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Information capacity of bosonic channels

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The capacity C for transmitting classical information is investigated for a vast class of noisy bosonic channel models. In the purely lossy case (in which signal photons may be lost during the communication) we calculate the exact value of C: we show that quantum entanglement is not necessary to achieve capacity and that a "classical" encoding procedure employing coherent states suffices. This means that the Holevo information of this channel is not superadditive. We also analyze some active channel models, in which noise photons are injected from an external environment or the signal is amplified with unavoidable quantum noise. In this case we provide some upper and lower bounds for the capacity, which are tight in many regimes.

A principal goal of quantum information theory is evaluating the information capacities of important communication channels. At present—despite the many efforts that have been devoted to this endeavor and the theoretical advances they have produced [1]—exact capacity results are known for only a handful of channels. Here we consider noisy Gaussian bosonic channels, and we study their *classical* capacity C, i.e., the number of bits that can be communicated reliably per channel use. This is a broad family of realistic channel models in which bosons, in particular electromagnetic radiation fields, are the information carriers and Gaussian input states evolve into Gaussian output states through interaction with a noisy environment. Among these channels we find accurate models for fiber-optic communication in the absence of nonlinear effects, and for free space radio communication in the presence of thermal noise. This study connects to a research line that began with the capacity derivation for the lossless (and hence noiseless) bosonic channel [2, 3], has continued with the capacity calculation for particular encodings (see for instance [4–6]), and only very recently has yielded the capacity of the lossy bosonic channel [7], on which this presentation is in part based.

In Sect. I, we analyze the purely lossy case, in which the exact capacity is derived. Here we show that entanglement is not required to achieve the capacity, proving that this channel's Holevo information is not superadditive. In Sect. II, we report our recent progress for a more general class of bosonic channels that include interaction with a thermal reservoir and amplification of the signal. In this case the value of C is not known, but useful upper and lower bounds are provided.

I. LOSSY BOSONIC CHANNEL

The lossy bosonic channel consists of a collection of bosonic modes that lose energy en route from the transmitter to the receiver. The received state is thus different from the transmitted state, and quantum mechanics requires that there be an accompanying quantum noise source. In this section we obtain the value of C when this noise source is in the vacuum state, i.e., when it injects the minimum amount of noise into the receiver. Our derivation proceeds by developing an upper bound for C and then showing that this bound coincides with the lower bound on C reported in [6, 8]. Our upper bound results from comparing the capacity of the lossy channel to that of the lossless channel whose average *input* energy matches the average *output* energy of the lossy case [4]. This argument is analogous to the derivation of the classical capacity of the erasure channel [9]. The lower bound comes from calculating the Holevo information for appropriately coded coherent-state inputs. Thus, because the two bounds coincide, we not only have the capacity of the lossy bosonic channel, but we also know that capacity can be achieved by transmitting coherent states.

The classical capacity of a channel can be expressed in terms of the Holevo information [10]

$$\chi(p_j,\sigma_j) \equiv S(\sum_j p_j\sigma_j) - \sum_j p_j S(\sigma_j) \;, \qquad (1)$$

where p_j are probabilities, σ_j are density operators and $S(\varrho) \equiv -\mathrm{Tr}[\varrho \ln \varrho]$ is the von Neumann entropy. Since it is not known whether χ is additive or whether entanglement among successive channel uses may play a role, C must be calculated by maximizing the Holevo information over successive uses of the channel. Hence, we must use $C = \sup_n (C_n/n)$ with

$$C_n = \max_{p_j,\sigma_j} \chi(p_j, \mathcal{N}^{\otimes n}[\sigma_j]) \;, \qquad (2)$$

where the states σ_j live in the Hilbert space $\mathcal{H}^{\otimes n}$ of n successive uses of the channel and $\mathcal{N}$ is the completely positive map that describes the channel. In our case, $\mathcal{H}$ is the Hilbert space associated with the bosonic modes used in the communication and $\mathcal{N}$ is the loss map. Because $\mathcal{H}$ is infinite dimensional, C_n diverges unless the maximization in Eq. (2) is constrained: here we assume that in each of the n realizations of the channel, the mean photon number of the k-th mode at the input is a fixed quantity N_k. For multimode bosonic channels, $\mathcal{N}$ is given by $\bigotimes_k \mathcal{N}_k$, where $\mathcal{N}_k$ is the loss map for the kth mode, which can be obtained tracing away the vacuum noise mode b_k from the Heisenberg evolution

$$a'_k = \sqrt{\eta_k}\; a_k + \sqrt{1-\eta_k}\; b_k \;, \tag{3}$$

with a_k and a'_k being the annihilation operators of the input and output modes and $0 \leq \eta_k \leq 1$ is the mode transmissivity (quantum efficiency).

Our main result is that the capacity of the lossy bosonic channel, in nats per channel use, is

$$C = \sum_k g(\eta_k N_k) \;, \tag{4}$$

where $g(x) \equiv (x+1)\ln(x+1) - x\ln x$. The right-hand side of Eq. (4) was shown in [8] to be a lower bound for C by generalizing the narrowband analysis of [6]. This expression was obtained from Eq. (2) by calculating χ for $n = 1$ under the following encoding: in every mode k we use a mixture of coherent states $|\mu\rangle_k$ weighted with the Gaussian probability distribution

$$p_k(\mu) = \exp[-|\mu|^2/N_k]/(\pi N_k) \;. \tag{5}$$

This corresponds to feeding the channel the input state

$$\varrho = \bigotimes_k \int d\mu\; p_k(\mu)\; |\mu\rangle_k\langle\mu| \;, \tag{6}$$

which is a thermal state that contains no entanglement or squeezing. The right-hand side of Eq. (4) is also an upper bound for C. To see that this is so, let $\bar{p}_j$, $\bar{\sigma}_j$ be the optimal encoding on n uses of the channel, which gives the capacity C_n of Eq. (2). The definition of χ and the subadditivity of the von Neumann entropy allow us to write

$$C_n \leqslant S(\mathcal{N}^{\otimes n}[\bar{\sigma}]) \leqslant \sum_{l=1}^{n} \sum_k S(\mathcal{N}_k[\varrho_k^{(l)}]) \;, \tag{7}$$

where $\bar{\sigma} \equiv \sum_j \bar{p}_j \bar{\sigma}_j$ and $\mathcal{N}_k[\varrho_k^{(l)}]$ is the reduced density operator of the kth mode in the lth realization of the channel, which is obtained from $\mathcal{N}^{\otimes n}[\bar{\sigma}]$ by tracing over all the other modes and over the other $n-1$ channel realizations. The first inequality in Eq. (7) comes from bounding C_n by the amount of information that can be transmitted through a lossless channel with input state $\mathcal{N}^{\otimes n}[\bar{\sigma}]$, viz., the output of the lossy channel with optimal input state $\bar{\sigma}$ [4]. Now let $N_k^{(l)}$ be the average photon number for the state $\varrho_k^{(l)}$; $\{N_k^{(l)}\}$ must satisfy the average input photon constraint, so that $N_k^{(l)} = N_k$ for all l, k. Moreover, the loss will leave only $\eta_k N_k^{(l)}$ photons, on average, in the corresponding output state $\mathcal{N}_k[\varrho_k^{(l)}]$. This implies that

$$S(\mathcal{N}_k[\varrho_k^{(l)}]) \leqslant g(\eta_k N_k^{(l)}) \;, \tag{8}$$

where the inequality follows from the fact that the term on the right is the maximum entropy associated with states that have $\eta_k N_k^{(l)}$ photons on average [3, 11]. Introducing Eq. (8) into (7), we obtain the desired result

$$C_n \leqslant \sum_{l=1}^{n} \sum_k g(\eta_k N_k^{(l)}) = n \sum_k g(\eta_k N_k) \;. \tag{9}$$

Because Eq. (9) holds for any n, we conclude that the right-hand side of (4) is indeed also an upper bound for C.

A. Discussion

Some important consequences derive from our analysis. First, capacity is achieved by a single use of the channel ($n = 1$) employing random coding—factorized over the channel modes—on coherent states as shown in Eq. (6). This means that, at least for this channel, entangled codewords are not necessary and that the Holevo information is not superadditive. Notice that the lossy bosonic channel can accommodate entanglement among successive uses of the channel, as well as entanglement among different modes in each channel use. Surprisingly, neither of these two strategies is necessary to achieve capacity. Nor is it necessary to use any non-classical state, such as a photon number state or a squeezed state, to achieve capacity; classical (coherent state) light is all that is needed. Classical light suffices because the loss map $\mathcal{N}$ simply contracts coherent-state codewords in phase space toward the vacuum state. Coherent states retain their purity in this process, and hence the non-positive part of the Holevo information—the second term of the right-hand side of Eq. (1)—retains its maximum value of zero. Despite the preceding properties, quantum effects are relevant to communication over the lossy bosonic channel. For example, our proof does not exclude the possibility of achieving capacity using quantum encodings, and such encodings may have lower error probabilities, for finite-length block codes, than those of the capacity-achieving coherent state encoding. This is certainly true for the lossless case. In particular, it was already known that C can be achieved with a number-state alphabet [2, 3]; our work shows that there is also a coherent-state encoding that achieves capacity for this case. These two procedures employ the same average input state, Eq. (6). However, the probability of the receiver confusing any two distinct finite-length number state codewords is zero in the lossless case, whereas it is positive for all pairs of finite-length coherent-state codewords.

How well can we approach this capacity using conventional decoding procedures? Using the coherent-state encoding of Eq. (6) with either heterodyne or homodyne detection, the amount of information that can be reliably transmitted is

$$I = \sum_k \xi \ln(1 + \eta_k N_k/\xi^2) \;, \tag{10}$$

where $\xi = 1/2$ for homodyne and $\xi = 1$ for heterodyne. Equation (10) has been obtained by summing over k the Shannon capacities for the appropriate detection procedure [3]. In general $I < C$: heterodyne or homodyne detection cannot be used to achieve the capacity. However, heterodyne is asymptotically optimal in the large photon-number limit for all modes, $N_k \to \infty$ for all k, because $g(x)/\ln(x) \to 1$ as $x \to \infty$.

B. An example: frequency-independent loss

As an application of the previous results, we now consider a broadband channel with uniform transmissivity, $\eta_k = \eta$, that employs a set of frequencies $\omega_k = k\ \delta\omega$ for $k \in \mathbb{N}$. Moreover, instead of fixing the average photon number in each mode, we will introduce a constraint on the average energy $\mathcal{E}$ of the input state, i.e. $\sum_k \hbar\omega_k N_k = \mathcal{E}$. In this case, using the Lagrange multiplier techniques, we find [7, 8]

$$C = \sqrt{\eta}\sqrt{\frac{\pi\mathcal{P}}{3\hbar}}\mathcal{T} \ , \tag{11}$$

where $\mathcal{T} = 2\pi/\delta\omega$ is the transmission time, and $\mathcal{P} = \mathcal{E}/\mathcal{T}$ is the average transmitted power. Equation (11) was derived for the lossless case ($\eta = 1$) in Ref. [2] and was shown to provide a lower bound on C in Ref. [8]. In order to show that the right-hand side of Eq. (11) is also an upper bound, consider the lossless broadband channel in which the average *input* power is equal to $\eta\mathcal{P}$, viz., the average *output* power of the lossy channel. According to [2], the capacity of this channel is $(\sqrt{\pi\eta\mathcal{P}/3})\mathcal{T}$, which coincides with the right-hand side of Eq. (11). The lossless channel cannot have a lower capacity than the lossy channel, because both have the same average received energy, and the set of receiver density operators achievable over the lossy channel is a proper subset of those achievable in the lossless system [4]. This implies that the lossless channel's capacity cannot be less than that of the lossy channel, thus completing the proof.

II. ACTIVE BOSONIC CHANNELS

In this section we study the capacity of the bosonic channels whose transmitted state, in addition to suffering loss is subject to photon injection from an external source. The noise map $\mathcal{N}$ for the channels of interest can be derived by replacing the evolution (3) with

$$a'_k = \sqrt{\eta_k}\ a_k + \sqrt{|1-\eta_k|}\ n_k + m_k \ . \tag{12}$$

where η_k is a coupling coefficient and m_k is a classical noise term. The noise operator n_k is given by b_k when $0 < \eta_k < 1$, and $b_k^\dagger$ when $\eta_k > 1$, where b_k is the annihilation operator of the environment which is in a thermal-like state (i.e., a zero-mean isotropic Gaussian state) with average photon number $\bar{N}_k$. The classical noise term m_k is a random complex variable which is distributed with an isotropic Gaussian probability with zero-mean and mean-square value $\bar{M}_k$. This model includes a wide variety of interesting channels. With $0 < \eta_k < 1$ we have propagation loss, η_k being the modal transmissivity. In this case $\bar{N}_k = \bar{M}_k = 0$ represents the lossy bosonic channel whose capacity was analyzed in Sect. I, and positive $\bar{N}_k$ with $\bar{M}_k = 0$ models the lossy channel in thermal equilibrium at a temperature $T > 0$. With $\eta_k > 1$ we have an amplifying channel with gain η_k. When $\bar{N}_k = 0$ the creation operator term in Eq. (12) injects the minimum quantum noise needed to enforce the Heisenberg uncertainty principle at the channel's output, while positive $\bar{N}_k$ models amplifier excess noise. For any value of η_k, a positive $\bar{M}_k$ accounts for the presence of classical additive isotropic Gaussian noise in the channel.

A simple upper bound for the capacity C follows from the Sect. I encoding, i.e. the one given by Eqs. (5) and (6). For this encoding, Eq. (12) implies that the resulting output state will be a coherent state $|\nu_k\rangle$, with ν_k being again a zero-mean isotropic Gaussian random variable whose mean-square value is now $\eta_k N_k + \bar{N}_k^o$, where $\bar{N}_k^o \equiv |1-\eta_k|\langle n_k^\dagger n_k\rangle + \bar{M}_k$. Using this property, and the fact [6] that $g(M)$ is the von Neumann entropy of a zero-mean isotropic Gaussian mixture of coherent states with mean-square value M, the Holevo information of the output can be calculated and shown to be

$$C \geqslant \sum_k \left[g(\eta_k N_k + \bar{N}_k^o) - g(\bar{N}_k^o) \right] \ . \tag{13}$$

In order to show that the right-hand side of Eq. (13) is the capacity itself, one would have to show that it is also an upper bound. As discussed in Ref. [12], this problem essentially reduces to proving that input coherent states give the minimal entropy at the output of the channel: this is quite reasonable, at least for $0 < \eta_k < 1$, because coherent states retain their purity after through a beam splitter transformation. As yet, however, we have not proved the preceding minimum-entropy conjecture, but we present below some preliminary results in support of this claim.

A. Channel output minimal entropy

For brevity, we will limit our treatment here to the classical-noise, single-mode channel, i.e., the channel that is described by the evolution (12) with $\eta = 1$, which gives rise to the unital map

$$\rho' \equiv \mathcal{N}(\rho) = \int d^2\mu\ P_{\bar{M}}(\mu)\ D(\mu)\rho D^\dagger(\mu), \tag{14}$$

where $P_{\bar{M}}(\mu) \equiv \exp[-|\mu|^2/\bar{M}]/(\pi\bar{M})$ and $D(\mu)$ is the displacement operator of the mode a. Since the map of Eq. (14) is covariant under the action of displacement operators, every coherent state has the same output entropy $g(\bar{M})$, which can be easily calculated as the output

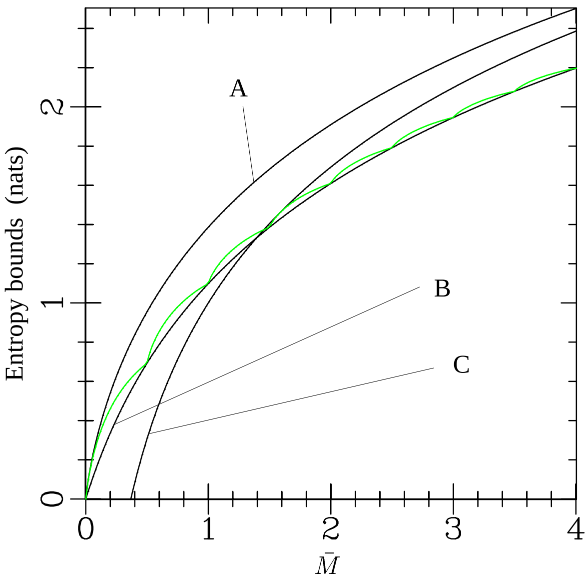

FIG. 1: Bounds for the minimal output entropy as a function of the classical noise parameter $\bar{M}$. Curve A is the function $g(\bar{M})$, i.e., the value of $S(\rho')$ achieved with a coherent state input, and hence an upper bound for the minimal output entropy. Curves B and C are, respectively, the lower bounds for $S(\rho')$ given by $\ln(2\bar{M}+1)$ and by $1+\ln(\bar{M})$. The bound B derives by comparing the von Neumann entropy with the minimal output Rényi entropy of order 2. The bound C derives by analyzing the concavity properties of the generalized Rényi entropies. The gray curve is a more sophisticated version of the lower bound B [12].

entropy of the vacuum state $|0\rangle$. Our claim is that the minimal entropy is achieved on coherent states, i.e.

$$\min_{\rho} S(\rho') = g(\bar{M}) \; . \tag{15}$$

Proving Eq. (15) is equivalent to proving that the right-hand side of Eq. (13) is the capacity C. By studying the directional derivative of $S(\rho')$ along linear trajectories, we have proven that coherent states are *local* minima for $S(\rho')$, Moreover, we have derived several lower bounds for the global minimum of this function, which we have plotted in Fig. 1. When any of these lower bounds is applied to the negative contribution in the Holevo quantity (1), we obtain an upper bound on the capacity C of the classical-noise channel. Because these minimum-entropy lower bounds exceed $g(\bar{M})$, we cannot yet conclude that the right-hand side of Eq. (13) is indeed the capacity. However the mismatch between the upper and lower capacity bounds is very small (especially in the high-noise regime), so that our estimation of the capacity is good for many practical purposes.

Additional support for our conjecture of Eq. (15) derives from the fact that integer-order output Rényi entropies (viz., the quantities $S_k(\rho') \equiv -[\ln \mathrm{Tr}(\rho'^k]/(k-1)$ for $k = 2, 3, \ldots$) are minimized by coherent-state inputs [12]. We have also studied the output Wehrl entropy [13],

$$W(\rho') \equiv -\int \frac{d^2\alpha}{\pi} \; \langle\alpha|\rho'|\alpha\rangle \;\; \ln\langle\alpha|\rho|\alpha\rangle, \tag{16}$$

where $|\alpha\rangle$ is the coherent state. This entropy, which gives a measure of the output state's phase-space localization, is also minimized by a coherent state input [12].

III. CONCLUSION

In conclusion, we have derived the classical capacity of the lossy multimode bosonic channel. Interestingly, quantum features of the signals (such as entanglement or squeezing) are not required to achieve capacity, because an optimal coherent-state encoding exists. At the decoding stage, however, quantum effects might still be necessary (e.g., in the form of joint measurements on the output) as standard homodyne and heterodyne measurements are not optimal, except for the high-power regime in which heterodyne detection is asymptotically optimal. We have also analyzed some active-noise channel models (in which the transmission modes are coupled to some external photon source), providing several lower bounds on the minimal channel-output entropy. These results, in turn, lead to upper bounds on the channel capacity which are close, in many cases, to a lower bound that we conjecture is the exact capacity.

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