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On the membership problem for the $\{0, 1/2\}$-closure

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ABSTRACT

In integer programming, $\{0, 1/2\}$-cuts are Gomory–Chvátal cuts that can be derived from the original linear system by using coefficients of value 0 or 1/2 only. The separation problem for $\{0, 1/2\}$-cuts is strongly NP-hard. We show that separation remains strongly NP-hard, even when all integer variables are binary.

1. Introduction

We consider rational polyhedra $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. Inequalities of the form

$$(\lambda^{\mathsf{T}}A)x \leq \lfloor \lambda^{\mathsf{T}} b \rfloor, \tag{1}$$

with $\lambda \in \mathbb{R}^m$, $\lambda^{\mathsf{T}}A \in \mathbb{Z}^n$, and $\lambda^{\mathsf{T}}b \notin \mathbb{Z}$ are commonly referred to as Gomory–Chvátal cuts; they were first mentioned in the work of Gomory [13] and Chvátal [7]. Gomory–Chvátal cuts are valid for the integer hull, $P_I = \operatorname{conv}\{x \in \mathbb{Z}^n : Ax \leq b\}$, of P. It is well known that it suffices to consider λ-vectors with small coefficients (see, e.g., [18]); more specifically,

$$\begin{aligned} P' &:= \{x : (\lambda^{\mathsf{T}}A)x \leq \lfloor \lambda^{\mathsf{T}} b \rfloor, \lambda \in \mathbb{R}^m, \lambda^{\mathsf{T}}A \in \mathbb{Z}^n\} \\ &= \{x : (\lambda^{\mathsf{T}}A)x \leq \lfloor \lambda^{\mathsf{T}} b \rfloor, \lambda \in [0, 1]^m, \lambda^{\mathsf{T}}A \in \mathbb{Z}^n\}, \end{aligned}$$

and this rational polyhedron is commonly referred to as the first Gomory–Chvátal closure. Geometrically speaking, P' arises from P by considering all inequalities that are valid for P and pushing the associated hyperplanes towards P_I until they contain some integer point. In particular, P' is a stronger relaxation of P_I than P, i.e., $P_I \subseteq P' \subseteq P$. There are several prominent explicit examples of Gomory–Chvátal cuts in polyhedral combinatorics, including the blossom inequalities of the matching polytope [10,7], the odd-cycle inequalities of the stable set polytope [17], the simple comb inequalities of the symmetric traveling salesman polytope [15,4], and the simple Möbius ladder inequalities of the acyclic subdigraph polytope [14,2], to name a few. Interestingly, the separation problem for all these families of inequalities (or classes containing them) can be solved in polynomial time. Moreover, all these cuts can be derived as in (1) with $\lambda \in \{0, 1/2\}^m$. This prompted Caprara and Fischetti [2] to introduce the family of all $\{0, 1/2\}$-cuts,

$$\mathcal{F}_{1/2}(A, b) := \{(\lambda^{\mathsf{T}}A)x \leq \lfloor \lambda^{\mathsf{T}} b \rfloor : \lambda \in \{0, 1/2\}^m, \lambda^{\mathsf{T}}A \in \mathbb{Z}^n\},$$

and to analyze the computational complexity of the following problem: Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, and $\hat{x} \in \mathbb{Q}^n$ with $A\hat{x} \leq b$, does $\hat{x}$ violate an inequality in $\mathcal{F}_{1/2}(A, b)$? This problem is, of course, equivalent to the membership problem for the $\{0, 1/2\}$-closure of $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, which is defined by the points in P that satisfy all inequalities in $\mathcal{F}_{1/2}(A, b)$. Caprara and Fischetti showed that checking whether $\hat{x}$ violates some inequality in $\mathcal{F}_{1/2}(A, b)$ is, in general, strongly NP-complete (and, therefore, the membership problem is strongly coNP-complete). However, the polytopes of interest in combinatorial optimization oftentimes have vertices with coordinates 0 or 1; that is, $P \subseteq [0, 1]^n$, which is not the case for the instances that occur in Caprara and Fischetti's proof. This provides the motivation for our work, in which we study the following problem.

> Given $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ such that $\{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0, 1]^n$, and $\hat{x} \in \mathbb{Q}^n$ with $A\hat{x} \leq b$, does $\hat{x}$ violate an inequality in $\mathcal{F}_{1/2}(A, b)$?

Our main result is that this problem is still strongly NP-complete, and we give two different proofs for it, each of which is interesting in its own right. One proof is a careful modification of Caprara

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and Fischetti's proof, and the other one shows a slightly stronger result in that the corresponding 0/1-polytopes that arise from our reduction are of set-packing type. That is, P is of the form $P = \{x \in \mathbb{R}^n_+ : Ax \le \mathbb{1}\}$, where $A \in \{0, 1\}^{m \times n}$ is a binary matrix, and $\mathbb{1}$ denotes the all-1 vector with m entries. Before we present the first proof in Section 2 and the second proof in Section 3, we briefly discuss some other work related to $\{0, 1/2\}$-cuts.

1.1. Related work

Caprara and Fischetti's original proof of their own hardness result (published in [3]) was, in some sense, stronger than that in [2], because it actually showed that checking whether a point violates some inequality in $\mathcal{F}_{1/2}(A, b)$ is strongly NP-hard, even when the non-negativity constraints $-x_i \le 0$ for all $i = 1, \dots, n$ are part of the system $Ax \le b$. A similar proof was later given in [5]. "Maximally violated" $\{0, 1/2\}$-cuts, however, can always be separated in polynomial time; in fact, this is true more generally for mod-k cuts, for any given $k \ge 2$ [4]. A mod-k cut is an inequality of the form (1) in which each component of λ is a multiple of $1/k$. A mod-k cut is maximally violated by a given point if the difference between its left-hand side and its right-hand side is equal to $(k - 1)/k$. Earlier, Caprara and Fischetti [2,3] had given a couple of sufficient conditions under which the separation problem for $\{0, 1/2\}$-cuts can be solved in polynomial time. Letchford [16] later introduced a superclass of $\{0, 1/2\}$-cuts, so-called binary clutter inequalities, which allowed him to describe broader classes of special cases in which $\{0, 1/2\}$-cuts can be separated efficiently. However, the computational complexity of separating $\{0, 1/2\}$-cuts for systems $Ax \le b$ with $P = \{x : Ax \le b\} \subseteq [0, 1]^n$ has remained open, and is the subject of this paper.

2. A reduction from DECODING OF LINEAR CODES

The following problem, known as DECODING OF LINEAR CODES, is NP-complete [12, Problem MS7].

> Given a matrix $Q \in \{0, 1\}^{r \times t}$, a vector $d \in \{0, 1\}^r$, and a positive integer K, is there a $z \in \{0, 1\}^t$ with no more than K entries equal to 1 such that $Qz \equiv d \bmod 2$?

Theorem 2.1. *The membership problem for the $\{0, 1/2\}$-closure of polytopes contained in the 0/1-hypercube is strongly coNP-complete.*

Our reduction carefully modifies the proof of Caprara and Fischetti in [2] so as to ensure that $P \subseteq [0, 1]^n$.

Proof. Membership testing is clearly in coNP. We give a reduction from DECODING OF LINEAR CODES to show completeness. Let Q, d, and K describe an instance of DECODING OF LINEAR CODES. We construct the following instance of the membership problem for the $\{0, 1/2\}$-closure:

$$A := \begin{pmatrix} Q^\top & \multirow{2}{*}{$2I_{t+1}$} \\ d^\top & \\ 2I_r & 0 \\ -2I_r & 0 \\ 0 & -3I_{t+1} \end{pmatrix}, \qquad b := \begin{pmatrix} 2 \cdot \mathbf{1}^t \\ 1 \\ 2 \cdot \mathbf{1}^r \\ \mathbf{0}^r \\ \mathbf{0}^{t+1} \end{pmatrix},$$

$$\hat{x} := \begin{pmatrix} \mathbf{0}^r \\ \mathbf{1}^t - \dfrac{1}{2} w \\ \dfrac{1}{2} \end{pmatrix},$$

where I_l is the identity matrix in dimension l, $\mathbf{0}^l$ is the all-0 vector in dimension l, $\mathbf{1}^l$ is the l-dimensional all-1 vector, $w := \frac{1}{K+1} \cdot \mathbf{1}^t$, and the 0s in the matrix represent all-0 submatrices of appropriate dimension. We first show that $P = \{x \mid Ax \le b\} \subseteq [0, 1]^{r+t+1}$. Consider row l of $Ax \le b$.

Case 1. $(t + 1) + 1 \le l \le (t + 1) + r$. We obtain the inequality $2x_{l-(t+1)} \le 2$ and, therefore, $x_{l-(t+1)} \le 1$. Put differently, $x_l \le 1$ for all $1 \le l \le r$.

Case 2. $(t+1)+r+1 \le l \le (t+1)+2r$. We have $-2x_{l-(t+1+r)} \le 0$ and, therefore, $x_{l-(t+1+r)} \ge 0$. We obtain $x_l \ge 0$ for all $1 \le l \le r$.

The first $t + 1$ rows of A correspond to inequalities of the form $\sum_{j=1}^r q_{ji} x_j + 2x_{r+l} \le 2$ for $1 \le l \le t$ and $\sum_{j=1}^r d_j x_j + 2x_{r+l} \le 1$ for $l = t + 1$. The non-negativity of the coefficients of Q and d together with $x_j \ge 0$ for all $j \in \{1, \dots, r\}$ implies that $x_{r+l} \le 1$ for all $1 \le l \le t + 1$.

Finally, consider row l with $(t+1)+2r+1 \le l \le 2(t+1)+2r$. The corresponding inequalities are of the form $-3x_{r+l-((t+1)+2r)} \le 0$ and, therefore, $x_{r+l} \ge 0$ for all $1 \le l \le t + 1$. It follows that $P \subseteq [0, 1]^{r+t+1}$. Moreover, $b - A\hat{x} = (w_1, \dots, w_t, 0, 2 \cdot \mathbf{1}^r, \mathbf{0}^r, 3 - \frac{3}{2} w_1, \dots, 3 - \frac{3}{2} w_t, \frac{3}{2})^\top$. In particular, $\hat{x} \in P$.

Note that $\hat{x}$ violates a $\{0, 1/2\}$-cut if and only if there exists $\mu \in \{0, 1\}^{2(t+1)+2r}$ such that $\mu^\top A \equiv 0 \pmod 2$, $\mu^\top b \equiv 1 \pmod 2$, and $\mu^\top (b - A\hat{x}) < 1$. To have $\mu^\top A \equiv 0 \pmod 2$, it is necessary that $\mu_l = 0$ for $(t + 2) + 2r \le l \le 2(t + 1) + 2r$. Furthermore, $\mu^\top b \equiv 1 \pmod 2$ if and only if $\mu_{t+1} = 1$. Consequently, there exists a $\mu \in \{0, 1\}^{2(t+1+r)}$ with $\mu^\top A \equiv 0 \pmod 2$ and $\mu^\top b \equiv 1 \pmod 2$ if and only if there exists a $z \in \{0, 1\}^t$ such that $Qz \equiv d \pmod 2$ with $z \in \{0, 1\}^t$. Indeed, $z_l = \mu_l$ for $1 \le l \le t$, and the remaining μ_l for the reverse direction can be chosen arbitrarily for those rows of A that are equal to $0 \pmod 2$.

Note that $w^\top z < 1$ if and only if no more than K entries of z are equal to 1. Thus, it remains to show that $\mu^\top (b - A\hat{x}) < 1$ if and only if $w^\top z < 1$ with z and μ as above. Assume first that $\mu^\top (b - A\hat{x}) < 1$. Recall that $b - A\hat{x} = (w_1, \dots, w_t, 0, 2 \cdot \mathbf{1}^r, \mathbf{0}^r, 3 - \frac{3}{2} w_1, \dots, 3 - \frac{3}{2} w_t, \frac{3}{2})^\top$. Therefore, $w^\top z < 1$ for $z \in \{0, 1\}^t$ with $z_l = \mu_l$ for $1 \le l \le t$. Conversely, let $w^\top z < 1$ for some $z \in \{0, 1\}^t$. Define $\mu \in \{0, 1\}^{2(t+1)+2r}$ by $\mu_l := z_l$ for $1 \le l \le t$, $\mu_{t+1} := 1$, and $\mu_l := 0$ otherwise. Then $1 > w^\top z = \mu^\top (w_1, \dots, w_t, 0, 2 \cdot \mathbf{1}^r, \mathbf{0}^r, 3 - \frac{3}{2} w_1, \dots, 3 - \frac{3}{2} w_t, \frac{3}{2})^T = \mu^\top (b - A\hat{x})$. So there is a violated $\{0, 1/2\}$-cut if and only if there is a solution to DECODING OF LINEAR CODES. $\square$

3. Reduction from EXACT 3-COVER

For a given $n \times m$ 0/1-matrix A, the intersection graph is an undirected graph with vertex set $V = \{1, \dots, n\}$, and an edge $\{i, j\}$ if and only if there is at least one row of A with a 1 in the ith and jth columns [17]. The edge $\{i, j\}$ represents the fact that x_i and x_j cannot take the value 1 simultaneously. The set-packing problem amounts to the problem of finding a maximum weight stable set (set of pairwise non-adjacent vertices) in the intersection graph. Padberg [17] showed that every clique C (i.e., every set of pairwise adjacent vertices) in the intersection graph yields a valid clique inequality $\sum_{j \in C} x_j \le 1$ for the set-packing polytope, and that such an inequality induces a facet of that polytope if and only if the clique is maximal.

In general, there may be many facet-inducing clique inequalities which are not represented in the system $Ax \le \mathbf{1}$. Indeed, the number of maximal cliques can be exponential in n and m. If, however, there is a one-to-one correspondence between the rows of A and the maximal cliques of the intersection graph (i.e., the system $Ax \le \mathbf{1}$ consists of the facet-inducing clique inequalities), then A is said to be a *clique matrix*.

We will find it helpful to write the $\{0, 1/2\}$-cuts of a clique matrix in a certain explicit form. Let $t \ge 1$ be an odd integer, and let $C_1, \dots, C_t$ be maximal cliques whose associated clique inequalities are to be used (receive a multiplier of 1/2) in the derivation of the cut. For $i = 1, \dots, n$, let ϕ_i represent the number of these cliques which contain i. That is, $\phi_i = |\{k \in \{1, \dots, t\} : i \in C_k\}|$. Then, we must use (set the multiplier to 1/2 for) a non-negativity inequality

$-x_i \leq 0$ for each $i \in V$ such that ϕ_i is odd. Thus, the cut takes the form

$$\sum_{i=1}^{n} \lfloor \phi_i/2 \rfloor x_i \leq \lfloor t/2 \rfloor.$$

Multiplying by 2, we see that this is equivalent to

$$\sum_{k=1}^{t} \sum_{i \in C_k} x_i - \sum_{\phi_i \text{ odd}} x_i \leq t - 1.$$

Following [2], we define the slack variables $s_k := 1 - \sum_{i \in C_k} x_i$ for $k = 1, \ldots, t$. The cut can then be written as

$$\sum_{k=1}^{t} s_k + \sum_{\phi_i \text{ odd}} x_i \geq 1.$$

Thus, we see that the $\{0, 1/2\}$-cut derived using cliques $C_1, \ldots, C_t$ is violated by a given $\hat{x}$ if and only if

$$\sum_{k=1}^{t} \hat{s}_k + \sum_{\phi_i \text{ odd}} \hat{x}_i < 1, \tag{2}$$

where $\hat{s}_k$ equals the slack of the kth clique inequality, computed with respect to $\hat{x}$.

We recall the definition of the NP-complete decision problem Exact 3-Cover [12, Problem SP2].

> Let s be a multiple of 3, and let $S_1, \ldots, S_q \subset \{1, \ldots, s\}$ be such that $|S_k| = 3$ for $k = 1, \ldots, q$. Is there some $R \subseteq \{1, \ldots, q\}$ with $|R| = s/3$ such that $\bigcup_{k \in R} S_k = \{1, \ldots, s\}$?

Theorem 3.1. *Testing whether a given* $\hat{x} \in P = \{x \mid Ax \leq b\}$ *violates a* $\{0, 1/2\}$*-cut is strongly NP-complete, even when the corresponding integer linear program is a set-packing problem, the matrix* A *is a clique matrix, and the intersection graph of* A *contains only* $O(n)$ *maximal cliques.*

Proof. Given an instance of Exact 3-Cover, we construct a graph with $2s+2+q$ vertices and $2q+3$ maximal cliques (see Fig. 1). For $i = 1, \ldots, s$, we have two vertices, u_i and v_i. For $k = 1, \ldots, q$ we have a vertex w_k. We also add two further vertices, u^* and v^*. Edges are put into the graph so that there are $2q + 3$ maximal cliques, as follows. The vertices of type u will be mutually adjacent and form the *u-clique*. The vertices of type v will likewise be mutually adjacent and form the *v-clique*. The two vertices u^* and v^* will also be connected by an edge, forming the *2-clique*. For $k = 1, \ldots, q$, we connect w_k to the three u-vertices representing S_k, thus forming q cliques of cardinality 4. We will call these the *upper 4-cliques*. Finally, for $k = 1, \ldots, q$, we connect w_k to the three v-vertices representing S_k, thus forming q more cliques of cardinality 4. We will call these the *lower 4-cliques*.

We now let A equal the clique matrix of this graph. (Note that A has $2q + 3$ rows and $2s + 2 + q$ columns.) We define a vector $\hat{x} \in P$ as follows. For $i = 1, \ldots, s$, we set the component of $\hat{x}$ corresponding to u_i to $2/(3s+3)$, and we do the same for v_i. We set the component of $\hat{x}$ corresponding to u^* to $(s+3)/(3s+3)$, and we do the same for v^*. Finally, for $k = 1, \ldots, q$, we set the component of $\hat{x}$ corresponding to w_k to $(3s - 6)/(3s + 3)$.

It is readily checked that the u-clique and the v-cliques have slack 0, the 2-clique has slack $(s - 3)/(3s + 3)$, and each of the upper and lower 4-cliques have slack $3/(3s + 3)$.

If the ϕ coefficient of a given vertex is odd, then we say that the vertex is *exposed*. Each w vertex is contained in exactly two cliques (an upper 4-clique and a lower 4-clique). An exposed w vertex contributes $(3s - 6)/(3s + 3)$ to the left-hand side of (2). Thus, there is at most one exposed w vertex.

Suppose there was *exactly one* exposed w vertex. As each upper and lower 4-clique used contributes $3/(s+3)$ to the left-hand side of (2), at most two of them could be used in the Gomory–Chvátal derivation. In fact, exactly one would have to be used, otherwise there would be either zero or two exposed w vertices. Moreover, the 2-clique could not be used either, because it would contribute $(s-3)/(3s+3)$ to the left-hand side of (2). Only the u and v cliques remain, and the $\{0, 1/2\}$-cut becomes vacuous. Therefore, there are no exposed w vertices.

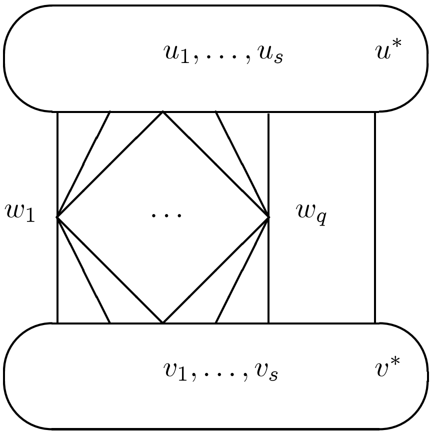

Fig. 1. Graph used in the proof.

Thus, we have shown that, if an upper 4-clique is used, the corresponding lower 4-clique must be used as well. That is, the 4-cliques come in *pairs*. Then, in order for the number of cliques used to be odd, we must use either one or three of the u-, v- and 2-cliques.

Suppose we use the u-clique but not the v- or 2-cliques. The vertex u^* is exposed, contributing $(s+3)/(3s+3)$ to the left-hand side of (2). Suppose that we use K *pairs*. Each pair contributes $6/(3s+3)$ to the left-hand side. Moreover, the number of exposed u_i vertices is at least $s - 3K$, and each contributes $2/(3s + 3)$ to the left-hand side. Thus, the left-hand side is at least $(s+3+6K+2s-6K)/(3s+3) = 1$, and the cut is not violated. By symmetry, we cannot use the v-clique without using the u- and 2-cliques. Moreover, we cannot use the 2-clique without using the u- and v-cliques, because this would immediately contribute 1 to the left-hand side of (2).

In order to obtain a violated cut, then, we must use the u-, v- and 2-cliques, together with a number of *pairs*. Suppose we use K *pairs*. Each *pair* contributes $6/(3s + 3)$ to the left-hand side of (2), and the 2-clique contributes $(s - 3)/(3s + 3)$. Moreover, the number of exposed u vertices is at least $\max\{0, s - 3K\}$, and the same holds for the number of exposed v vertices. Thus, the left-hand side of (2) is at least

$$6K/(3s + 3) + (s - 3)/(3s + 3) + \max\{0, 4s - 12K\}/(3s + 3).$$

It is readily checked that this is less than 1 if and only if $K = s/3$. Thus, there is a violated $\{0, 1/2\}$-cut if and only if $K = s/3$ and there are no exposed vertices at all. This is true if and only if, for $i = 1, \ldots, s$, vertex u_i appears in exactly one of the $s/3$ upper 4-cliques and vertex v_i appears in exactly one of the $s/3$ lower 4-cliques. Thus, there is a violated $\{0, 1/2\}$-cut if and only if there is a solution to Exact 3-Cover. □

4. Concluding remarks

It is not difficult to see that finding a stable set of maximum weight in graphs of the type used in the proof of Theorem 3.1 can be performed in polynomial time (by enumerating over all possible choices of a u-vertex, and all possible choices of a v-vertex). Therefore, the hardness result holds even if the associated integer linear program itself is polynomially solvable. On the other hand, Caprara and Salazar [6] consider an interesting class of NP-hard set-packing problems for which the separation of {0, 1/2}-cuts is polynomially solvable. So the complexity of a class of integer linear programs is not related to the complexity of the separation problem for the associated {0, 1/2}-cuts. See also [5,9].

It is worth pointing out that the hardness proof of Section 3 can easily be adapted to set-partitioning and set-covering problems. This is interesting because Bienstock and Zuckerberg [1] have recently shown that, in the case of set covering, one can separate over *all* Gomory–Chvátal-cuts to an arbitrary fixed precision in polynomial time.

Naturally, our results imply that it is NP-hard to optimize a linear function over the {0, 1/2}-closure of a polyhedron $P \subseteq [0, 1]^n$. This provides an interesting contrast to the fact that one can optimize in polynomial time over the elementary closures associated with lift-and-project, Sherali–Adams, Lovász–Schrijver, and Lasserre cuts (see, e.g., [8]).

For Caprara and Fischetti's second proof of their hardness result (in [2]), it is not difficult to see that the {0, 1/2}-closure and the Gomory–Chvátal closure coincide [11]. In particular, testing membership (or separation) over the Gomory–Chvátal closure is NP-hard in general. However, in spite of the results provided herein, it remains unknown whether testing membership for the Gomory–Chvátal closure remains NP-hard for rational polytopes contained in the unit cube.

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