Testing membership in the $\{0, 1/2\}$-closure is strongly NP-hard, even for polytopes contained in the n-dimensional 0/1-cube

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Abstract Caprara and Fischetti introduced a class of cutting planes, called $\{0, 1/2\}$-cuts, which are valid for arbitrary integer linear programs. They also showed that the associated separation problem is strongly NP-hard. We show that separation remains strongly NP-hard, even when all integer variables are binary, even when the integer linear program is a set packing problem, and even when the matrix of left-hand side coefficients is the clique matrix of a graph containing a small number of maximal cliques. In fact, we show these results for the membership problem, which is weaker than separation.

1 Introduction

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron defined by an integer matrix $A \in \mathbb{Z}^{m\times n}$ and an integer vector $b \in \mathbb{Z}^m$. Various schemes are known for finding a linear description of its integer hull, $P_I = \text{conv}\{x \in \mathbb{Z}^n : Ax \leq b\}$. Several of these convexification procedures are confined to polyhedra whose integer hulls are 0/1-polytopes; i.e., they have vertices with coordinates 0 or 1 only. Among them are the lift and project, Sherali and Adams, Lovász and Schrijver, and Lasserre hierarchies [1,18,15,14]. One convexification method that works for arbitrary polyhedra is based on Gomory-Chvátal cuts [13,6]. A Gomory-Chátal cut is an inequality of the form

$$(\lambda^T A)x \leq \lfloor \lambda^T b \rfloor,$$

where the multiplier vector $\lambda \in [0, 1)^m$ is chosen such that $\lambda^T A \in \mathbb{Z}^n$ and $\lambda^T b \notin \mathbb{Z}$. The elementary closure, P', is the set of points in P that satisfy all Gomory-Chvátal cuts. Obviously $P_I \subseteq P' \subseteq P$. It is well known that $P_I = P^{(k)}$ for some finite k, where $P^{(i+1)} = (P^{(i)})'$ [6,17]. Moreover, $k = \mathrm{O}(n^2 \log n)$ for $P \subseteq [0, 1]^n$ [11].

Noticing the prevalence of Gomory-Chvátal cuts for which $\lambda \in \{0, 1/2\}^m$, Caprara and Fischetti studied this special class of cuts, which they called $\{0, 1/2\}$-cuts [3]. The $\{0, 1/2\}$-closure, $P_{1/2}(A, b)$, is the set of all solutions to $Ax \leq b$ that also satisfy all $\{0, 1/2\}$-cuts. Clearly, $P' \subseteq P_{1/2}(A, b) \subseteq P$. The membership problem for the $\{0, 1/2\}$-closure is:

> Given an integral matrix $A \in \mathbb{Z}^{m\times n}$, an integral vector $b \in \mathbb{Z}^m$, and a rational vector $\hat{x} \in \mathbb{Q}^n$, is $\hat{x} \in P_{1/2}(A, b)$?

We may assume that $A\hat{x} \leq b$, so the problem is to test whether $\hat{x}$ violates a $\{0, 1/2\}$-cut. This problem is closely related to the separation problem, which has the same input, and requires to find a $\{0, 1/2\}$-cut that is violated by $\hat{x}$, if one exists. The relevance of this problem is due to the fact that many classes of inequalities in polyhedral studies of combinatorial optimization problems are $\{0, 1/2\}$-cuts. Caprara and Fischetti showed that the membership problem (and, therefore, the separation problem) for $\{0, 1/2\}$-cuts is strongly NP-hard in general (though polynomially solvable in certain cases). This result was strengthened by Caprara and Letchford, who showed that it remains strongly NP-hard, even when all variables are assumed to be non-negative (i.e., the inequalities $-x_i \leq 0$ for all $i = 1, \ldots, n$ are part of the system $Ax \leq b$) [4]. However, it was open whether this remains true for relaxations of 0/1-polytopes, the class of polytopes most relevant to combinatorial optimization.

We will show that membership testing stays strongly NP-hard even when $Ax \leq b$ defines a polytope P that is contained in the 0/1-cube; i.e., $P \subseteq [0, 1]^n$. In fact, we can establish the same result for fractional set packing polytopes, that is, polytopes of the form

$$P = \{x \in \mathbb{R}^n_+ : Ax \leq \mathbb{1}\},$$

where $A \in \{0, 1\}^{m\times n}$ is a binary matrix and $\mathbb{1}$ denotes the all-ones vector with m entries. We actually show this for a very special class of set packing problems, in which A is the clique matrix of a graph containing a small (linear) number of maximal cliques. These results provide an interesting contrast to the fact that one can optimize in polynomial time over the elementary closures associated with lift-and-project, Sherali-Adams, Lovász-Schrijver, and Lasserre cuts (see, e.g., [7]).

This paper is organized as follows. In Section 2, we will show how to refine the reduction in [3] to obtain the stronger result for polytopes contained in $[0, 1]^n$. We provide a completely different reduction in Section 3, which gives rise to the set packing result. Section 4 contains our concluding remarks.

2 A reduction from Decoding of Linear Codes

The following problem, known as Decoding of Linear Codes, is NP-complete [12, Problem MS7]:

> Given a matrix $Q \in \{0, 1\}^{r\times t}$, a vector $d \in \{0, 1\}^r$, and a positive integer K, is there a $z \in \{0, 1\}^t$ with no more than K ones such that $Qz \equiv d \mod 2$?

Theorem 1. *The membership problem for the* $\{0, 1/2\}$*-closure is coNP-complete.*

Our reduction carefully modifies the original proof of Caprara and Fischetti so as to ensure that $P \subseteq [0, 1]^n$.

Proof. Membership testing is clearly in coNP. We give a reduction from Decoding of Linear Codes to show its completeness. Let Q, d, and K describe an instance of Decoding of Linear Codes. We construct the following instance of the membership problem for the $\{0,1/2\}$-closure:

$$A = \begin{pmatrix} Q^\intercal & 2I_{t+1} \\ d^\intercal & \\ 2I_r & 0 \\ -2I_r & 0 \\ 0 & -3I_{t+1} \end{pmatrix},$$

$$b = (2 \cdot \mathbf{1}^t, 1, 2 \cdot \mathbf{1}^r, \mathbf{0}^r, \mathbf{0}^{t+1})^\intercal,$$
$$\hat{x} = (\mathbf{0}^r, \mathbf{1}^t - \frac{1}{2}w^\intercal, \frac{1}{2})^\intercal,$$

where $\mathbf{0}^l = \{0\}^l$, $\mathbf{1}^l = \{1\}^l$, and $w = (1/(K+1), \ldots, 1/(K+1)) \in \mathbb{Q}^t$. As a first step we prove that $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \subseteq [0,1]^n$. Consider row l of $Ax \leq b$:

Let $(t+1)+1 \leq l \leq (t+1)+r$. We obtain the inequality $2x_{l-(t+1)} \leq 2$ and therefore $x_{l-(t+1)} \leq 1$. After shifting indices we obtain $x_l \leq 1$ for all $1 \leq l \leq r$.

Let $(t+1)+r+1 \leq l \leq (t+1)+2r$. We have the inequality $-2x_{l-(t+1+r)} \leq 0$ and therefore $x_{l-(t+1+r)} \geq 0$. After shifting indices we obtain $x_l \geq 0$ for all $1 \leq l \leq r$.

We have thus established that the first r variables are indeed in $[0,1]$. We further get inequalities of the form $\sum_{j=1}^r q_{ji}x_j + 2x_{r+l} \leq 2$ for $1 \leq l \leq t$ and $\sum_{j=1}^r d_j x_j + 2x_{r+l} \leq 1$ for $l = t+1$. As $Q \in \{0,1\}^{r \times t}$ it suffices to observe that $x_j \geq 0$ for all $j \in \{1, \ldots, r\}$; therefore, $x_{r+l} \leq 1$ for all $1 \leq l \leq t+1$.

Finally, consider row l with $(t+1)+2r+1 \leq l \leq (t+1)+2r$. The corresponding inequalities are of the form $-3x_{r+l-((t+1)+2r)} \leq 0$ and therefore $x_{r+l} \geq 0$ for all $1 \leq l \leq t+1$. Hence, $P \subseteq [0,1]^n$.

Note that $b - A\hat{x} = (w_1, \ldots, w_t, 0, 2 \cdot \mathbf{1}^r, \mathbf{0}^r, 3 - \frac{3}{2}w_1, \ldots, 3 - \frac{3}{2}w_t, \frac{3}{2})^\intercal$. In particular, $\hat{x} \in P$.

Moreover, $\hat{x} \notin P_{1/2}(A,b)$ if and only if there exists $\mu \in \{0,1\}^{2(t+1)+2r}$ such that $\mu^\intercal A \equiv 0 \pmod 2$, $\mu^\intercal b \equiv 1 \pmod 2$, and $\mu^\intercal(b - A\hat{x}) < 1$. (In this case, $\frac{1}{2}(\mu^\intercal A)x \leq \lfloor \frac{1}{2}\mu^\intercal b \rfloor$ cuts of $\hat{x}$.) Observe that for $\mu^\intercal A \equiv 0 \pmod 2$ it is necessary that $\mu_l = 0$ for $(t+2)+2r \leq l \leq 2(t+1)+2r$. Furthermore, $\mu^\intercal b \equiv 1 \pmod 2$ if and only if $\mu_{t+1} = 1$. Having established this, we get that $\mu^\intercal A \equiv 0 \pmod 2$ and $\mu^\intercal b \equiv 1 \pmod 2$ if and only if $Qz \equiv d \pmod 2$ with $z \in \{0,1\}^t$, $z_l = \mu_l$ for $1 \leq l \leq t$. Note that the remaining μ_l for the reverse direction can be chosen arbitrarily for the rows l such that $A_l \equiv 0 \pmod 2$; the other entries of μ are already fixed by the conditions we just stated.

It remains to show that $\mu^\intercal(b - A\hat{x}) < 1$ if and only if $w^\intercal z < 1$ with $z \in \{0,1\}^t$, $z_l = \mu_l$ for $1 \leq l \leq t$ chosen as above. Let $\mu^\intercal(b - A\hat{x}) < 1$. Then $\mu^\intercal(w_1, \ldots, w_t, 0, 2 \cdot \mathbf{1}^r, \mathbf{0}^r, 3 - \frac{3}{2}w_1, \ldots, 3 - \frac{3}{2}w_t, \frac{3}{2})^\intercal < 1$. Therefore, $w^\intercal z < 1$ for $z \in \{0,1\}^t$ with $z_l = \mu_l$ for $1 \leq l \leq t$. Conversely, let $w^\intercal z < 1$ for some $z \in \{0,1\}^t$. Define $\mu \in \{0,1\}^{2(t+1)+2r}$ via $\mu_l = z_l$ for $1 \leq l \leq t$, $\mu_{t+1} = 1$, and $\mu_l = 0$ otherwise. Then $1 > w^\intercal z = \mu^\intercal(w_1, \ldots, w_t, 0, 2 \cdot \mathbf{1}^r, \mathbf{0}^r, 3 - \frac{3}{2}w_1, \ldots, 3 -$

$\frac{3}{2}w_t, \frac{3}{2})^\intercal = \mu^\intercal(b - A\hat{x})$. Thus there is a violated $\{0, 1/2\}$-cut if and only if there is a solution to Decoding of Linear Codes. □

3 Reduction from Exact 3-Cover

For a given 0-1 matrix A, the *intersection graph* or *conflict graph* is an undirected graph with vertex set $V = \{1, \ldots, n\}$, and an edge $\{i, j\}$ if and only if there is at least one row of A with a '1' in the ith and jth columns [16]. The edge $\{i, j\}$ represents the fact that x_i and x_j cannot take the value 1 simultaneously. Clearly, the set packing problem amounts to the problem of finding a maximum weight stable set (set of pairwise non-adjacent vertices) in the intersection graph. Padberg [16] showed that every clique C (i.e., every set of pairwise adjacent vertices) in the intersection graph yields a valid *clique inequality* $\sum_{j \in C} x_j \leq 1$ for the set packing polytope, and that such an inequality induces a facet of if and only if the clique is maximal.

In general, there may be many facet-inducing clique inequalities which are not represented in the system $Ax \leq \mathbf{1}$. Indeed, the number of maximal cliques can be exponential in both n and m. If, however, there is a one-to-one correspondence between the rows of A and the maximal cliques of the intersection graph (i.e., the system $Ax \leq \mathbf{1}$ consists of the facet-inducing clique inequalities), then A is said to be a *clique matrix*. When A is a clique matrix, the $\{0, 1/2\}$-cuts can be shown to include many interesting valid inequalities, such as the *odd hole* and *odd antihole* inequalities of Padberg, and some of the *web* and *antiweb* inequalities of Trotter [19]. It also follows from matching theory that the $\{0, \frac{1}{2}\}$-cuts give a complete description of P_I when A is the clique matrix of a line graph [9]. Here, however, we are concerned with separation.

We will find it helpful to write the $\{0, 1/2\}$-cuts in a certain explicit form. Let $t \geq 1$ be an odd integer and let $C_1, \ldots, C_t$ be maximal cliques whose associated clique inequalities are to be used (receive a multiplier of 1/2) in the derivation of the cut. For $i = 1, \ldots, n$, let ϕ_i represent the number of these cliques which contain i. That is, $\phi_i = |\{k \in \{1, ..., t\} : i \in C_k\}|$. Then, we must use (set the multiplier to 1/2 for) a non-negativity inequality $-x_i \leq 0$ for each $i \in V$ such that ϕ_i is odd. Thus, the cut takes the form:

$$\sum_{i=1}^{n} \lfloor \phi_i/2 \rfloor x_i \leq \lfloor t/2 \rfloor.$$

Multiplying by two, we see that this is equivalent to

$$\sum_{k=1}^{t} \sum_{i \in C_k} x_i - \sum_{\phi_i \text{ odd}} x_i \leq t - 1.$$

Following [3], we define the *slack variables* $s_k = 1 - \sum_{i \in C_k} x_i$ for $i = 1, \ldots, t$. The cut can then be written as

$$\sum_{k=1}^{t} s_k + \sum_{\phi_i \text{ odd}} x_i \geq 1.$$

Thus, we see that the $\{0, 1/2\}$-cut derived using cliques $C_1, \ldots, C_t$ is violated by a given $\hat{x}$ if and only if

$$\sum_{k=1}^{t} \hat{s}_k + \sum_{\phi_i \text{ odd}} \hat{x}_i < 1, \tag{1}$$

where $\hat{s}_k$ equals the slack of the kth clique inequality, computed with respect to $\hat{x}$.

We recall the definition of the NP-complete decision problem Exact 3-Cover [12, Problem SP2]:

> Let s be a multiple of three and let $S_1, \ldots, S_q \subset \{1, \ldots, s\}$ be such that $|S_k| = 3$ for $k = 1, \ldots, q$. Is there some $R \subseteq \{1, \ldots, q\}$ with $|R| = s/3$ such that $\bigcup_{k \in R} S_k = \{1, \ldots, s\}$?

Theorem 2. *Testing whether a given $\hat{x} \in P$ violates a $\{0, 1/2\}$-cut is strongly $\mathcal{NP}$-complete, even when the corresponding integer linear program is a set packing problem, the matrix A is a clique matrix, and the intersection graph of A contains only $O(n)$ maximal cliques.*

Proof. Given an instance of Exact 3-Cover, we construct a graph with $2s + 2 + q$ vertices and $2q + 3$ maximal cliques (see Figure 1). For $i = 1, \ldots, s$, we have two vertices u_i and v_i. For $k = 1, \ldots, q$ we have a vertex w_k. We also add two further vertices u^* and v^*. Edges are put into the graph so that there are $2q + 3$ maximal cliques, as follows. The vertices of type 'u' will be mutually adjacent and form the *u-clique*. The vertices of type 'v' will likewise be mutually adjacent and form the *v-clique*. The two vertices u^* and v^* will also be connected by an edge, forming the *2-clique*. For $k = 1, \ldots, q$, we connect w_k to the three u-vertices representing S_k, thus forming q cliques of cardinality 4. We will call these the *upper 4-cliques*. Finally, for $k = 1, \ldots, q$, we connect w_k to the three v-vertices representing S_k, thus forming q more cliques of cardinality 4. We will call these the *lower 4-cliques*.

We now let A equal the clique matrix of this graph. (Note that A has $2q + 3$ rows and $2s+2+q$ columns.) We define a vector $\hat{x} \in P$ as follows. For $i = 1, \ldots, s$, we set the component of $\hat{x}$ corresponding to u_i to $2/(3s + 3)$, and we do the same for v_i. We set the component of $\hat{x}$ corresponding to u^* to $(s + 3)/(3s + 3)$. Finally, for $k = 1, \ldots, q$, we set the component of $\hat{x}$ corresponding to w_k to $(3s - 6)/(3s + 3)$.

It is readily checked that the u-clique and the v-cliques have slack zero, the 2-clique has slack $(s - 3)/(3s + 3)$, and each of the upper and lower 4-cliques have slack $3/(3s + 3)$.

If the ϕ coefficient of a given vertex is odd, then we say that the vertex is *exposed*. Each w vertex is contained in exactly two cliques (an upper 4-clique and a lower 4-clique). An exposed w vertex contributes $(3s - 6)/(3s + 3)$ to the left-hand side of (1). Thus, there is at most one exposed w vertex.

Suppose there was *exactly one* exposed w vertex. As each upper and lower 4-clique used contributes $3/(s + 3)$ to the left-hand side of (1), at most two

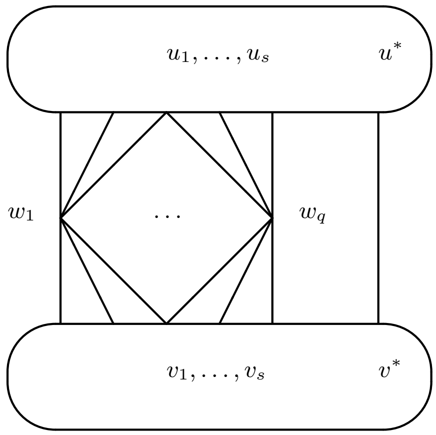

Figure 1. Graph used in proof

of them could be used in the Gomory-Chvátal derivation. In fact, exactly one would have to be used, otherwise there would be either zero or two exposed w vertices. Moreover, the 2-clique could not be used either, because it would contribute $(s-3)/(3s+3)$ to the left-hand side of (1). Only the u and v cliques remain, and the $\{0, 1/2\}$-cut becomes vacuous. Therefore, there are no exposed w vertices.

Thus, we have shown that if an upper 4-clique is used, the corresponding lower 4-clique must be used as well. That is, the 4-cliques come in *pairs*. Then, in order for the number of cliques used to be odd, we must use either one or three of the u-, v- and 2-cliques.

Suppose we use the u-clique but not the v- or 2-cliques. The vertex u^* is exposed, contributing $(s+3)/(3s+3)$ to the left-hand side of (1). Suppose we use K *pairs*. Each pair contributes $6/(3s+3)$ to the left-hand side. Moreover, the number of exposed u vertices is at least $s - 3K$ and each contributes $2/(3s+3)$ to the left-hand side. Thus, the left-hand side is at least $(s + 3 + 6K + 2s - 6K)/(3s + 3) = 1$ and the cut is not violated. By symmetry, we cannot use the v-clique without using the u- and 2-cliques. Moreover, we cannot use the 2-clique without using the u- and v-cliques because this would immediately contribute 1 to the left-hand side of (1).

In order to obtain a violated cut, then, we must use the u- v- and 2-cliques, together with a number of *pairs*. Suppose we use K *pairs*. Each *pair* contributes $6/(3s+3)$ to the left-hand side of (1) and the 2-clique contributes $(s-3)/(3s+3)$. Moreover, the number of exposed u vertices is at least $\max\{0, s - 3K\}$ and the same holds for the number of exposed v vertices. Thus, the left-hand side of (1) is at least

$$6K/(3s + 3) + (s - 3)/(3s + 3) + \max\{0, 4s - 12K\}/(3s + 3).$$

It is readily checked that this is less than one if and only if $K = s/3$. Thus, there is a violated $\{0, 1/2\}$-cut if and only if $K = s/3$ and there are no exposed vertices at all. This is true if and only if, for $i = 1, \ldots, s$, vertex u_i appears in exactly one of the $s/3$ upper 4-cliques and vertex v_i appears in exactly one of the $s/3$ lower 4-cliques. Thus, there is a violated $\{0, 1/2\}$-cut if and only if there is a solution to EXACT 3-COVER. □

4 Concluding Remarks

It is not difficult to see that finding a stable set of maximum weight in graphs of the type used in the proof of Theorem 2 can be performed in polynomial time. Therefore the hardness result holds even if the associated integer linear program itself is polynomially solvable. On the other hand, Caprara and Salazar [5] consider an interesting class of NP-hard set packing problems for which the separation of $\{0, 1/2\}$-cuts is polynomially solvable. So the complexity of a class of integer linear programs is not related to the complexity of the separation problem for the associated $\{0, 1/2\}$-cuts. See also Caprara and Letchford [4] and Cornuéjols and Li [8].

It is worth pointing out that the hardness proof of Section 3 can easily be adapted to set partitioning and set covering problems. This is also interesting, because Bienstock and Zuckerberg [2] have recently shown that, in the case of set covering, one can separate over *all* Gomory-Chvátal-cuts to an arbitrary fixed precision in polynomial time.

Our results imply that it is NP-hard to optimize a linear function over the $\{0, 1/2\}$-closure of a polyhedron $P \subseteq [0, 1]^n$. Eisenbrand [10] had previously observed that for the instances used in the original proof of Caprara and Fischetti [3], $P_{1/2}(A, b) = P'$. In particular, optimizing a linear function over the first elementary Gomory-Chvátal closure is NP-hard in general. However, Eisenbrand's observation does not apply to the proofs provided herein. In particular, it is unknown whether testing membership (and, hence, optimization) for the elementary Gomory-Chvátal closure remains NP-hard for polyhedra P contained in $[0, 1]^n$.

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