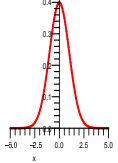
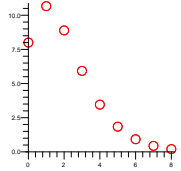
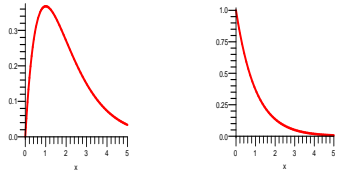
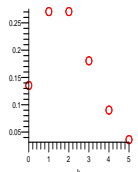
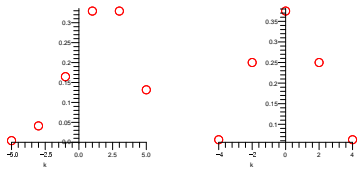
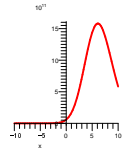


Free Meixner distributions and random matrices

Michael Anshelevich

July 13, 2006

Some common distributions first...

<p style="text-align: center;">Gaussian</p>	<p style="text-align: center;">Negative binomial Pascal geometric</p>	<p style="text-align: center;">Gamma chi-square exponential</p>
$\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$  <p style="text-align: center;">$(-\infty, \infty)$</p>	$\sum_{n=0}^{\infty} \binom{t+n-1}{t-1} (1-\mu)^t \mu^n \delta_n$  <p style="text-align: center;">$\{0, 1, 2, \dots\}$</p>	$\frac{1}{\Gamma(t)} e^{-x} x^{t-1} \mathbf{1}_{[0, \infty)}(x)$  <p style="text-align: center;">$[0, \infty)$</p>
<p style="text-align: center;">Poisson</p>	<p style="text-align: center;">Binomial</p>	<p style="text-align: center;">Hyperbolic tangent</p>
$e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta_n$  <p style="text-align: center;">$\{0, 1, 2, \dots\}$</p>	$\sum_{n=0}^T \binom{T}{n} p^n (1-p)^{T-n} \delta_{2n-T}$  <p style="text-align: center;">$\{-T, -T+2, \dots, T-2, T\}$</p>	$C_{t,\zeta} e^{(\pi-2\zeta)x} \Gamma(t+ix) ^2$  <p style="text-align: center;">$(-\infty, \infty)$ ₂</p>

all shifted to have mean zero.

Meixner family of probability distributions.

What do these have in common?

Look at their orthogonal polynomials.

For the Gaussian distribution, Hermite polynomials are orthogonal:

$$\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} H_n(x, t) H_k(x, t) e^{-x^2/2t} dx = 0 \text{ if } n \neq k.$$

$$H_1 = x, \quad H_2 = x^2 - t, \quad H_3 = x^3 - 3xt, \dots$$

Distributions	Polynomials
Gaussian	Hermite
Poisson	Charlier
Gamma (chi-square, exponential)	Laguerre
Negative binomial (geometric)	Meixner
Binomial	Krawtchouk
Hyperbolic tangent	Meixner-Pollaczek

Generating functions for polynomials

$$F(x, t, z) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x, t) z^n.$$

Polynomials	Generating function
Hermite	$\exp(xz - tz^2/2)$
Charlier	$(1 + z)^{x+t} e^{-tz}$
Laguerre	$(1 + z)^{-t} \exp - \left(\frac{x+t}{1+z} \right)$
Meixner	$(1 - \mu)^{(\mu^{1/2} - \mu^{-1/2})^2} ((1 - \mu) + z)^{x - (1 - \mu)t}$ $\times ((1 - \mu) + \mu z)^{-x + \frac{1 - \mu}{\mu} t}$
Krawtchouk	$(1 + (1 - p)z)^{pT+x} (1 - pz)^{(1-p)T-x}$
Meixner-Pollaczek	$\left(1 - \frac{ze^{i\zeta}}{2 \sin \zeta} \right)^{-t+ix} \left(1 - \frac{ze^{-i\zeta}}{2 \sin \zeta} \right)^{-t-ix}$

All of the form $e^{xu(z) - tv(z)}$.

Meixner distributions μ have orthogonal polynomials with generating functions $e^{xu(z)-tv(z)}$.

Theorem. (Meixner 1934) Up to re-scalings, translations, they are the only ones.

Another way to see: recursion relations.

Hermite

$$xH_n = H_{n+1} + ntH_{n-1}.$$

Charlier

$$xC_n = C_{n+1} + nC_n + ntC_{n-1}.$$

Laguerre

$$xL_n = L_{n+1} + 2nL_n + n(t + (n - 1))L_{n-1}.$$

Meixner

$$xM_n = M_{n+1} + \frac{1 + \mu}{1 - \mu}nM_n + n \left(t + \frac{\mu}{(1 - \mu)^2}(n - 1) \right) M_{n-1}.$$

Krawtchouk

$$xK_n = K_{n+1} + (1 - 2p)nK_n + n(T - p(1 - p)(n - 1))K_{n-1}.$$

Meixner-Pollaczek

$$xP_n(x, t) = P_{n+1}(x, t) + 2(\cos \zeta)nP_n(x, t) + n(t + (n - 1))P_{n-1}(x, t).$$

Thus all of the Meixner form

$$xM_n(x, t) = M_{n+1}(x, t) + bnM_n(x, t) + n(t + c(n - 1))M_{n-1}(x, t).$$

Generating function

$$H(x, z, t) = \exp \left(x \int \frac{dz}{1 + bz + cz^2} - t \int \frac{z dz}{1 + bz + cz^2} \right).$$

More details:

Wim Schoutens, STOCHASTIC PROCESSES AND ORTHOGONAL POLYNOMIALS, Springer 2000.

P. Feinsilver, R. Schott, ALGEBRAIC STRUCTURES AND OPERATOR CALCULUS I. REPRESENTATIONS AND PROBABILITY THEORY, Kluwer 1993.

Other properties:

Natural exponential families with quadratic variance function (Morris '83); generalized linear models.

Quadratic regression property (Laha, Lukacs '60).

Distributions of stochastic processes with linear conditional expectations and quadratic conditional variances (Wesołowski '93).

Polynomial basis for representations of \mathfrak{sl}_2 .

Positive integer linearization coefficients (Kim, Zeng 2001).

Combinatorics: Rota, Sheffer, Viennot, etc.

FREE ANALOGS.

In $e^{xu(z)-tv(z)}$, replace e^z with $\frac{1}{1-z}$.

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots, \quad \frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

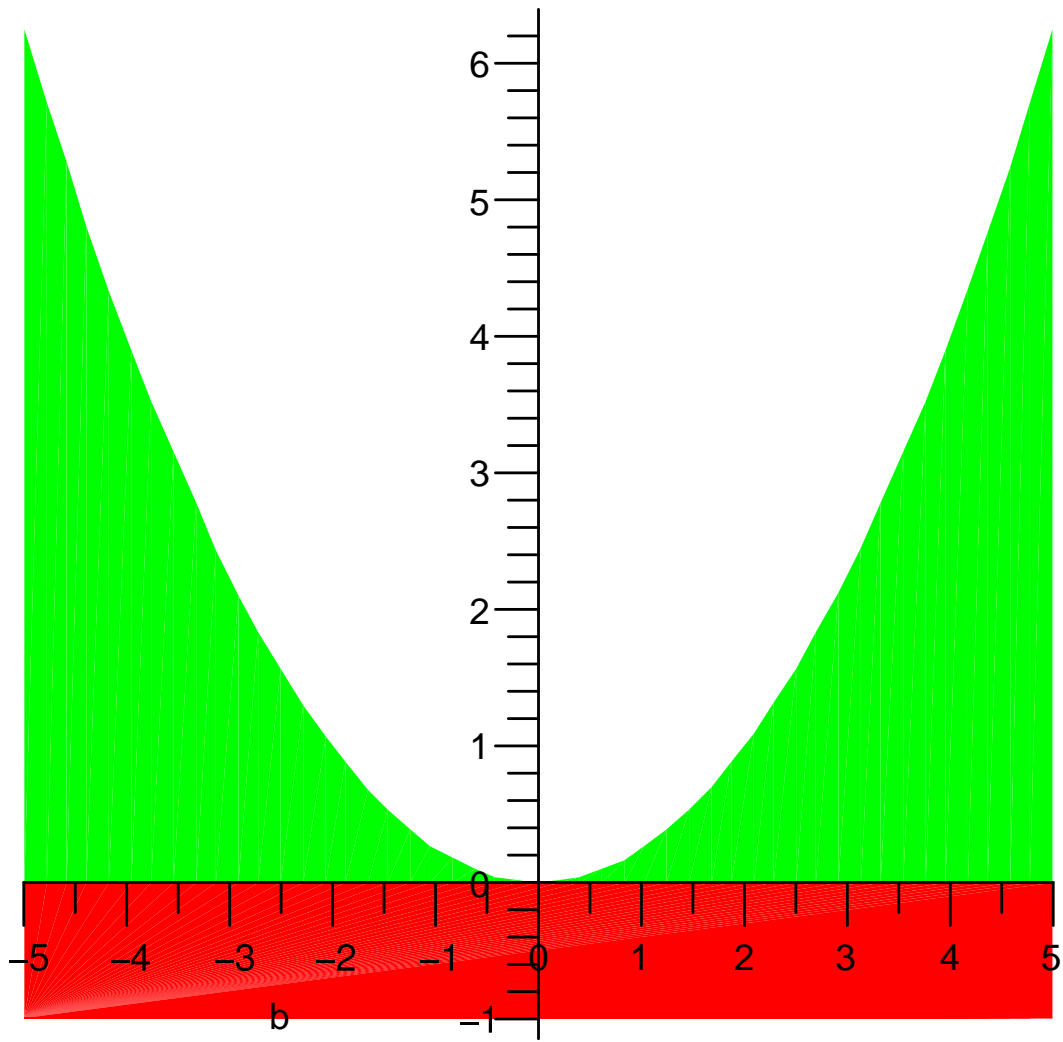
Distributions with orthogonal polynomials $\{P_n\}$,

$$\sum_{n=0}^{\infty} P_n(x)z^n = \frac{1}{1 - xu(z) + tv(z)}.$$

Again can describe completely (M.A. 2003): up to re-scaling

$$\frac{1}{2\pi} \frac{t \sqrt{4(t+c) - (x-b)^2}}{t^2 + tbx + cx^2} + \text{zero, one, or two atoms.}$$

- $b = c = 0$ (“Gaussian”) no atoms.
- $c = 0$ (“Poisson”) at most one atom.
- $c > 0, b^2 > 4c$ (“Negative binomial”) at most one atom.
- $c > 0, b^2 = 4c$ (“Gamma”) no atoms.
- $c > 0, b^2 < 4c$ (“Hyperbolic tangent”) no atoms.
- $-1 \leq c < 0$ (“Binomial”) at most two atoms.



$$\frac{1}{2\pi} \frac{t \sqrt{4(t+c) - (x-b)^2}}{t^2 + tbx + cx^2} + \text{zero, one, or two atoms.}$$

APPEARANCE IN RANDOM MATRIX THEORY.

1. $b = c = 0$. Semicircular distribution. Limit of spectral distribution of GUE.
2. $c = 0$. Marchenko-Pastur distribution. Limit of Wishart matrices.
3. $-1 \leq c < 0$. Limit of Jacobi or Beta (Wachter '80; Capitaine, Casalis 2002).
4. $c > 0, b^2 \geq 4c$. Wireless communications (Gaussian matrices with correlated entries)?

Jacobi(n, α, β) has distribution

$$(Z_n^{\alpha, \beta})^{-1} \det(1 - M)^\alpha \det(M)^\beta \mathbf{1}_{0 \leq M \leq 1} dM$$

Also,

$$J = (X + X')^{-1/2} X (X + X')^{-1/2}$$

for some independent Wishart matrices. Limiting distribution

$$\frac{1}{2\pi} \frac{\sqrt{(x - \lambda_-)(\lambda_+ - x)}}{x - x^2} + \max(0, 1 - \alpha)\delta_0 + \max(0, 1 - \beta)\delta_1.$$

for some λ_\pm .

These are all free Meixner distributions.

FREE PROBABILITY.

Random $n \times n$ matrices $A, B \longrightarrow$ non-commuting objects x, y .

Expectation $\frac{1}{n} \text{Tr } E \longrightarrow$ state φ .

A, B diagonal, U random unitary matrix with uniform (Haar) distribution.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Tr } E \left[A^2 (UBU^{-1}) A (UBU^{-1})^3 A^2 \right] = \varphi \left[x^2 y x y^3 x^2 \right], \text{ etc.}$$

Then x, y freely independent. Explicit formulas to calculate all moments.

Free central limit theorem. Let x_1, x_2, \dots be identically distributed, mean zero, variance one, freely independent. Then

$$\text{dist} \left(\frac{x_1 + x_2 + \dots + x_k}{\sqrt{k}} \right) \xrightarrow{k \rightarrow \infty} \text{Semicircular.}$$

Free Poisson limit theorem. Let $\{x_{i,k} : 1 \leq i \leq k, k = 1, 2, \dots\}$ have distributions

$$\text{dist}(x_{i,k}) = \left(1 - \frac{1}{k}\right) \delta_0 + \frac{1}{k} \delta_1$$

and be freely independent. Then

$$\text{dist}(x_{1,k} + x_{2,k} + \dots + x_{k,k}) \xrightarrow{k \rightarrow \infty} \text{Marchenko-Pastur.}$$

APPEARANCE IN FREE PROBABILITY.

1. $b = c = 0$. Appears in the free central limit theorem, free analog of Gaussian.
2. $c = 0$. Appears in the free Poisson limit theorem, free analog of the Poisson distribution.
3. $-1 \leq c < 0$. Free binomial distributions.

$P_\alpha = n \times n$ diagonal matrix with αn 1's, $(1 - \alpha)n$ 0's on the diagonal.

Asymptotic distribution of P_α is Bernoulli $(1 - \alpha)\delta_0 + \alpha\delta_1$.

$U_1, U_2, \dots, U_T = n \times n$ independent random unitary matrices with uniform (Haar) distribution.

$\lim_{n \rightarrow \infty} \text{dist}\left(U_1 P_\alpha U_1^* + U_2 P_\alpha U_2^* + \dots + U_T P_\alpha U_T^*\right) = \text{FreeBinomial}(\alpha, T)$
 (up to a shift).

In fact, can do this for any *real* $T \geq 1$: for $\beta = \frac{1}{T}$,

$$\text{dist}\left(P_\beta U P_\alpha U^* P_\beta\right) \rightarrow \text{shift of FreeBinomial}(\alpha, T) \stackrel{d}{=} \text{lim Jacobi}$$

True for finite matrices (Collins 2005).

4. $c > 0, b^2 \geq 4c$. S -transform is a rational function.

OTHER PROPERTIES.

Polynomial recursion.

$$xP_0 = P_1,$$

$$xP_1 = P_2 + bP_1 + P_0,$$

$$xP_n = P_{n+1} + bP_n + (t + c)P_{n-1}.$$

“almost constant coefficients”

Stieltjes transform.

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{z - x}.$$

Modified R -transform

$$G\left(\frac{1 + R(z)}{z}\right) = z.$$

$$\frac{R(z)}{z^2} = 1 + b\frac{R(z)}{z} + c\left(\frac{R(z)}{z}\right)^2.$$

cf.

$$(\log \mathcal{F})'' = 1 + b(\log \mathcal{F})' + c[(\log \mathcal{F})']^2$$

Second order difference equation, with $f'(z)$ replaced by $\frac{f(z)-f(0)}{z}$.

Rao, Edelman.

CONDITIONAL CHARACTERIZATION (Bożejko, Bryc 2006).

Let \mathbb{X} , \mathbb{Y} be freely independent, with conditional expectation

$$E[\mathbb{X}|\mathbb{X} + \mathbb{Y}] = \alpha(\mathbb{X} + \mathbb{Y}) + \alpha_0 I$$

and conditional variance

$$\text{Var}[\mathbb{X}|\mathbb{X} + \mathbb{Y}] = C(I + b(\mathbb{X} + \mathbb{Y}) + c(\mathbb{X} + \mathbb{Y})^2).$$

Then both \mathbb{X} and \mathbb{Y} have free Meixner distributions.

Matrix version?

GAMMA CHARACTERIZATIONS.

Classical: Suppose X, Y are non-degenerate, independent non-negative random variables, and $S = X + Y$ positive. Let $Z = X/S$. S and Z are independent if and only if X, Y have gamma-type distributions. (Lukacs 1955).

Wishart: Suppose \mathbb{X}, \mathbb{Y} are non-degenerate, independent positive random matrices, and $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ is strictly positive. Let $\mathbb{Z} = \mathbb{S}^{-1/2}\mathbb{X}\mathbb{S}^{-1/2}$. \mathbb{S} and \mathbb{Z} are independent if and only if \mathbb{X} and \mathbb{Y} have Wishart distributions. (Olkin, Rubin 1962).

Symmetric cones: Casalis, Letac 1996.

Wishart: gamma-like but limit free Poisson.

Suppose \mathbb{X}, \mathbb{Y} are non-degenerate, freely independent non-commutative random variables, and $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ is strictly positive. Let $\mathbb{Z} = \mathbb{S}^{-1/2}\mathbb{X}\mathbb{S}^{-1/2}$. \mathbb{S} and \mathbb{Z} are freely independent if and only if \mathbb{X} and \mathbb{Y} have free Poisson-type distributions.

Suppose \mathbb{X}, \mathbb{Y} are non-degenerate, freely independent, identically distributed, and strictly positive non-commutative random variables. Let $\mathbb{Z} = \mathbb{S}^{-1}\mathbb{X}^2\mathbb{S}^{-1}$. If \mathbb{Z} and \mathbb{S} are free, \mathbb{X} and \mathbb{Y} are free gamma-type. Do not know if free gamma has this property.

No matrix version (Letac)

Questions.

Realize free Meixner distributions for general b, c by random matrices

Realize free Meixner distributions for general b, c in free probability.

Appearance in wireless communications?

Meixner distributions for matrices of finite size?

Multivariate orthogonal polynomials in
non-commuting variables

`math.CO/0410482`

Trans. AMS 2008?

Michael Anshelevich

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Polynomials in non-commuting variables.

$$\mathbb{R}\langle x_1, x_2, \dots, x_n \rangle.$$

Example.

$$P(x_1, x_2, x_3) = 3x_1^2x_2x_1 - x_2x_1x_3^2x_2.$$

ORTHOGONAL POLYNOMIALS. Not with respect to a measure on \mathbb{R}^n , but a state φ .

Linear, $\varphi[1] = 1$,

$$\varphi[P(\mathbf{x})^*P(\mathbf{x})] \geq 0$$

and equals to zero only if $P(\mathbf{x}) = 0$.

$P(\mathbf{x}), Q(\mathbf{x})$ orthogonal if

$$\langle P(\mathbf{x}), Q(\mathbf{x}) \rangle = \varphi [P(\mathbf{x})^* Q(\mathbf{x})] = 0.$$

Commutative case:

$$\langle P(\mathbf{x}), Q(\mathbf{x}) \rangle = \varphi [P(\mathbf{x})^* Q(\mathbf{x})] = \int P(\mathbf{x})^* Q(\mathbf{x}) d\mu(\mathbf{x}) = 0$$

A polynomial family $\{P_{\vec{u}}\}$, for all choices of multi-indices \vec{u} .

$$P_{\vec{u}}(\mathbf{x}) = x_{\vec{u}} + \dots$$

Example.

$$P_{(1,2,1,1)}(\mathbf{x}) = x_1 x_2 x_1^2 + \dots$$

Monic, cf. polynomials in commuting variables.

Proposition. Monic polynomials are orthogonal with respect to some state if and only if they satisfy a recursion

$$x_i P_{\vec{u}} = P_{(i, \vec{u})} + \sum_{|\vec{w}|=|\vec{u}|} B_{i, \vec{w}, \vec{u}} P_{\vec{w}} + \sum_{|\vec{v}|=|\vec{u}|-1} C_{i, \vec{v}, \vec{u}} P_{\vec{v}}$$

with

(a) $C_{i, \vec{s}, \vec{u}} = 0$ unless $\vec{u} = (i, \vec{s})$, and $C_{i, \vec{s}, (i, \vec{s})} > 0$,

(b) Denoting $\vec{s}_j = (s(j), \dots, s(k))$,

$$B_{i, \vec{s}, \vec{u}} \prod_{j=1}^k C_{s(j), \vec{s}_{j+1}, \vec{s}_j} = B_{i, \vec{u}, \vec{s}} \prod_{j=1}^k C_{u(j), \vec{u}_{j+1}, \vec{u}_j}.$$

In 2 variables:

$$x_1 P_{2,1,2} = P_{1,2,1,2} + \sum_{|\vec{w}|=3} B_{\vec{w}} P_{\vec{w}},$$
$$x_2 P_{2,1,2} = P_{2,2,1,2} + \sum_{|\vec{w}|=3} B_{\vec{w}} P_{\vec{w}} + C P_{1,2}.$$

Generating functions of special form

$$F(\mathbf{x}, \mathbf{z}) = \sum P_{\vec{u}}(\mathbf{x}) z_{\vec{u}} = \frac{H(\mathbf{z})}{1 - \mathbf{x} \cdot \mathbf{V}(\mathbf{z})}.$$

Example.

$$\frac{\frac{1}{z_1^2 + z_2^2}}{1 - \left(x_1 \frac{z_1}{z_1^2 + z_2^2} + x_2 \frac{z_2}{z_1^2 + z_2^2} \right)} = \frac{1}{1 - (x_1 z_1 + x_2 z_2) + (z_1^2 + z_2^2)}.$$

Free Meixner states = states whose orthogonal polynomials have such a generating function.

Theorem. Start with $\{P_{\vec{u}}\}$ as above. Define a functional φ by

$$\varphi [1] = 1, \quad \varphi [P_{\vec{u}}(\mathbf{x})] = 0.$$

Not always positive.

(1) Necessarily

$$F(\mathbf{x}, \mathbf{z}) = \frac{1}{1 - \mathbf{x} \cdot \mathbf{U}(\mathbf{z}) + R(\mathbf{U}(\mathbf{z}))}.$$

$R(w) = R_{\varphi}(w) =$ multivariate R -transform.

Can be calculated using the implicit relation with the moment series

$$R\left(w_1(1 + \mathbf{M}(\mathbf{w})), \dots, w_n(1 + \mathbf{M}(\mathbf{w}))\right) = \mathbf{M}(\mathbf{w}).$$

Thus from $\{P_{\vec{u}}\}$, can calculate φ .

Now suppose that $\{P_{\vec{u}}\}$ are orthogonal (so φ positive).

(2) U is determined by the relation

$$D_i(R)(U_1(\mathbf{z}), U_2(\mathbf{z}), \dots, U_n(\mathbf{z})) = z_i.$$

$D_i =$ left partial derivative

$$D_1(z_1^2 z_2 z_1) = z_1 z_2 z_1,$$

$$D_2(z_1^2 z_2 z_1) = 0.$$

(3)

$$x_i P_{(t, \vec{u})} = P_{(i, t, \vec{u})} + \sum_j B_{ij}^t P_{(j, \vec{u})} + \delta_{it} (1 + C_{i, u(1)}) P_{\vec{u}},$$

where

(a) $C_{ij} > -1$.

(b) $B_{ij}^t = B_{it}^j$.

(c) For each j, t , either $B_{ij}^t = 0$ for all i , or $C_{ju} = C_{tu}$ for all u .

Equivalently,

$$D_i D_j R = \delta_{ij} + \sum_t B_{ij}^t D_t R + C_{ij} D_i R D_j R,$$

i.e. R -transform satisfies a second order PDE (“partial difference equation”)

In 2 variables:

$$x_1 P_{2,1,2} = P_{1,2,1,2} + B_1 P_{1,1,2} + B_2 P_{2,1,2},$$

$$x_2 P_{2,1,2} = P_{2,2,1,2} + B_1 P_{1,1,2} + B_2 P_{2,1,2} + C P_{1,2}.$$

Example. One-dimensional free Meixner states.

$$\frac{R(z)}{z^2} = 1 + b \frac{R(z)}{z} + c \left(\frac{R(z)}{z} \right)^2.$$

One variable \Rightarrow commutative \Rightarrow states are measures.

$$\frac{1}{2\pi} \frac{\sqrt{4(1+c) - (x-b)^2}}{1+bx+cx^2} + \text{zero, one, or two atoms.}$$

Semicircular=free Gaussian

Marchenko-Pastur=free Poisson

Jacobi=free binomial.

Free product of n one-dimensional states = state on $\mathbb{R}\langle x_1, x_2, \dots, x_n \rangle$.

Example. Two states φ_1 on $\mathbb{R}[x_1]$, φ_2 on $\mathbb{R}[x_2]$. Corresponding polynomials $\{P_n^{(i)}(x_i)\}$, $i = 1, 2$.

$$\varphi \left[5 - 2P_3^{(2)}(x_2) + P_1^{(1)}(x_1)P_4^{(2)}(x_2) + 3P_2^{(1)}(x_1)P_2^{(2)}(x_2)P_3^{(1)}(x_1) \right] = 5.$$

Free products of free Meixner states are again free Meixner.

$$\frac{R(z)}{z^2} = 1 + b \frac{R(z)}{z}.$$

Semicircular, Marchenko-Pastur. Free products still satisfy a first-order equation.

Rotation $y = O\mathbf{x}$, O orthogonal still OK.

φ a tracial state if

$$\varphi [ABC] = \varphi [CAB].$$

(note: not $\varphi [BAC]$).

Proposition. If φ is a tracial free Meixner state with

$$D_i D_j R(z) = \delta_{ij} + B_{ij}^t D_t R(z),$$

then φ is a rotation of a free product of semicircular and Marchenko-Pastur distributions.

Can construct a state φ so that

$$R_\varphi(\mathbf{z}) = \sum_i z_i M_{\text{free product of shifted scaled semicircular}}(\mathbf{z}) z_i$$

φ will be free Meixner, non-tracial, freely infinitely divisible.

Non-freely-infinitely-divisible examples? Free multinomial?