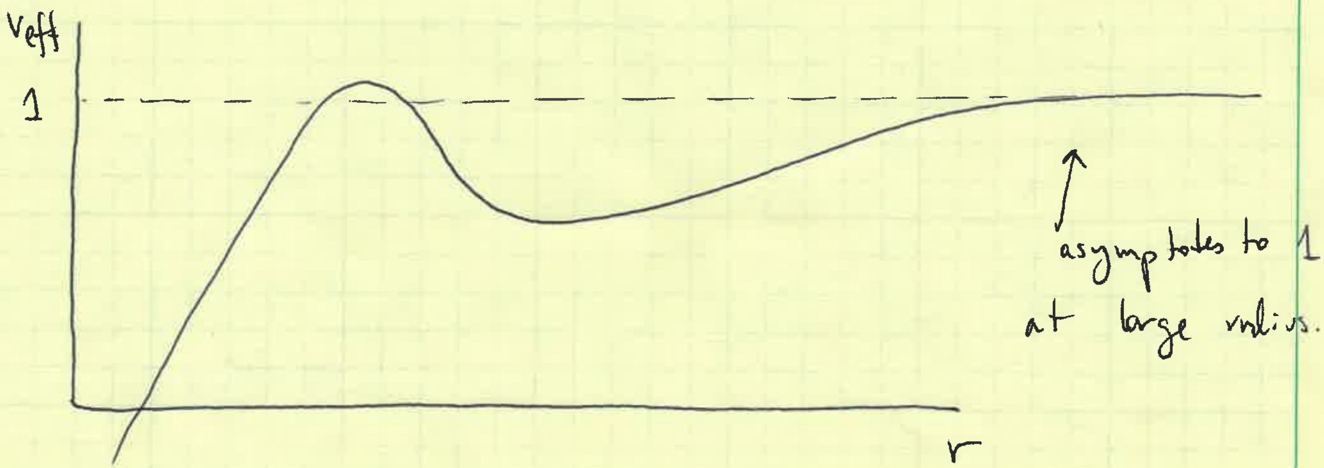
$$\frac{d\phi}{d\tau} = \frac{\hat{L}}{r^2}$$

$$\frac{dt}{d\tau} = \frac{\hat{E}}{1 - 2GM/r}$$

All of the interesting behavior is bound up in the function

$$V_{eff} = \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right)$$

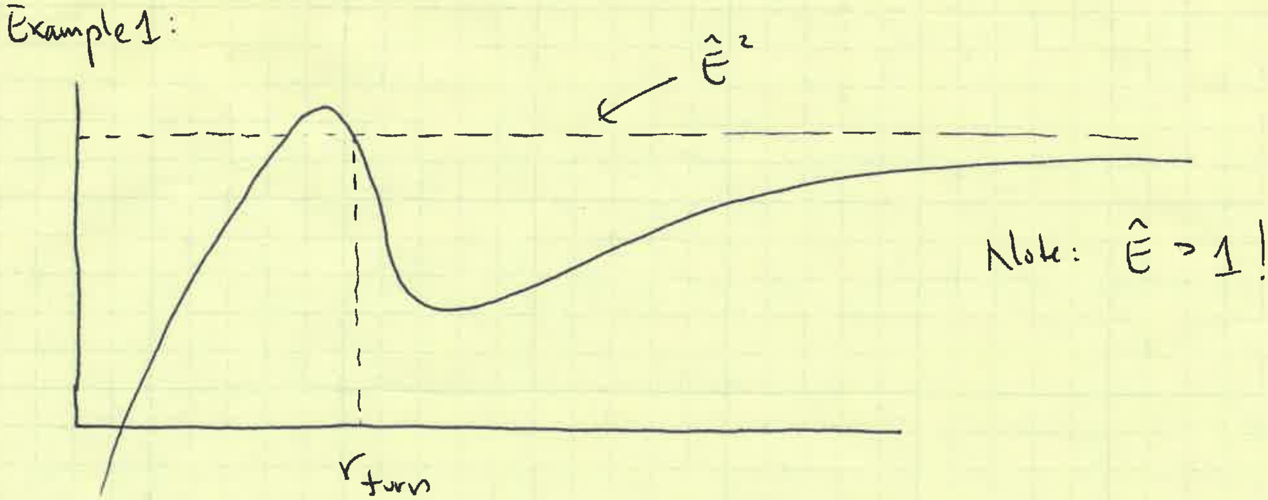
Typical shape of "potential" given $\hat{L}$:



$\hat{E}^2$ has the same units as V_{eff}. Can plot them together to understand radial motion.

$$\left(\frac{dr}{d\tau}\right)^2 = \hat{E}^2 - V_{eff}(r, \hat{L})$$

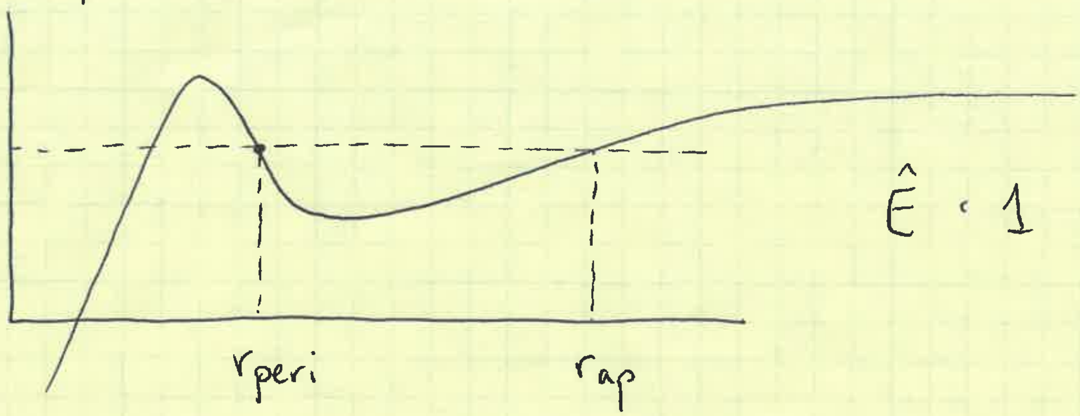
Example 1:



For this $\hat{E}$, body comes in from infinity, turns around at r_{turn} [$dr/d\tau = 0$ at turning point: $\hat{E} = \sqrt{V_{eff}(r_{Turn})}$], goes back out to infinity.

Relativistic generalization of "hyperbolic orbit".

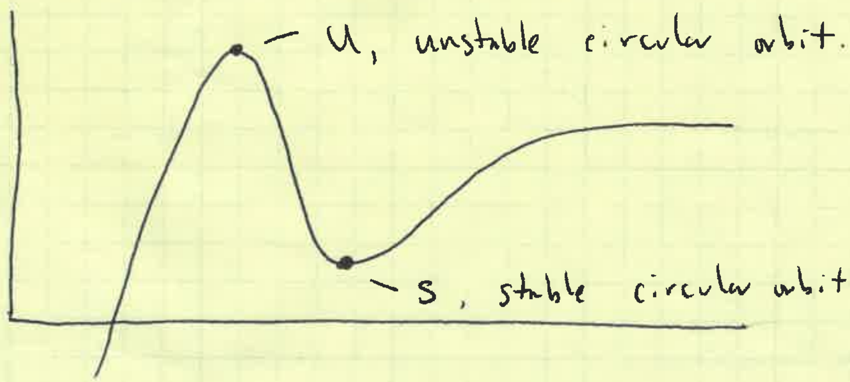
Example 2:



$\hat{E} < 1$

Orbits only have $\hat{E}^2 > V_{eff}$ in the range $r_p \le r \le r_a$; otherwise, $(dr/d\tau)^2 < 0$.

These orbits give relativistic generalization of eccentric orbits. Suggests that for each potential there should be circular orbits as well:



Conditions for circularity:

$$\frac{dr}{d\tau} = 0 \rightarrow \hat{E} = \sqrt{V_{eff}}$$

$$V_{eff} = \text{min or max} \rightarrow \frac{\partial V_{eff}}{\partial r} = 0$$

$$\frac{\partial V_{eff}}{\partial r} = 0 \rightarrow 2\hat{L}^2 (r - 3GM) = 2GMr^2$$

$$\rightarrow \boxed{\hat{L} = \pm\sqrt{\frac{GMr}{1 - 3GM/r}}}$$

Asymptotes to $\hat{L} = \pm\sqrt{GMr}$ for large r → same as Newtonian value.

Enforce $\hat{E} = \sqrt{V_{eff}}$ for this value of $\hat{L}$:

$$\boxed{\hat{E} = \frac{1 - 2GM/r}{\sqrt{1 - 3GM/r}}}$$

Notice. $\hat{E} < 1$. Intuitively, can regard

$$\hat{E} = \frac{E}{m} = \frac{(m + E_{kinetic} + E_{potential})}{m}$$

Bound orbits have $|E_{pot}| > E_{kin}$; and $E_{pot} < 0$.

Hence, $\hat{E} < 1$. Note that

$$\hat{E} \rightarrow 1 - \frac{GM}{2r} \quad \text{as} \quad r \rightarrow \infty \;-$$

Newtonian value for circular orbits.

Useful quantities: the angular velocity of circular orbits as seen by distant observers.

$$\Omega \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{\hat{L}/r^2}{\hat{E}}\left(1 - \frac{2GM}{r}\right)$$

$$= \frac{1}{r^2}\cdot\sqrt{GMr} = \sqrt{\frac{GM}{r^3}} \rightarrow \text{same as Newtonian value!}$$

Orbits are stable if $\partial^2 V/\partial r^2 > 0$, unstable if $\partial^2 V/\partial r^2 < 0$.

What if stable & unstable orbits coincide? Only have a marginally stable orbit:

$$\frac{\partial^2 V}{\partial r^2} = 0 = \frac{6\hat{L}^2(r-4GM) - 4GMr^2}{r^5}$$

$$= \frac{2GM(r-6GM)}{r^3(r-3GM)} \qquad \text{(subst. our solution for } \hat{L}\text{)}$$

$$\rightarrow \quad \boxed{r = 6GM}$$

No stable circular orbit exists inside $r = 6GM$!

Very non-Newtonian behavior.

Photon orbits, Schwarzschild:

Recall that for null geodesics, we define the affine parameter such that $\vec{p} = d\vec{x}/d\lambda$. Null means $\vec{p}\cdot\vec{p} = 0$; leads to a rather different potential describing the motion.

$$\vec{p}\cdot\vec{p} = 0 \rightarrow 0 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

Still have conserved energy + angular momentum:

$$E = -p_t = \left(\frac{dt}{d\lambda}\right)\left(1 - \frac{2GM}{r}\right)$$

$$L = p_\phi = r^2 \frac{d\phi}{d\lambda}$$

$$\rightarrow \left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{L^2}{r^2}\left(1 - \frac{2GM}{r}\right)$$

$$\frac{E}{1 - 2GM/r} = \left(\frac{dt}{d\lambda}\right) = \cancel{\frac{E}{1 - 2GM/r}} \qquad \left(\frac{d\phi}{d\lambda}\right) = \frac{L}{r^2}$$

Problem: trajectory should not depend on energy!

Geometric optics limit: Gamma rays + radio waves follow the same null paths.

Redefine affine parameter: Put $\lambda \to L\lambda$, define

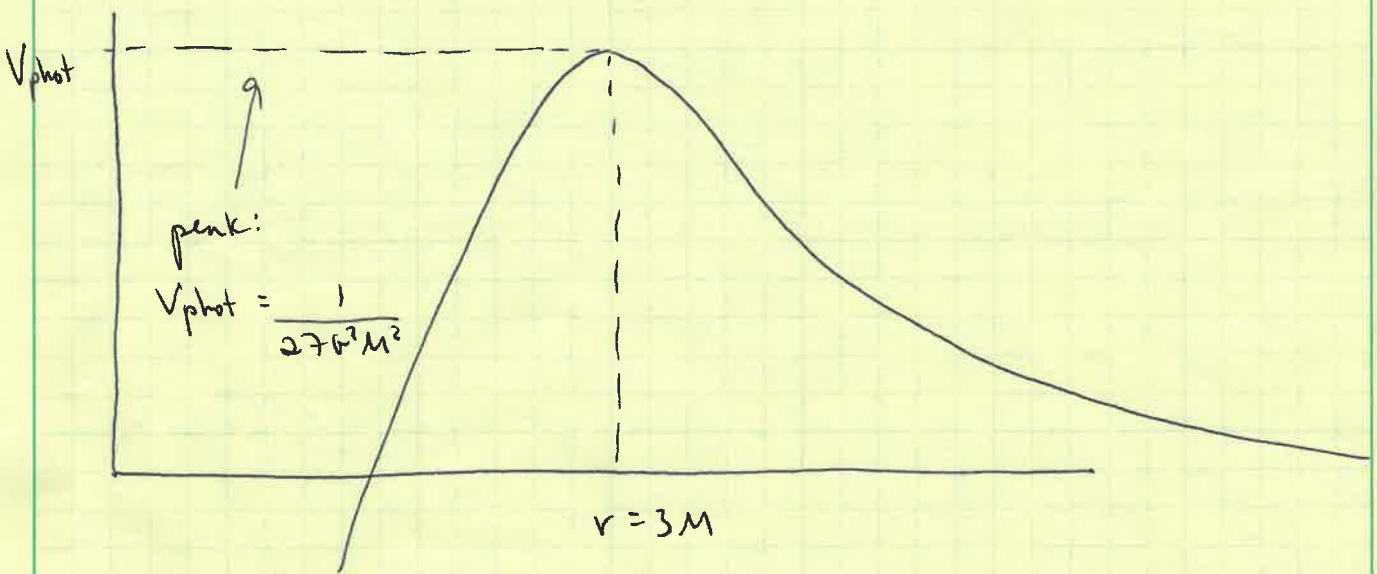
$b \equiv L/E$ "Impact parameter"

$$\rightarrow \quad \left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

$$= \frac{1}{b^2} - V_{phot}(r)$$

$$\frac{dt}{d\lambda} = \frac{1}{b(1-2GM/r)} , \quad \frac{d\phi}{d\lambda} = \frac{1}{r^2}$$

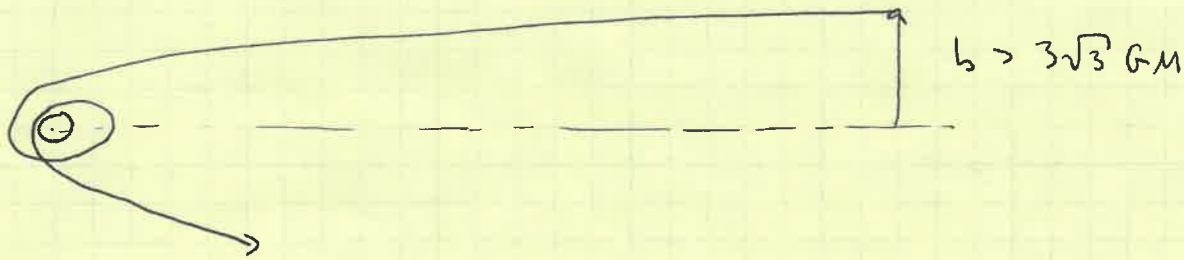
The photon potential doesn't depend on any parameters:



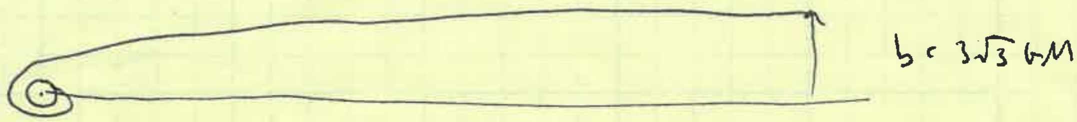
Range of motion follows by comparing $1/b^2$ with $V_{phot}(r)$, particularly the peak.

Results:

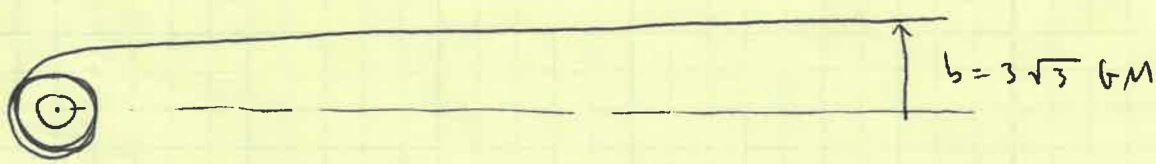
$b > 3\sqrt{3}\,MG$: Ray comes in, bends, goes back out.



$b < 3\sqrt{3}\,MG$: Ray goes into event horizon.



$b = 3\sqrt{3}\,GM$: Ray is pulled into a circular orbit of radius $3\,GM$



"Capture cross section": $\sigma = 27 G^2 M^2$.