rLe AT MIT

July 2, 2003

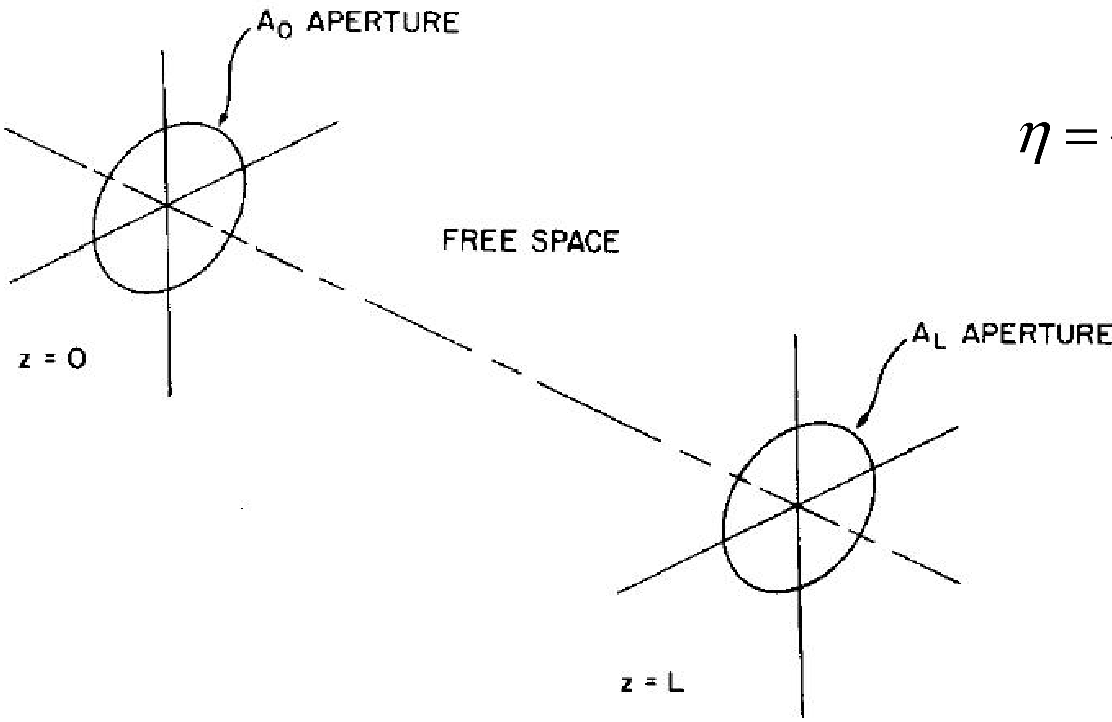
Capacity of the Single-Mode Free Space Quantum Optical Channel

Saikat Guha and Jeffrey H. Shapiro

Outline

- Introduction
  - Free space Propagation Geometry
  - Shannon Capacity of a Quantum Channel
  - Blahut-Arimoto Algorithm
  - Lossless Power-constrained Bosonic Channel
- Number States, Direct Detection Capacity
  - Narrowband, Low Input Power
  - Wideband, Low Input Power
  - Simulation Results for Arbitrary Input Power
- Conclusions and Future Work
  - Comparison with Lossless Number-State Capacity
  - Far-Field, Number-State Capacity
  - Other Classical Capacities for Quantum Channels

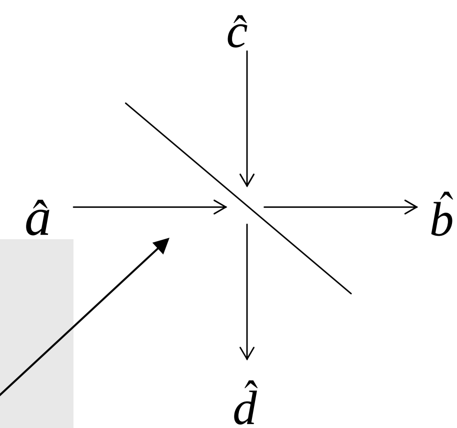
Free Space Propagation Geometry



$$\eta = \frac{A_0 A_L}{\lambda^2 L^2} \quad \text{(Fresnel Number)}$$

$\eta << 1$ (far field)

$\eta > 1$ (near field)



- Quantized Transmitter - $\mathbf{E}(x,y,t)$; $(x,y) \in A_0$, $t \in T$
- Single-Mode Propagation - Far Field at freq f
- Propagation Model - Beamsplitter with transmittivity η

$$\hat{b} = \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{c}$$

[Yuen, Shapiro *IEEE Trans. of Inf. th.* No.6, Nov.'78]

Shannon Capacity of a Quantum Channel

- $S(\boldsymbol{H})$: Convex Set of all states in Hilbert Space $\boldsymbol{H}$
- π : A classical probability assignment $\{\pi_i\}$ to all states S_i in $\boldsymbol{H}$
- X : A decision rule (POVM) at detector $\{X_j\}$
- Φ : Quantum channel (a linear completely positive trace preserving map)

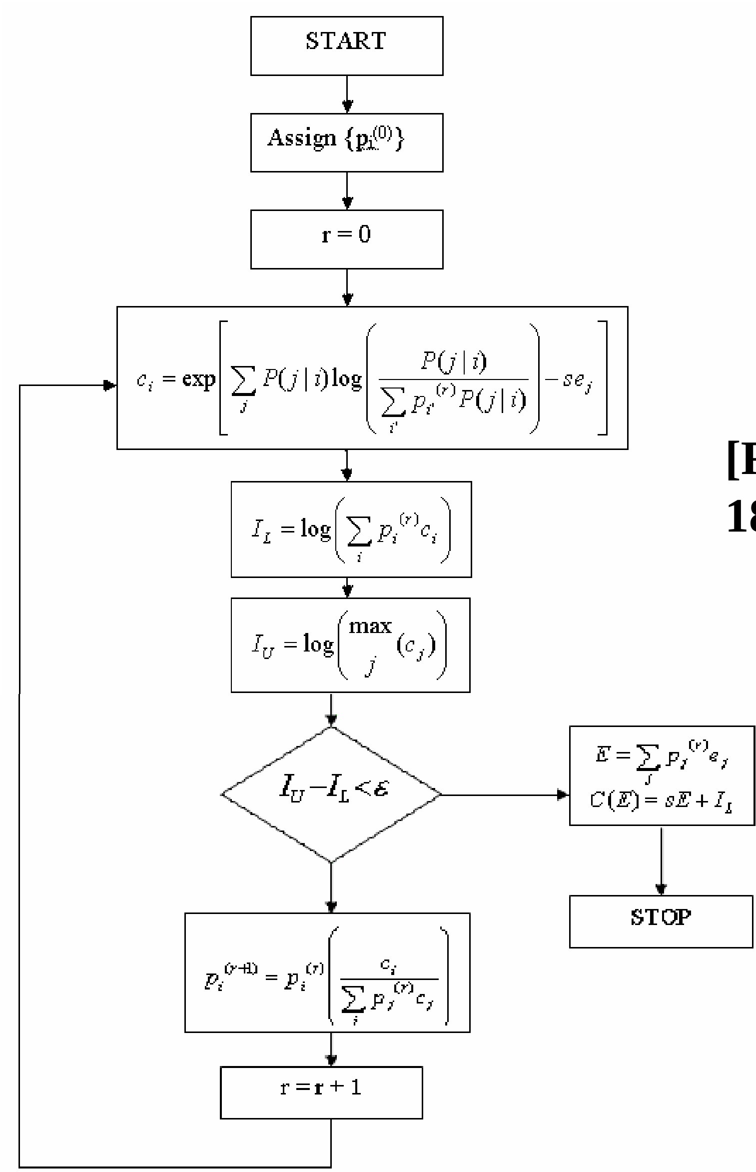
Classical transition probabilities: $P(j|i) = \mathrm{Tr}(\Phi(S_i)X_j)$

Classical mutual information: $$I(\pi,\Phi,X) = \sum_{j,i} \pi_i P(j\,|\,i) \log\left(\frac{P(j\,|\,i)}{\sum_k \pi_k P(j\,|\,k)} \right)$$

Capacity $$C = \sup_{\pi,X}\left(I(\pi,\Phi,X) \right)$$

[Capacity in bits/use of channel – with product-state inputs and product POVMs over multiple channel uses.]

Blahut-Arimoto Algorithm



An efficient algorithm to numerically compute Shannon Capacity of a classical discrete memoryless channel with transition probabilities P(j|i)

[Richard E. Blahut, *IEEE Trans. of Inf. th.* Vol 18, No.4, July 1972]

$$p \in P_E ; P_E = \left\{ p : \sum_j p_j e_j \le E \right\}$$

Expense of using the letter j

Lossless Power Constrained Channel

[Caves and Drummond, *PRA*, 1994]

- Assumptions
  - Lossless Propagation: Φ = I (identity)
  - Transmitter Average Power Constraint *P*
  - Product State Inputs over Repeated Channel Uses
  - POVM Detection across Channel Uses
- Narrowband Capacity
  - $C_{NB} = 2\sqrt{\frac{\eta_0 P}{h}}$
  - η_0 = Bandwidth to Center Frequency Ratio
- Wideband Capacity
  - $C_{WB} = \frac{\pi}{\ln 2}\sqrt{\frac{2P}{3h}}$

Lossy Number State Channel

- Number-State Transmitter $\hat{\rho}_T = |n\rangle\langle n|$ transmitted with prob π_n
- Received State $\hat{\rho}_R = \sum_{k=0}^{n} \binom{n}{k} \eta^k (1-\eta)^{n-k} |k\rangle\langle k|$
- Direct-Detection Measurement $X_m = |m\rangle\langle m|$

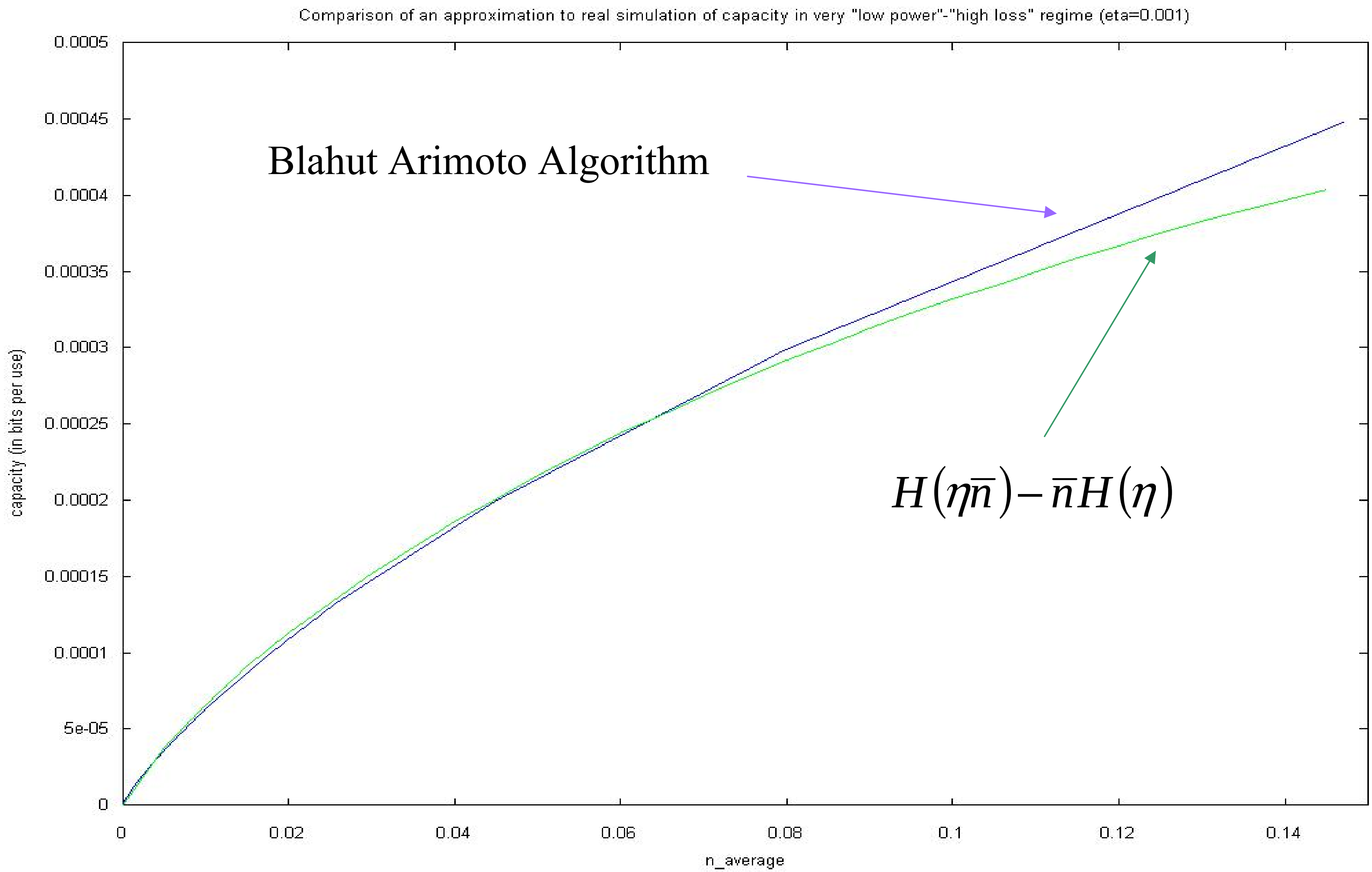
Narrowband: Low Input Power

- $\eta\bar{n} << 1$
- $C_{NB} = B\left(H(\eta\bar{n}) - \bar{n}H(\eta)\right)$ bits/sec
  - $H(p)$ = binary entropy function
  - B = bandwidth

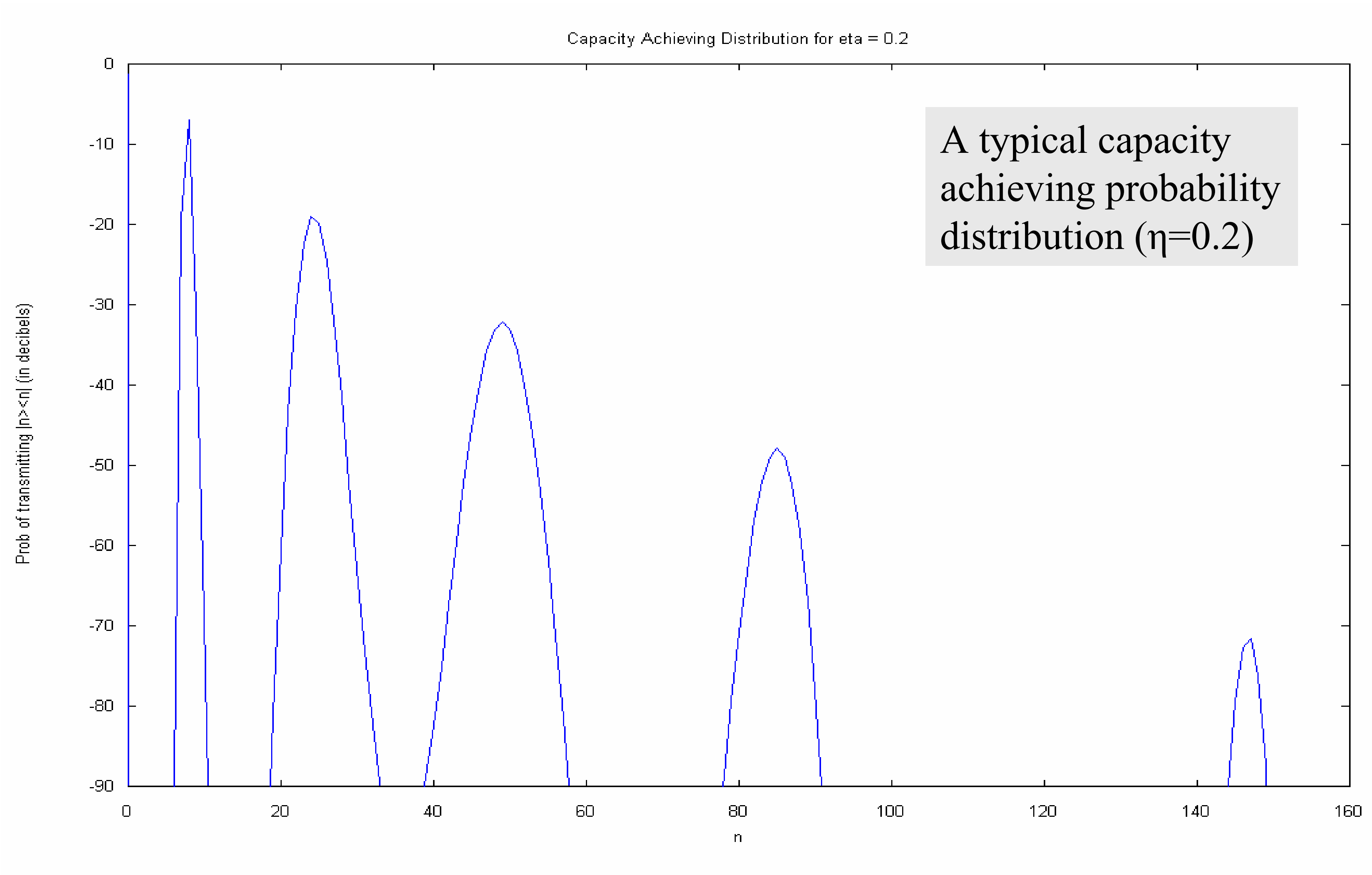
Wideband: Low Input Power

- $\eta\bar{n} << 1$ in each freq bin
- Loss independent of freq
- $C_{WB} = \frac{2\eta}{\ln 2}\sqrt{\frac{eP}{h}}$ bits/sec

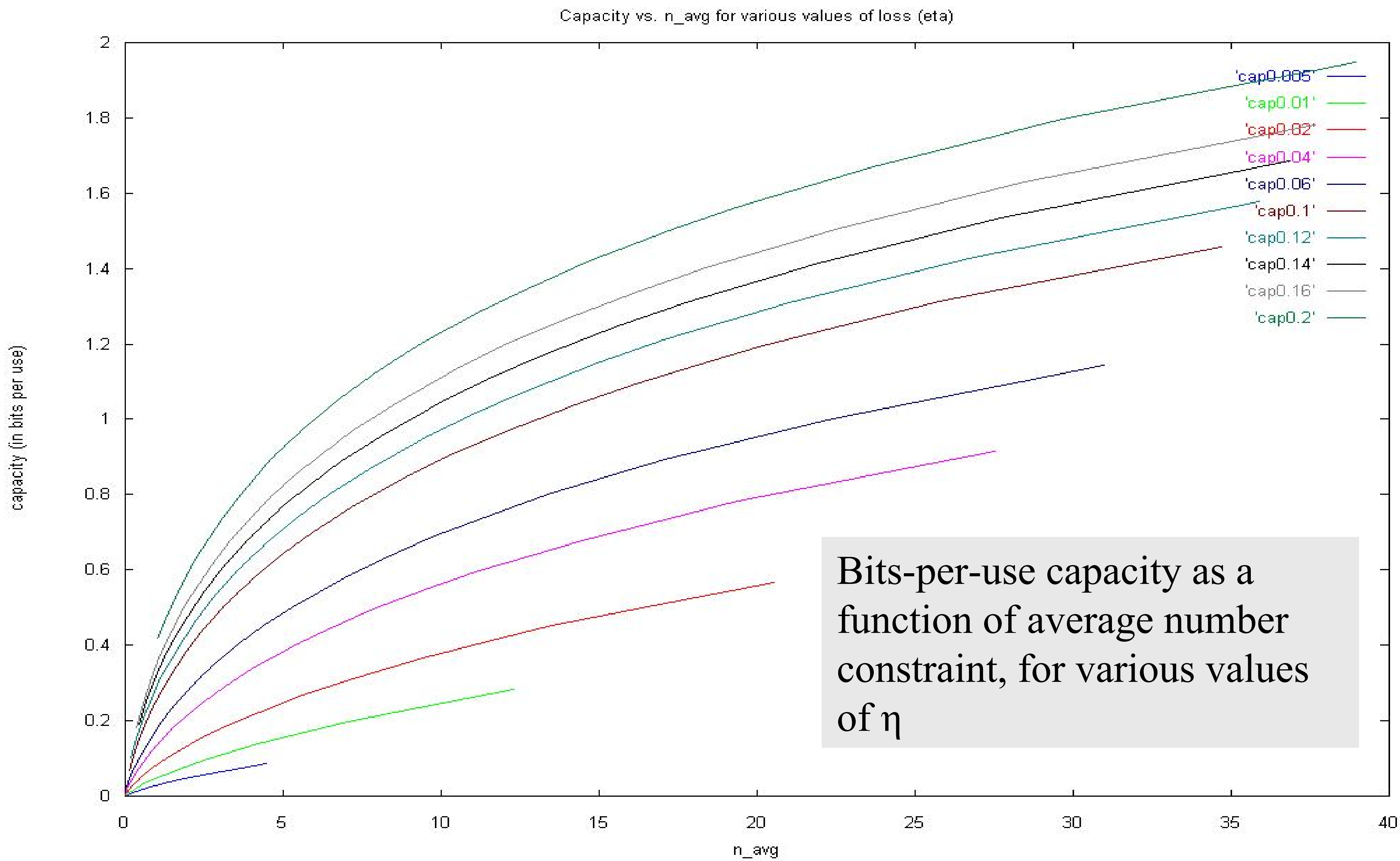
Low-Power Approximation versus Numerical Result



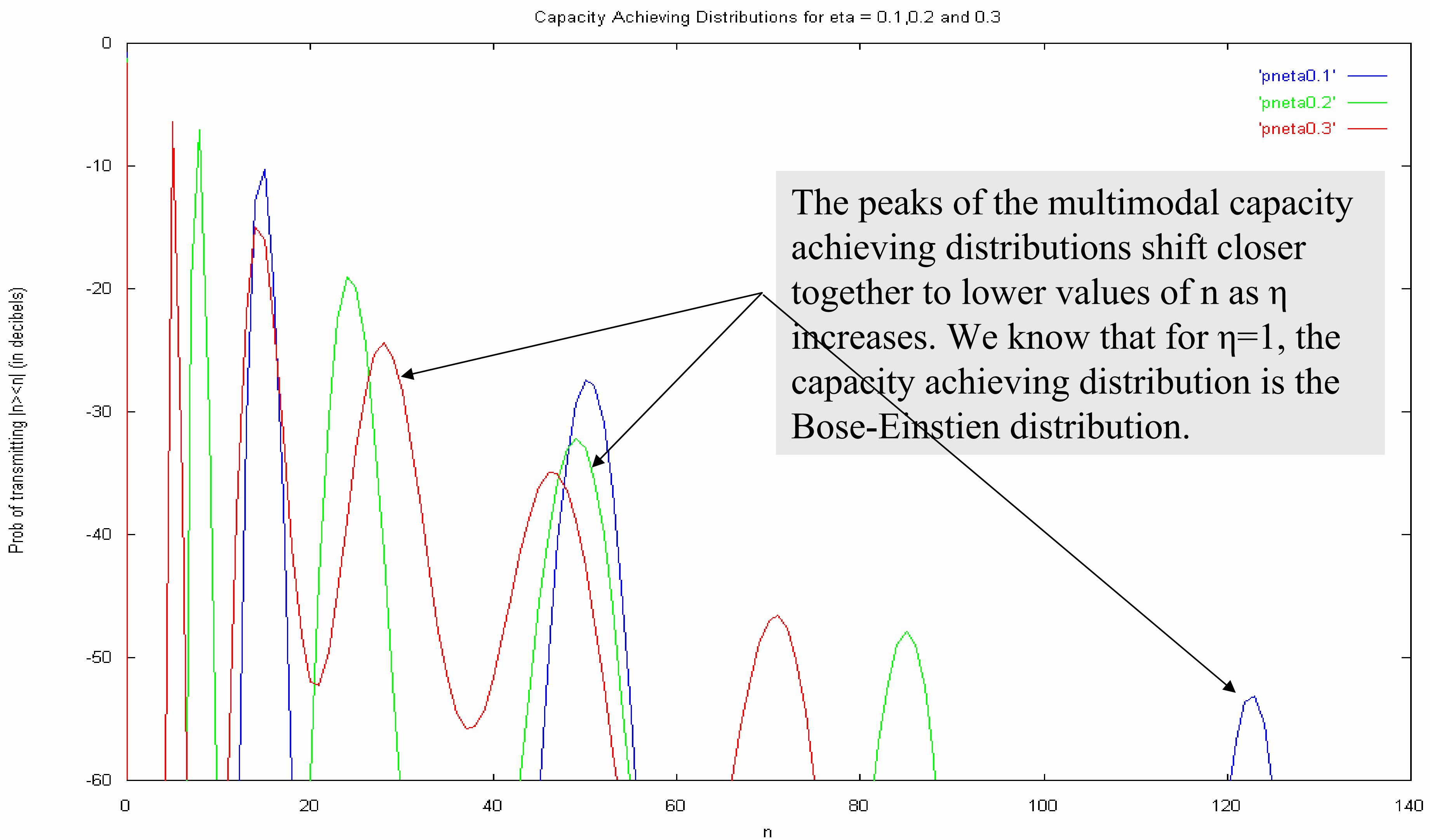
Narrowband Capacity for Arbitrary Input Power



Narrowband Capacity for Arbitrary Input Power



Comparison of Capacity Achieving Distributions for Different Values of Loss



The peaks of the multimodal capacity achieving distributions shift closer together to lower values of n as η increases. We know that for η=1, the capacity achieving distribution is the Bose-Einstien distribution.

Wideband Capacity for Arbitrary Input Power

- Assumption: Loss independent of Frequency
- Wideband Capacity $$C_{WB} = \sqrt{\frac{P}{h}}\left(\frac{\int_0^{\infty} C(z)dz}{\sqrt{\int_0^{\infty} m(z)dz}} \right)$$
  - where, $$z = \beta f$$ $$\beta^2 = \frac{h}{P}\int_0^{\infty} m(z)dz$$ $$m(z) = z[\bar{n}(z)]$$
- For η = 0.2, numerically interpolating and integrating Blahut-Arimoto calculations gives a wideband capacity:

$$C_{WB} = 0.5877\sqrt{\frac{P}{h}}$$

Conclusions and Future Work

- Lossless versus Lossy Capacity: Frequency-Independent η
  - C_{NB} and C_{WB} proportional to η
  - $$\frac{C_{WB(\eta=0.2)}}{C_{WB(\eta=1)}} = \frac{0.5877\sqrt{\frac{P}{h}}}{\frac{\pi}{\ln 2}\sqrt{\frac{2P}{3h}}} = 0.16 \approx \{\eta = 0.2\}$$
- Far-Field, Number-State Capacity
  - C_{CQ} capacity: Maximizing Holevo Information for number-state inputs yields the Capacity for any general POVM detection scheme, [Holevo, Schumacher, Westmoreland (HSW)]
    - $C_{CQ} = \sup_{\pi}\left(S\left(\sum_n \pi_n |n\rangle\langle n|\right) - \sum_n \pi_n S(|n\rangle\langle n|)\right)$
  - Wideband Capacity for Real Loss (Freq dependent Loss)
    - $\eta \propto f^2$ as per our model
  - Entanglement-Assisted Capacity for a Real Lossy Channel