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Note

The permutahedron of series-parallel posets

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Abstract

The permutahedron *Perm*(*P*) of a poset *P* is defined as the convex hull of those permutations that are linear extensions of *P*. Von Arnim et al. (1990) gave a linear description of the permutahedron of a series-parallel poset. Unfortunately, their main theorem characterizing the facet defining inequalities is only correct for not series-decomposable posets. We do not only give a proof of the revised version of this theorem but also extend it partially to the case of arbitrary posets and obtain a new complete and minimal description of *Perm*(*P*) if *P* is series-parallel. Furthermore, we summarize briefly results about the corresponding separation problem.

1. The permutahedron of a poset

We follow to a large extent the notation and terminology introduced in [2]. Let $P = (N, <_P)$ be a partially ordered set (poset) on $N = \{1, \dots, n\}$. The permutahedron of P is defined as the convex hull of those permutations $\pi : N \to \{1, \dots, n\}$ that are linear extensions of P,

$$Perm(P) := \operatorname{conv}\{(\pi(1), \dots, \pi(n)) \colon \pi \text{ permutation with } \pi(u) < \pi(v) \text{ if } u <_P v\}.$$

The poset P has an unique series decomposition $P = P_1 * \cdots * P_k$ where $P_1, \dots, P_k$ are neither empty nor series-decomposable. The dimension of the polytope $Perm(P)$ is determined by the number of these suborders.

Theorem 1.1. *Let P be a poset with series decomposition $P_1 * \cdots * P_k$. Then*

$$x(P_i) = f(P_i) + |P_i|(|P_1| + \cdots + |P_{i-1}|), \quad i = 1, \dots, k \tag{1}$$

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is a maximal irredundant linear equation system for Perm(P), where $f(S) := \frac{1}{2}|S|(|S| + 1)$ *for* $S \subseteq P$.

Proof. First, observe that each linear extension π of P is the concatenation of linear extensions π_i of P_i, i.e. $\pi = (\pi_1, \ldots, \pi_k)$ where each component of π_i is shifted by $|P_1| + \cdots + |P_{i-1}|$. Therefore, equation system (1) is valid with regard to *Perm(P)*. Obviously, the matrix of this equation system has full row rank. Thus, it remains to be shown that for any equation $ax = \alpha$ satisfied by all linear extensions of P, there exists $\lambda \in \mathbb{R}^k$ such that $\lambda_i = a_u$ for all $u \in P_i$, $i \in \{1, \ldots, k\}$. Let $ax = \alpha$ be such an equation and let u, v for $i \in \{1, \ldots, k\}$ be two distinct elements of P_i. We have to show that $a_u = a_v$. We consider two cases.

(i) Let u and v be incomparable. Then there exists a linear extension π of P such that $\pi(v) = \pi(u) + 1$. Let $\bar{\pi}$ be the adjacent transposition of π with $\bar{\pi}(u) = \pi(v)$ and $\bar{\pi}(v) = \pi(u)$. Then $\bar{\pi}$ is also a linear extension of P. Therefore, $0 = a\pi - a\bar{\pi} = (a_v - a_u)(\pi(v) - \pi(u))$, i.e. $a_u = a_v$.

(ii) Now assume that u and v are comparable. It suffices to consider the case that u is covered by v, i.e. there does not exist an element $w \in P_i \setminus \{u, v\}$ with $u <_P w <_P v$. Let $U \subset P_i \setminus \{u\}$ be the set of elements that are incomparable with u. Notice that $|U| \geqslant 1$ since P_i is not series-decomposable. If there exists an element $\tilde{u} \in U$ being also incomparable with v, we get $a_u = a_{\tilde{u}} = a_v$ applying (i). Otherwise, each element of U precedes v. Then let V be the set of those elements in $P_i \setminus \{v\}$ that are incomparable with v. All elements of V are successors of u. Since P_i is not series-decomposable, there exist two incomparable elements $\tilde{u} \in U$ and $\tilde{v} \in V$. Thus, using (i) again, we obtain $a_u = a_{\tilde{u}} = a_{\tilde{v}} = a_v$. □

Note that Theorem 1.1 holds for arbitrary posets. Therefore, it implies the following extension of Theorem 4.3 of [2].

Corollary 1.2. *Let P be a poset with series decomposition $P_1 * \cdots * P_k$. Then*

$$\dim(Perm(P)) = |P| - k$$

and there exist $|P| - k + 1$ linearly independent linear extensions of P.

2. Facet inducing ideal constraints

Von Arnim et al. [2] derived the following two classes of valid inequalities for *Perm(P)*: *ideal constraints*

$$x(I) \geqslant f(I), \quad I \subset P \text{ is an ideal of } P$$

and *convex set constraints*

$$|A|x(B) - |B|x(A) \geqslant \tfrac{1}{2}|A||B|(|A| + |B|), \quad A * B \subset P \text{ convex.}$$

These two classes of inequalities together with the equation system (1) are sufficient to describe $Perm(P)$ completely if P is series-parallel (cf. [2]).

The following theorem corrects and extends Theorem 4.6(i) in [2] that is only correct for posets that are not series-decomposable. Before stating it, we observe that any ideal $I \subseteq P$ is always of the form $I = P_1 * \cdots * P_i * \hat{I}$ for some $i \in \{0, \ldots, k-1\}$ where $\hat{I}$ is an ideal of the suborder P_{i+1} and P has the series decomposition $P_1 * \cdots * P_k$. Furthermore, such an ideal I induces a nontrivial and proper face of $Perm(P)$ if and only if $\emptyset \subset \hat{I} \subset P_{i+1}$.

Theorem 2.1. *Let P be a poset with series decomposition $P_1 * \cdots * P_k$. An ideal $I = P_1 * \cdots * P_i * \hat{I}, i \in \{0, \ldots, k-1\}$, defines a facet of $Perm(P)$ if and only if $\emptyset \subset \hat{I} \subset P_{i+1}$ and both $\hat{I}$ and $P_{i+1} \backslash \hat{I}$ are not series-decomposable.*

Proof. The most important observation for proving this theorem is the fact that the face induced by an ideal I is itself the permutahedron of an appropriate poset. More precisely, let Q be the poset defined by $Q := P_1 * \cdots * P_i * \hat{I} * (P_{i+1} \backslash \hat{I}) * P_{i+2} * \cdots * P_k$. Then

$$\{x \in Perm(P)\colon x(I) = f(I)\} = Perm(Q).$$

Now Corollary 1.2 completes the proof. □

The characterization of the facet defining ideal constraints as stated in Theorem 2.1 was independently obtained by von Arnim [1].

3. The series-parallel case

Although it is possible to characterize those series-decomposable convex sets that induce facets of the permutahedron of an arbitrary poset (cf. [3]), we restrict ourselves here to the series-parallel case, just correcting Theorem 4.6(ii) of [2]. A series-decomposable set $C = A * B \subseteq P$ is called *bipartite* if neither A nor B is itself series-decomposable.

Theorem 3.1. *Let P be a series-parallel poset with series decomposition $P_1 * \cdots * P_k$. A bipartite convex set $C = A * B$ induces a facet of $Perm(P)$ if and only if*

(a) *$C \subset P_i$, for some $i \in \{1, \ldots, k\}$, or*

(b) *$A = P_i$, $B \subset P_{i+1}$, for some $i \in \{1, \ldots, k-1\}$ and $P_{i+1} \backslash B$ is not series-decomposable, or*

(c) *$A \subset P_i$, $B = P_{i+1}$, for some $i \in \{1, \ldots, k-1\}$ and $P_i \backslash A$ is not series-decomposable.*

Proof. Let $C = A * B$ be a bipartite convex set that induces a facet of $Perm(P)$. Suppose that neither the set inclusion of (a) nor the set relationships of (b) nor those of (c) are valid. Not (a) implies that $C \not\subseteq P_i$ for all $i \in \{1, \dots, k\}$. P_i not series-decomposable implies that $C \neq P_i$. Since C is convex and bipartite, we have $A \subseteq P_i$ and $B \subseteq P_{i+1}$ for some $i \in \{1, \dots, k-1\}$. Regarding not (b) and not (c), there are only two possibilities, namely $A \subset P_i$ and $B \subset P_{i+1}$, or $A = P_i$ and $B = P_{i+1}$. From the latter we obtain by Theorem 1.1 that every $x \in Perm(P)$ satisfies the inequality under consideration with equality and therefore a contradiction. In order to lead the other case to a contradiction we use

$$\{x \in Perm(P)\colon |A|x(B) - |B|x(A) = \tfrac{1}{2}|A||B|(|A| + |B|)\} = Perm(Q), \tag{2}$$

where the extension Q of P is defined as $Q := P_1 * \cdots * (P_i \backslash A) * A * B * (P_{i+1} \backslash B) * \cdots * P_k$. Equation (2) was already observed by von Arnim et al. Hence, using Corollary 1.2 we obtain $\dim(Perm(Q)) \leqslant n - k - 2$, contradicting our assumption. Thus we have proved that C has to satisfy one of the set relationships (a)–(c). Therefore, the last observation needed is that $P_{i+1} \backslash B$ in case (b) and $P_i \backslash A$ in case (c) are not series-decomposable, respectively. This follows also from Eq. (2), since otherwise Corollary 1.2 implies again that the respective convex set constraint defines no facet.

For the converse direction we already know that (a) implies that C defines a facet. This is due to von Arnim et al. In case (b) and (c) we consider once more the corresponding poset Q that satisfies (2). Since $P_{i+1} \backslash B$ and $P_i \backslash A$ are not series-decomposable, Corollary 1.2 implies that these convex set constraints are facet defining ones, respectively. $\square$

Note that Theorem 3.1 does not exclude that there are series-decomposable convex subsets of P which are not bipartite but facet defining. However, it is quite easy to show that these facets are identical to some of those mentioned in Theorem 3.1.

There is another important remark on the facet defining inequalities of Theorems 2.1 and 3.1. They do not necessarily induce distinct facets. First, notice that $C = P_i * B$ is convex and bipartite with $P_{i+1} \backslash B$ not series-decomposable if and only if $I = P_1 * \cdots * P_i * B$ is an ideal with B and $P_{i+1} \backslash B$ not series-decomposable. Analogously, $D = A * P_{i+1}$ is convex and bipartite with $P_i \backslash A$ not series-decomposable if and only if $J = P_1 * \cdots * P_{i-1} * (P_i \backslash A)$ is an ideal and both A and $P_i \backslash A$ are not series-decomposable. If one remembers the proofs of Theorems 2.1 and 3.1, it is not hard to see that the facets induced by C and I as well as those induced by D and J, respectively, are identical. Observe further that all the other inequalities characterized in these theorems induce mutually distinct facets.

Given a polyhedron $T = \{x \in \mathbb{R}^n\colon Ax \geqslant b\}$, there is at least one inequality $ax \geqslant \beta$ of the system $Ax \geqslant b$ with $F = \{x \in T : ax = \beta\}$ for each facet F of T. Furthermore, T is completely described by the system which is obtained by taking one of these inequalities for each facet together with a maximal irredundant equation system for T.

Therefore, the above observations together with Theorem 3.3 of [2] imply the following theorem.

Theorem 3.2. *Let P be a series-parallel poset with series decomposition* $P_1 * \cdots * P_k$. *Then the following linear system is a complete and minimal linear description of Perm(P):*

$$x(P_i) = f(P_i) + |P_i|(|P_1| + \cdots + |P_{i-1}|), \quad i = 1, \ldots, k;$$

$$x(I) \geqslant f(I), \quad \textit{for all ideals } I (= P_1 * \cdots * P_i * \hat{I} \textit{ for some } i \in \{0, \ldots, k-1\}), \hat{I} \textit{ and } P_{i+1} \backslash \hat{I} \textit{ not series-decomposable}, \emptyset \subset \hat{I} \subset P_{i+1};$$

$$|A|x(B) - |B|x(A) \geqslant \tfrac{1}{2}|A||B|(|A| + |B|), \quad A * B \subset P_i \textit{ convex and bipartite for some } i \in \{1, \ldots, k\}.$$

4. The separation problem

The complete description of *Perm(P)* by means of linear equations and inequalities leads naturally to the question how to solve the separation problem for a given point in $\mathbb{R}^n$ and *Perm(P)*. Queyranne and Wang (cf. [4, 5]) studied a closely related full-dimensional single machine scheduling polyhedron $P(N)$. They derived inequalities corresponding to initial (ideal) and intermediate (convex) sets, respectively, which turn out to be identical to those mentioned above when all job processing times are equal to one. In this case *Perm(P)* is a proper face of $P(N)$, and it is a facet if and only if the poset P is not series-decomposable.

Queyranne and Wang [5] proposed an $O(n \log n)$ separation algorithm for the class of ideal constraints that is based on sorting the components of the given point for which the separation problem has to be solved. They adapted this algorithm to the class of those convex sets $A * B$, where A or B is a singleton. It is even possible to extend this idea to the case $|A| = p$ or $|B| = p$ where p is fixed, providing an $O(n^{p+1})$ algorithm. Nevertheless, in the case of a series-parallel poset P it is easy to include the whole class of convex set constraints using the binary decomposition tree of P as follows.

We are interested in series-decomposable sets $A * B$. Thus, it is obviously sufficient to consider only those tree nodes corresponding to a series composition. We consider these nodes separately. The basic idea is now to fix the cardinalities of A and B and to compute for each possible combination of these cardinalities at the actual tree node the difference between the left-hand side and the right-hand side of the convex set constraint built by the set of the biggest elements of the left subtree and the set of the smallest elements of the right subtree. This is quite natural and the reader should be able to fill in the details.

However, determining the computational complexity of the separation problem for the class of convex set constraints in the case of an arbitrary poset P remains an open problem.

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