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On the Chvátal rank of polytopes in the 0/1 cube

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Abstract

Given a polytope $P \subseteq \mathbb{R}^n$, the Chvátal–Gomory procedure computes iteratively the integer hull P_I of P. The Chvátal rank of P is the minimal number of iterations needed to obtain P_I. It is always finite, but already the Chvátal rank of polytopes in $\mathbb{R}^2$ can be arbitrarily large. In this paper, we study polytopes in the 0/1 cube, which are of particular interest in combinatorial optimization. We show that the Chvátal rank of any polytope $P \subseteq [0,1]^n$ is $O(n^3 \log n)$ and prove the linear upper and lower bound n for the case $P \cap \mathbb{Z}^n = \emptyset$.

1. Introduction

The Chvátal rank of a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, was introduced by Chvátal in [1] as an indicator for the "degree of discreteness" and thus the complexity of an integer linear program of the form $\max\{c^{\mathrm{T}}x \mid x \in P \cap \mathbb{Z}^n\}$, $c \in \mathbb{R}^n$. An inequality $c^{\mathrm{T}}x \leqslant \lfloor\delta\rfloor$, with $c \in \mathbb{Z}^n$ and $\delta = \max\{c^{\mathrm{T}}x \mid x \in P\}$, is called a Chvátal–Gomory *cutting plane* (here $\lfloor\delta\rfloor$ denotes the greatest integer less than or equal to δ). The set of vectors P' satisfying all cutting planes for P is called the *elementary closure* of P. Let $P^{(0)} = P$ and $P^{(i+1)} = (P^{(i)})'$, for $i \geqslant 0$. Then the *Chvátal rank* of P is the smallest number t such that $P^{(t)} = P_I$, where P_I is the *integer hull* of P, i.e., the convex hull of $P \cap \mathbb{Z}^n$.

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Chvátal [1] showed that every bounded polyhedron $P \subseteq \mathbb{R}^n$ has finite rank. Schrijver [11] extended this result to possibly unbounded, but rational polyhedra $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$, where $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$. Cook et al. [3] and Gerards [5] proved that for every matrix $A \in \mathbb{Z}^{m\times n}$ there exists $t \in \mathbb{N}$ such that for all right-hand sides $b \in \mathbb{Z}^m$, the Chvátal rank of $P_b = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$ is bounded by t. More details on these results can be found in [12,9].

Already in dimension 2, there exist rational polyhedra of arbitrarily large Chvátal rank [1]. Lower bounds for the Chvátal rank of special polytopes arising in combinatorial optimization were given in [1,2,4,6,7,13], among others. Hartmann et al. [8] give conditions under which an inequality has Chvátal rank greater than one. In this paper, we study the Chvátal rank of polytopes contained in the n-dimensional 0/1 cube $[0,1]^n$. We show that the Chvátal rank of a polytope $P \subseteq [0,1]^n$ is $\mathrm{O}(n^3 \log n)$ and prove the linear upper and lower bound n for the case $P \cap \mathbb{Z}^n = \emptyset$.

The organization of the paper is as follows. We start with some preliminaries in Section 2. In Section 3, we consider polytopes in the n-dimensional 0/1 cube whose integer hull is empty and show that their Chvátal rank is at most n. In Section 4, we first recall that each integral 0/1 polytope can be described by a system of integral inequalities $Ax \leqslant b$ such that each absolute value of an entry in A is bounded by $n^{n/2}$. We then use this fact to derive an $\mathrm{O}(dn^2 \log n)$ upper bound for the Chvátal rank of d-dimensional rational polytopes in the 0/1 cube. Here, the basic idea is to use scaling of the row vectors a^{T} of A. The sequence of integral vectors obtained from a^{T} by dividing it by decreasing powers of 2 followed by rounding gives a better and better approximation of a^{T} itself. One estimates the number of iterations of the Chvátal–Gomory rounding procedure needed until the face given by some vector in the sequence contains integer points, using the fact that the face given by the previous vector in the sequence also contains integer points. Although the size of the vector is doubled every time, the number of iterations of the Chvátal–Gomory rounding procedure in each step is at most quadratic.

2. Preliminaries

A *polyhedron* P is a set of vectors of the form $P = \{x \in \mathbb{R}^n \mid Ax \leqslant b\}$, for some matrix $A \in \mathbb{R}^{m\times n}$ and some vector $b \in \mathbb{R}^m$. The polyhedron is *rational* if both A and b can be chosen to be rational. If P is bounded, then P is called a *polytope*. An *integral* 0/1 *polytope* is a polytope that is the convex hull of a set of 0/1 vectors $S \subseteq \{0,1\}^n$. The *integer hull* P_I of a polytope P is the convex hull of the integral vectors in P. The *dimension* $\dim(P)$ of P is the dimension of its affine hull and $P \subseteq \mathbb{R}^n$ is *full-dimensional* if $\dim(P) = n$. An inequality $c^{\mathrm{T}}x \leqslant \delta$ *defines a facet* $F = \{x \in P \mid c^{\mathrm{T}}x = \delta\}$ of P, if $\delta = \max\{c^{\mathrm{T}}x \mid x \in P\}$ and $\dim(F) = \dim(P) - 1$.

A *rational half-space* is of the form $H = \{x \in \mathbb{R}^n \mid c^{\mathrm{T}}x \leqslant \delta\}$, for some non-zero vector $c \in \mathbb{Q}^n$. The corresponding *hyperplane*, denoted by $(c^{\mathrm{T}}x = \delta)$, is the set $\{x \in \mathbb{R}^n \mid c^{\mathrm{T}}x = \delta\}$. A rational half-space always has a representation in which the components of c are relatively prime integers. That is, we can chose $c \in \mathbb{Z}^n$ with $\gcd(c) = 1$, in which case $H_I = \{x \in \mathbb{R}^n \mid c^{\mathrm{T}}x \leqslant \lfloor \delta \rfloor\}$. The *elementary closure* of a polyhedron P is the set

$$P' = \bigcap_{H \supseteq P} H_I,$$

where the intersection ranges over all rational half-spaces containing P. If $P^{(0)} = P$ and $P^{(i+1)} = (P^{(i)})'$, for $i \geqslant 0$, then the *Chvátal rank* of P is the smallest number t such that $P^{(t)} = P_I$. We refer to an application of the $'$ operation as one iteration of the Chvátal–Gomory procedure. One has the following facts (see, e.g., [12, Section 23.1]).

Theorem 1. *For each polytope P, there exists a number t such that $P^{(t)} = P_I$.*

Lemma 2. *Let F be a face of a rational polyhedron P. Then $F' = P' \cap F$.*

The l_∞-norm $||c||_\infty$ of the vector $c \in \mathbb{R}^n$ is the largest absolute value of its entries: $||c||_\infty = \max\{|c_i| \mid i = 1, \ldots, n\}$.

We define the function $\lg : \mathbb{N} \to \mathbb{N}$ as

$$\lg n = \begin{cases} 1 & \text{if } n = 0, \\ 1 + \lfloor \log_2(n) \rfloor & \text{if } n > 0, \end{cases}$$

where $\lfloor y \rfloor$ denotes the largest integer smaller than or equal to y. Note that $\lg n$ is the number of bits in the binary representation of n.

For $x \in \mathbb{R}$ we define

$$\lfloor x \rceil = \begin{cases} \lfloor x \rfloor & \text{if } x \geqslant 0, \\ \lceil x \rceil & \text{if } x < 0. \end{cases}$$

For $w \in \mathbb{R}^n$, let $\lfloor w \rceil \in \mathbb{Z}^n$ be the vector obtained by component-wise application of $\lfloor \cdot \rceil$.

3. Polytopes in the 0/1 cube without integral points

For our main result (see Theorem 11), one has to consider the rank of faces of polytopes in the 0/1 cube with empty integer hull.

Lemma 3. *Let $P \subseteq [0,1]^n$ be a d-dimensional rational polytope in the* 0/1 *cube with $P_I = \emptyset$. If $d = 0$, then $P' = \emptyset$; if $d > 0$, then $P^{(d)} = \emptyset$.*

Proof. The case $d = 0$ is obvious.

If $d = 1$, then P is the convex hull of two points $a, b \in [0,1]^n$, $a \neq b$. Since $P \cap \mathbb{Z}^n = \emptyset$, there exists an $i \in \{1, \ldots, n\}$ such that $0 < a_i < 1$. If $a_i \leqslant b_i$ (resp. $a_i \geqslant b_i$), then $x_i \geqslant a_i$ (resp. $x_i \leqslant a_i$) is valid for P and $P' \subseteq (x_i = 1)$ (resp. $P' \subseteq (x_i = 0)$). Since $0 < a_i < 1$ and $\dim(P) = 1$, it follows $P' \subseteq \{b\}$. Likewise, we can show in the same way that $P' \subseteq \{a\}$. Together, we obtain $P' \subseteq \{a\} \cap \{b\} = \emptyset$.

The general case is proven by induction on d and n. If P is contained in $(x_n=0)$ or $(x_n=1)$, we are done by induction on n. Otherwise, the dimension of $P_0=P\cap(x_n=0)$ and $P_1=P\cap(x_n=1)$ is strictly smaller than d. By the induction hypothesis and Lemma 2 we get $P_0^{(d-1)}=P^{(d-1)}\cap(x_n=0)=\emptyset$ and $P_1^{(d-1)}=P^{(d-1)}\cap(x_n=1)=\emptyset$. It follows $0<\min\{x_n \mid x\in P^{(d-1)}\}\leqslant\max\{x_n \mid x\in P^{(d-1)}\}<1$, which implies $P^{(d)}=\emptyset$. □

Irrational polytopes. For each polytope $P\subseteq[0,1]^n$, there exists a rational polytope $P^*\supseteq P$ in the 0/1 cube with the same integer hull (see [12], proof of Corollary 23.2a). Indeed, for each 0/1 point $y\notin P$, there exists a rational half space H_y containing P but not containing y. So

$$P^*=[0,1]^n\cap\bigcap_{\substack{y\in\{0,1\}^n\\ y\notin P}} H_y \tag{1}$$

has the desired properties. As $P^*\supseteq P$ implies $(P^*)^{(t)}\supseteq P^{(t)}$ we have proved the following corollary.

Corollary 4. *The Chvátal rank of polytopes $P\subseteq[0,1]^n$ with $P_I=\emptyset$ is at most n.*

The next lemma implies that the bound from above is tight. Its proof follows immediately from the proof of Lemma 7.2 in [2].

Lemma 5. *Let F_j be the set of all vectors y in $\mathbb{R}^n$ such that j components of y are 1/2 and each of the remaining $n-j$ components are equal to 0 or 1. If a polyhedron P contains F_1, then $F_j\subseteq P^{(j-1)}$, for all $j=1,\dots,n$.*

If we define P_n as the convex hull of F_1, then one has

$$P_n=\left\{x\in\mathbb{R}^n \,\middle|\, \sum_{j\in J}x_j+\sum_{j\notin J}(1-x_j)\geqslant\frac{1}{2},\ \text{for all } J\subseteq\{1,\dots,n\}\right\},$$

$(P_n)_I=\emptyset$ and $F_n=\{(1/2,\dots,1/2)\}\subseteq P_n^{(n-1)}$. Thus n is the smallest number such that $P_n^{(n)}=(P_n)_I=\emptyset$. We therefore, have the following proposition.

Proposition 6. *There exist rational polytopes $P\subseteq[0,1]^n$ with $P_I=\emptyset$ and Chvátal rank n.*

4. A polynomial upper bound in the dimension

Unless explicitly stated, we assume throughout this section that $P\subseteq[0,1]^n$, $n\geqslant1$, is a rational polytope with $P_I\neq\emptyset$ and $\dim(P)=d$.

Lemma 7. *For $0\neq c\in\mathbb{Z}^n$ let $\gamma=\max\{c^{\mathrm{T}}x \mid x\in P\}$ and $\delta=\max\{c^{\mathrm{T}}x \mid x\in P_I\}$. Then $c^{\mathrm{T}}x\leqslant\delta$ is valid for $P^{(k)}$, for all $k\geqslant d\lceil\gamma-\delta\rceil$.*

Intuitively, the lemma says that any face-defining inequality $c^{\mathrm{T}}x \leqslant \delta$ of P_I can be obtained from P by at most $d\lceil d_c \rceil$ iterations of the Chvátal–Gomory procedure, where $d_c = \gamma - \delta$ is the integrality gap of P with respect to c. A related result can be found in [1, Section 4], see also [6, Lemma 2.2.7].

Proof. If $d = 0$, then $P_I = P$ and the claim follows trivially. If $d = 1$ and $P \neq P_I$, then P is the convex hull of a 0/1 point a and some non-integral point $b \in [0,1]^n$. An argument similar to the one in Lemma 3 shows that $P' = \{a\} = P_I$, which implies the claim for $d = 1$, too.

So assume that $d \geqslant 2$. The proof is by induction on $\lceil \gamma - \delta \rceil$. The case $\lceil \gamma - \delta \rceil = 0$ is trivial, so suppose $\lceil \gamma - \delta \rceil > 0$.

If $\gamma \notin \mathbb{Z}$, then $c^{\mathrm{T}}x \leqslant \lfloor \gamma \rfloor = \lceil \gamma \rceil - 1$ is valid for P'.

If $\gamma \in \mathbb{Z}$, then $F = (c^{\mathrm{T}}x = \gamma) \cap P$ is a face of P without any integral points and $\dim(F) < d$. With Lemma 3 and since $d \geqslant 2$, we get $F^{(d-1)} = \emptyset$. Since $F^{(d-1)} = P^{(d-1)} \cap F$, we have $\max\{c^{\mathrm{T}}x \mid x \in P^{(d-1)}\} < \gamma$, which implies that $c^{\mathrm{T}}x \leqslant \gamma - 1$ is valid for $P^{(d)}$.

So in any case we see that $c^{\mathrm{T}}x \leqslant \lceil \gamma \rceil - 1$ is valid for $P^{(d)}$. Let $\gamma' = \max\{c^{\mathrm{T}}x \mid x \in P^{(d)}\}$. Then $\gamma' \leqslant \lceil \gamma \rceil - 1$ and since $\delta \in \mathbb{Z}$, it follows by induction that $c^{\mathrm{T}}x \leqslant \delta$ is valid for $(P^{(d)})^{(k')}$, for all $k' \geqslant d(\lceil \gamma - \delta \rceil - 1) \geqslant d\lceil \gamma' - \delta \rceil$. This implies the claim. □

Hadamard's inequality can be used to show that an integral 0/1 polytope can be described by inequalities with integer normal vectors whose l_∞-norm is only exponential in n (see, e.g, [10, Theorem 2]).

Theorem 8. *An integral* 0/1 *polytope* P *can be described by a system of integral inequalities* $Ax \leqslant b$ *with* $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ *such that each absolute value of an entry in* A *is bounded by* $n^{n/2}$.

We now use a scaling argument to obtain our main result. Let $c^{\mathrm{T}}x \leqslant \delta$ be a face-defining inequality of P_I from the description of Theorem 8 and let $c^{\mathrm{T}}x \leqslant \gamma$ be valid for P. From Lemma 7 we know that $c^{\mathrm{T}}x \leqslant \delta$ is valid for $P^{(k)}$ for all $k \geqslant d\lceil \gamma - \delta \rceil$. Since $||c||_\infty \leqslant n^{n/2}$, an exponential upper bound of the Chvátal rank follows immediately.

Instead of using the possibly large vector c from the beginning one can use smaller vectors close to c first, estimate the number of Chvátal–Gomory steps until they "touch" P_I and perform only the last steps of the Chvátal–Gomory rounding with c itself. These vectors are obtained by scaling.

Lemma 9. *If* $c \neq 0$ *is an integral vector with* $\lg(||c||_\infty) \leqslant k$ *and if* $c^{\mathrm{T}}x \leqslant \delta$ *is valid for* P_I, *then* $c^{\mathrm{T}}x \leqslant \delta$ *is valid for* $P^{(k\,d\,n)}$.

Proof. Assume that $\delta = \max\{c^{\mathrm{T}}x \mid x \in P_I\}$. We proceed by induction on k.

For $k = 1$ note that $c \in \{-1, 0, 1\}^n$, so for $\gamma = \max\{c^{\mathrm{T}}x \mid x \in P\}$ one has $\gamma - \delta \leqslant n$ and the claim follows with Lemma 7.

Now let $k > 1$ and write c as the sum $2c_1+c_2$ with $c_1=\lfloor c/2 \rceil$. Note that $\lg(||c_1||_\infty)<\lg(||c||_\infty)$ and that $c_2\in\{-1,0,1\}^n$. Let $c_1^{\mathrm{T}}x\leqslant\delta_1$ be a face-defining inequality for P_I. By the induction hypothesis it follows that $c_1^{\mathrm{T}}x\leqslant\delta_1$ is valid for $P^{((k-1)dn)}$. Let $x_I\in P_I$ satisfy $c_1^{\mathrm{T}}x_I=\delta_1$. Let $\gamma'=\max\{c^{\mathrm{T}}x\mid x\in P^{((k-1)dn)}\}$. We will conclude that $\gamma'-\delta\leqslant n$ and the claim then follows again from Lemma 7. Let $\hat{x}\in P^{((k-1)dn)}$ satisfy $c^{\mathrm{T}}\hat{x}=\gamma'$. Clearly $c^{\mathrm{T}}(\hat{x}-x_I)$ is an upper bound on the integrality gap $\gamma'-\delta$. But

$$c^{\mathrm{T}}(\hat{x}-x_I)=2c_1(\hat{x}-x_I)+c_2(\hat{x}-x_I)\leqslant c_2(\hat{x}-x_I)\leqslant n.$$

This follows since x_I maximizes $\{c_1^{\mathrm{T}}x\mid x\in P^{((k-1)dn)}\}$ and since c_2 and $\hat{x}-x_I$ are in $[-1,1]^n$. □

A polynomial upper bound on the Chvátal rank now follows easily.

Theorem 10. *Let $P\subseteq[0,1]^n$, $P_I\neq\emptyset$, be a d-dimensional rational polytope in the* 0/1 *cube. The Chvátal rank of P is at most* $(\lfloor n/2\log_2 n\rfloor+1)nd$.

Proof. P_I is obtained by i iterations of the Chvátal–Gomory procedure if each inequality $c^{\mathrm{T}}x\leqslant\delta$ out of the description delivered by Theorem 8 is valid for $P^{(i)}$. With Lemma 9 this is true for all $i\geqslant\lg(n^{n/2})dn=(\lfloor n/2\log_2 n\rfloor+1)dn$ □

We can now conclude with our main result.

Theorem 11. *The Chvátal rank of any polytope $P\subseteq[0,1]^n$ in the n-dimensional* 0/1 *cube is at most* $(\lfloor n/2\log_2 n\rfloor+1)n^2$.

Proof. Let P^* be the construction from Eq. (1) in Section 3. The rank of P^* is an upper bound on the rank of P. Since P^* is rational either Lemma 3 or Theorem 10 applies to P^* and the result follows. □

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References

[1] V. Chvátal, Edmonds polytopes and a hierarchy of combinatorial problems, Discrete Math. 4 (1973) 305–337.
[2] V. Chvátal, W. Cook, M. Hartmann, On cutting-plane proofs in combinatorial optimization, Linear Algebra Appl. 114/115 (1989) 455–499.
[3] W. Cook, A.M.H. Gerards, A. Schrijver, E. Tardos, Sensitivity theorems in integer linear programming, Math. Programming 34 (1986) 251–264.

[4] W. Cook, M. Hartmann, On the complexity of branch and cut methods for the traveling salesman problem, in: W. Cook, P.D. Seymour (Eds.), Polyhedral Combinatorics, DIMACS, 1990, pp. 75–81.

[5] A.M.H. Gerards, On cutting planes and matrices, in: W. Cook, P.D. Seymour (Eds.), Polyhedral Combinatories, DIMACS, 1990, pp. 29–32.

[6] M. Hartmann, Cutting planes and the complexity of the integer hull, Technical Report 819, School of Operations Research and Industrial Engineering, Cornell University, September 1988.

[7] M. Hartmann, On the Chvátal rank of the subtour polytope for almost all graphs, Technical Report UNC/OR/TR-90/24, Department of Operations Research, University of North Carolina, December 1990.

[8] M. Hartmann, M. Queyranne, Y. Wang, On the Chvátal rank of certain inequalities, Manuscript, 1998.

[9] G.L. Nemhauser, L.A. Wolsey, Integer and Combinatorial Optimization, Wiley, New York, 1988.

[10] M.W. Padberg, M. Grötschel, Polyhedral computations, in: E.L. Lawler, J.K. Lenstra, A.H.G. Rinnoy Kan, D.B. Shmoys (Eds.), The Traveling Salesman Problem, Wiley, New York, 1985, pp. 307–360.

[11] A. Schrijver, On cutting planes, Ann. Discrete Math. 9 (1980) 291–296.

[12] A. Schrijver, Theory of Linear and Integer Programming, Wiley, New York, 1986.

[13] A.S. Schulz, Polytopes and Scheduling, Ph.D. Thesis, Technical University of Berlin, Berlin, Germany, 1996.