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Classical Capacity of Far-Field Quantum Optical Communication

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Classical Capacity of Far-Field Quantum Optical Communication

- Quantum Communication and Information Theory
  - Density operators
  - Quantum channels
  - Classical information capacity of a quantum channel
- Classical Capacity of the Bosonic Channel
  - Channel model
  - Capacity of lossless and lossy Bosonic channels
- Free-Space Quantum Optical Communication
  - Far-field propagation model
  - Capacities with structured transmitters and receivers
  - Ultimate capacity of the far-field free space channel
- Conclusions and Ongoing Research

Density Operators and Quantum Channels

- State of a Quantum System
  - Complete knowledge is a pure state $|\psi\rangle$
  - Incomplete knowledge is a classically-random mixture of pure states
  - Density operator is a positive, unit-trace operator

$$\hat{\rho} \equiv \sum_i p(i)|\psi_i\rangle\langle\psi_i| = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$\{p(i)\}$ and $\{\lambda_i\}$ are probability distributions, $\{|\lambda_i\rangle\}$ are orthonormal vectors

- Quantum Channels
  - Mapping of input density operators to output density operators

$$\hat{\rho}_{\text{in}} \rightarrow \hat{\rho}_{\text{out}} = \Phi(\hat{\rho}_{\text{in}})$$

Φ is a completely-positive, trace-preserving map

Classical Information over Quantum Channels

- Signal States: $\rho \equiv \{\hat{\rho}_i\}$

  $i \in$ discrete classical symbol set, prior probability distribution $p \equiv \{p(i)\}$
- Quantum Channel: Φ
- Quantum Measurement: $\Pi \equiv \{\hat{\Pi}_j\}$

  $j \in$ discrete classical symbol set

$$\hat{\Pi}_j^{+} = \hat{\Pi}_j \text{ (Hermitian)}$$

$$\hat{\Pi}_j \geq 0 \text{ (positive)}$$

$$\sum_j \hat{\Pi}_j = \hat{I} \text{ (complete)}$$

- Classical Symbol Transition Probabilities: $P \equiv \{P(j \mid i)\}$

$$P(j \mid i) \equiv \mathrm{Tr}\left(\Phi(\hat{\rho}_i)\hat{\Pi}_j\right)$$

Classical and Quantum Information Measures

- Shannon Mutual Information: Single Channel Use

$$I(p,\rho,\Phi,\Pi) \equiv \sum_i \sum_j p(i)P(j\,|\,i)\log_2\left(\frac{P(j\,|\,i)}{\sum_k p(k)P(j\,|\,k)}\right) = H(J) - H(J\,|\,I)$$

- Holevo Information: Single Channel Use

$$\chi(p,\rho,\Phi) \equiv S\left(\sum_i p(i)\Phi(\hat{\rho}_i)\right) - \sum_i p(i)S(\Phi(\hat{\rho}_i))$$

  - Von Neumann Entropy:

$$S(\hat{\rho}) \equiv -\mathrm{Tr}(\hat{\rho}\log_2(\hat{\rho})) = H(\lambda)$$

Classical Capacity of a Quantum Channel

- Capacity with Structured Transmitter and Receiver

$$C_{structured}(\rho, \Phi, \Pi) = \sup_{p} I(p, \rho, \Phi, \Pi)$$

- Ultimate Capacity: Must Include Multiple Channel Uses

$$C_{ultimate}(\Phi) = \sup_{n} \frac{C_n(\Phi^{\otimes n})}{n}$$

$$C_n(\Phi^{\otimes n}) \equiv \sup_{p_n, \rho_n} \chi(p_n, \rho_n, \Phi^{\otimes n})$$

Holevo,
***IEEE Trans. Info. Thy.* 1998**

Schumacher, Westmoreland,
***PRA* 1997**

  - Allows for source entanglement across multiple channel uses
  - Allows for measurement entanglement across multiple channel uses

Single-Mode and Wideband Bosonic Channels

- Single-Mode Channel Model

$$\hat{c} = \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{b}$$

  - Photon annihilation operator of transmitted mode: $\hat{a}$
  - Photon annihilation operator of received mode: $\hat{c}$
  - Channel transmissivity: η, $\quad (0 \le \eta \le 1)$
  - Photon annihilation operator of environment: $\hat{b}$
  - Average photon number constraint: $\langle \hat{a}^{+}\hat{a} \rangle \le \bar{n}$
  - Minimum-noise environment: $\hat{b}$ in vacuum state

- Wideband Channel Model
  - Collection of single-mode channels at frequencies $f_i = iB$
  - Average power constraint: $\sum_i hf_i \langle \hat{a}_i^{+}\hat{a}_i \rangle B \le P_s$

Capacity of Lossless and Lossy Bosonic Channels

- Ultimate Capacity of the Lossless Channel: $\eta = 1$

$$C_{\text{single-mode}}(\bar{n}) = g(\bar{n}) = (1+\bar{n})\log(1+\bar{n}) - \bar{n}\log(\bar{n})$$

$$C_{\text{wideband}}(P_s) = \frac{\pi}{\ln 2}\sqrt{\frac{2P_s}{3h}}$$

Yuen & Ozawa
***PRL* 1992**

  - Achieved with photon-number state source and photon counting

- Ultimate Capacity of the Lossy Channel: $\eta(f) = \eta < 1$,

$$C_{\text{single-mode}}(\bar{n},\eta) = g(\eta\bar{n}) = C_{\text{single-mode}}(\eta\bar{n},1)$$

$$C_{\text{wideband}}(P_s,\eta) = \frac{\pi}{\ln 2}\sqrt{\frac{2\eta P_s}{3h}} = C_{\text{wideband}}(\eta P_s,1)$$

Giovannetti, Guha, Lloyd, Maccone, Shapiro, & Yuen
***PRL* 2004**

  - Achieved with coherent-state source

Structured Transmitters and Receivers

- Transmitter States
  - Photon number state $|n\rangle$ has exactly *n* photons
  - Coherent state $|\alpha\rangle$ is classical state of complex amplitude α
  - Squeezed state $|\beta;\mu,\nu\rangle$ is nonclassical minimum-uncertainty state

- Receiver Measurements
  - Direct detection measures photon number

$$\hat{\Pi}_n = |n\rangle\langle n|, \ \ n = 0,1,2,\ldots$$

  - Heterodyne detection measures field

$$\hat{\Pi}_\alpha = \frac{|\alpha\rangle\langle\alpha|}{\pi}, \ \ \alpha \in C$$

  - Homodyne detection measures single field quadrature

$$\hat{\Pi}_{\alpha_1} = |\alpha_1\rangle\langle\alpha_1|, \ \ \alpha_1 \in \Re$$

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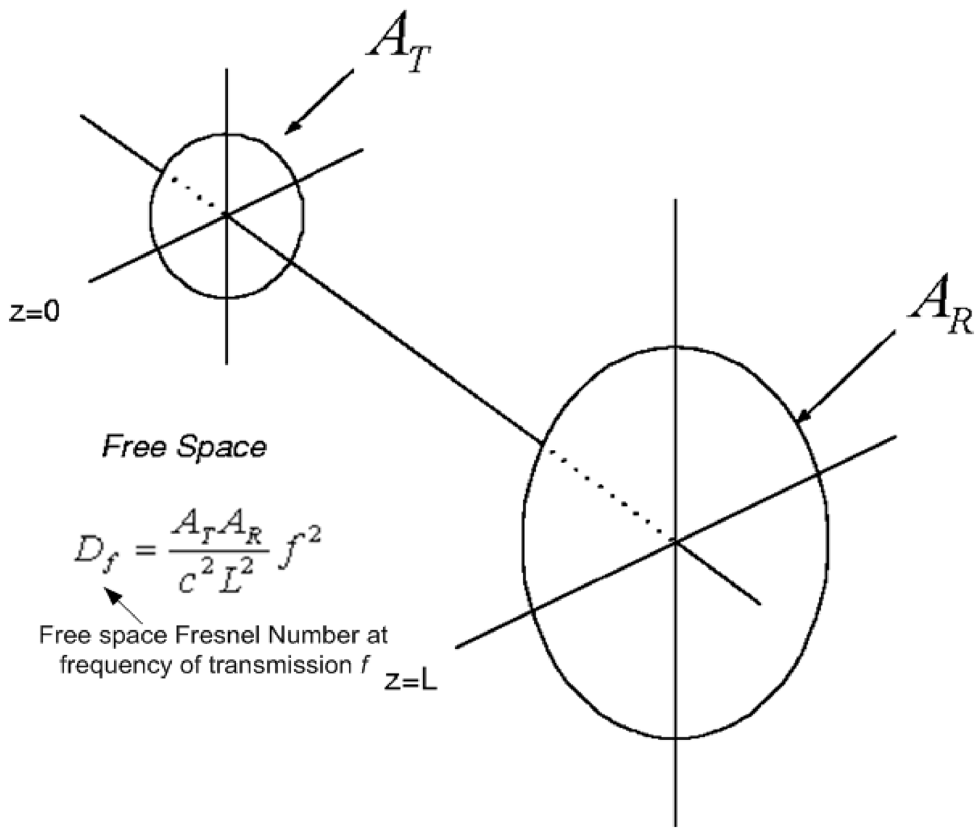
Free-Space Quantum Optical Communication

- Far-Field Propagation: $D_f << 1$



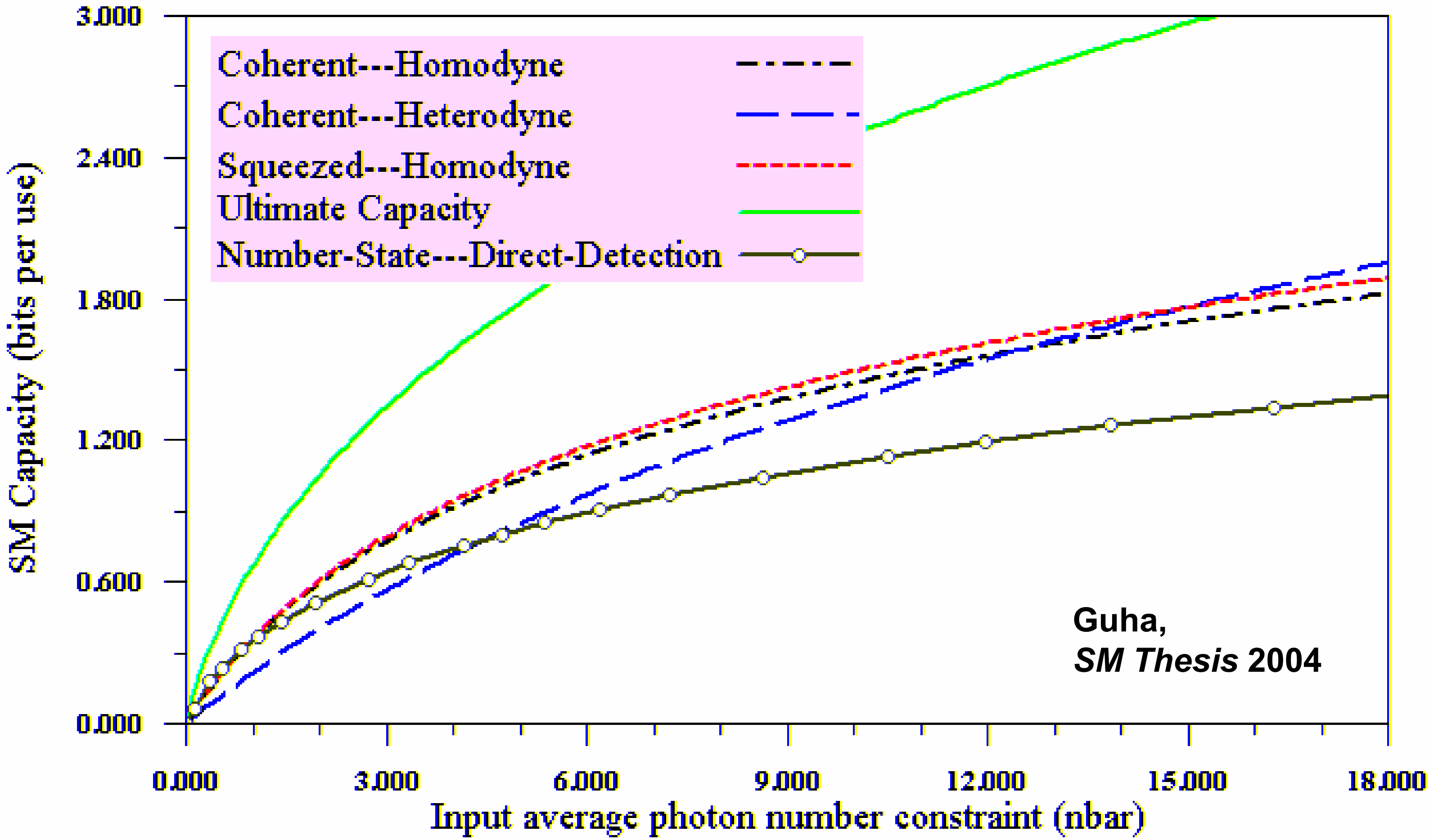
Yuen & Shapiro
IEEE Trans.
***Info. Thy.* 1978**

- Single-Spatial Mode Couples Power to Receiver for $f \leq f_{max}$:

$$\eta(f) \approx D_f = \frac{A_T A_R}{c^2 L^2} f^2 \leq \frac{A_T A_R}{c^2 L^2} f_{max}^2 << 1$$

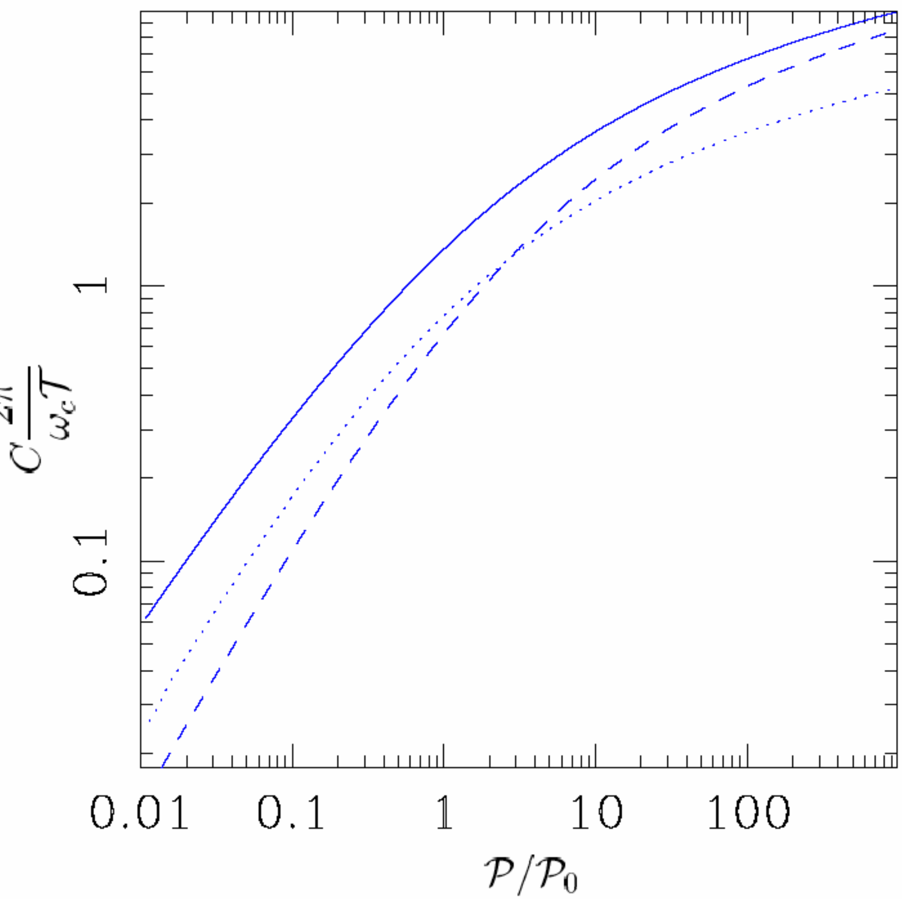
Single-Mode Free Space Channel Capacity

- Far-Field Propagation Assumed: $\eta = 0.16$



Far-Field Wideband Channel Capacity

- Coherent-State Source



- Ultimate capacity (solid)
- Homodyne capacity (dotted)
- Heterodyne capacity (dashed)

It can be shown analytically that

$$\lim_{P_s \to \infty} \frac{C_{\text{ultimate}}(P_s)}{C_{\text{coherent-heterodyne}}(P_s)} = 1$$

Giovannetti, Guha, Lloyd, Maccone, Shapiro, & Yuen *PRL* 2004

Conclusions and Ongoing Research

- Capacity of the Lossy Bosonic Channel Derived
  - Coherent-state encoding achieves ultimate capacity
- Far-Field Free Space Quantum Optical Channel
  - Coherent-state encoding and heterodyne detection achieves capacity in the limit of high input power
- Extension to Lossy Channel with Thermal Noise
  - Environment mode now in thermal (non-vacuum) state
  - Structured transmitter/receiver capacities have been derived
  - A lower bound on ultimate capacity has been derived
  - Conjecture: the lower bound is the lossy channel capacity