Airplane Boarding, Spacetime and RMT

E. Bachmat, D. Berend, L. Sapir and S. Skiena

The boarding process

We represent passengers by pairs (q, r) in the unit square $[0, 1]^2$, q represents the time in which the passenger joined the queue, r represent the row number.

The parameters for the boarding process are

D – A delay distribution. The amount of time a passenger needs to get organized and clear the aisle will be sampled from the distribution D.

h – The number of passengers per row.

l – Distance between rows (leg room).

W – A width distribution. The values w_i will be sampled from W.

F – An airline boarding policy. We represent an airline boarding policy by a function F. $F(r)$ indicates the first time at which passengers from row r

are allowed to join the boarding queue.

Ω – A passenger's reaction model. We need to make an assumption as to the nature of the reaction of passengers to the airline policy. The reaction determines the effect of the boarding policy on the boarding process.

$p(q,r)$ We combine the passenger reaction model Ω with the airline boarding policy F to produce a joint distribution $p(q,r)dqdr$ on passengers' row number and queueing time.

Modeling

We associate a Lorentzian metric on the unit square to the airplane boarding process with parameters

$D, h, l, W, p(q,r)$.

Put

$$\alpha(q,r) = \int_r^1 p(q,z)dz$$

and

$$k = hE(W)/l$$

where $E(W)$ denotes the expectation of W.

We define a Lorentzian metric by:

$$g_{p,p} = 4D^2 p(q,r)k\alpha,\ g_{p,q} = g_{q,p} = 2D^2 p(q,r),\ g_{q,q} = 0.$$

Equivalently, the length (proper time) element ds is given by:

$$ds^2 = 4D^2 p(q,r)(dqdr + \alpha k dq^2).$$

The following facts serve as motivation for the introduction of the metric.

A) The volume form associated with the metric g, is proportional to the passenger density distribution $p(q,r)dqdr$.

B) Asymptotically, the causal relation among space-time points with respect to the metric and the blocking relation in the airplane boarding process coincide locally.

Given a point x, let the *(proper time) level* $\tau(x)$ of x be the maximum of the lengths of time-like curves in ending at x. The *diameter* T is the maximum of all $\tau(x)$.

$\tau(x)\sqrt{n}$ is the approximate boarding time of a passenger represented by x. $T\sqrt{n}$ is therefore the boarding time.

No policy

Since p is uniform we have

$$\alpha = 1 - r$$

the squared length element thus being

$$ds^2 = dqdr + k(1-r)dq^2$$

.

This metric is flat !!

The Euler-Lagrange equation for curve length degenerates to the Beltrami equation

$$\frac{r'}{2\sqrt{r' + k(1-r)}} - \sqrt{r' + k(1-r)} = const.$$

After placing the boundary conditions $r(0) = 0, r(1) = 1$, we obtain the solution

$$r = \frac{e^{2kq}}{e^k - 1} - \frac{e^k}{e^k - 1} e^{kq} + 1.$$

The solution remains within the unit square when $k \leq \ln 2$.

When $k > \ln 2$, the solution is not contained in the unit square anymore. The maximal curve first "crawls" along the bottom.

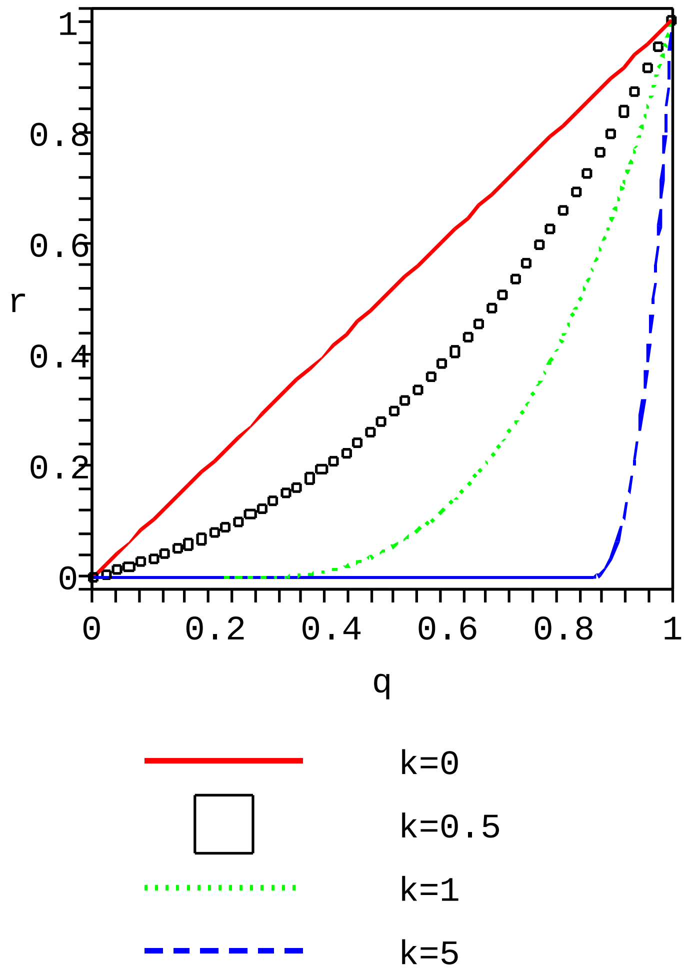
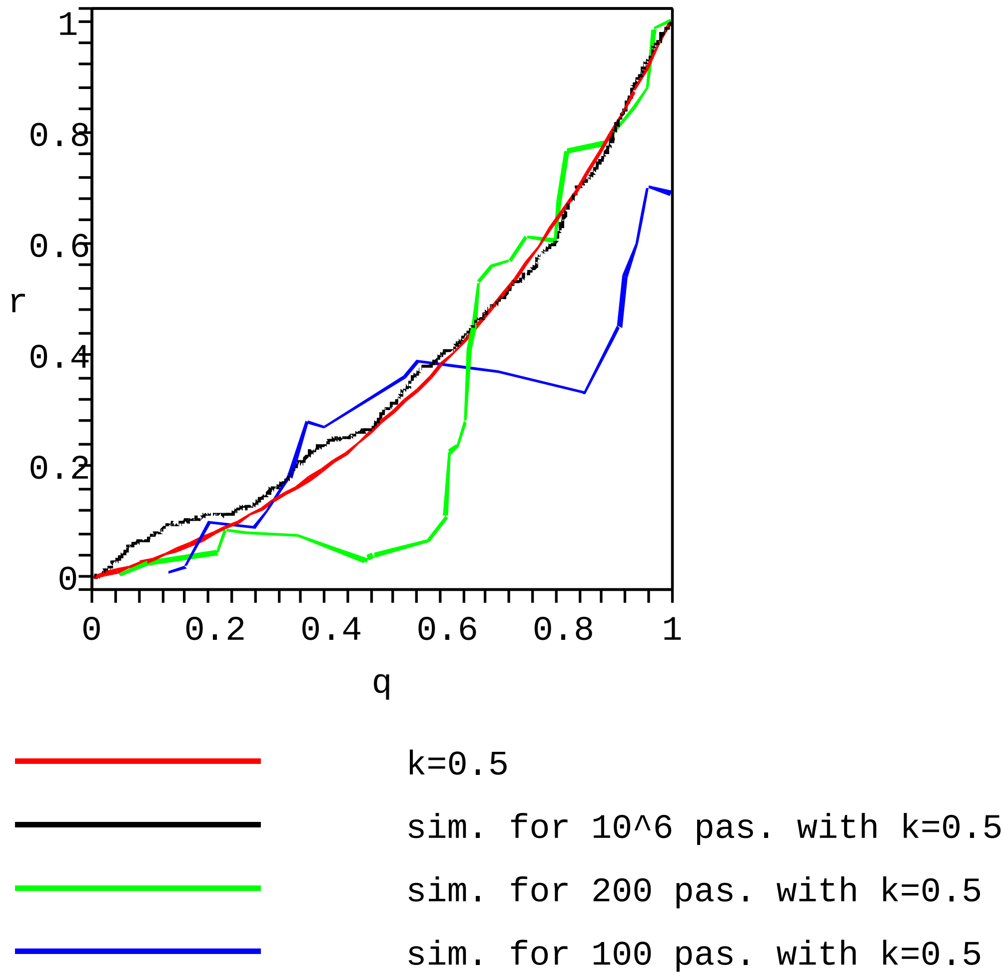
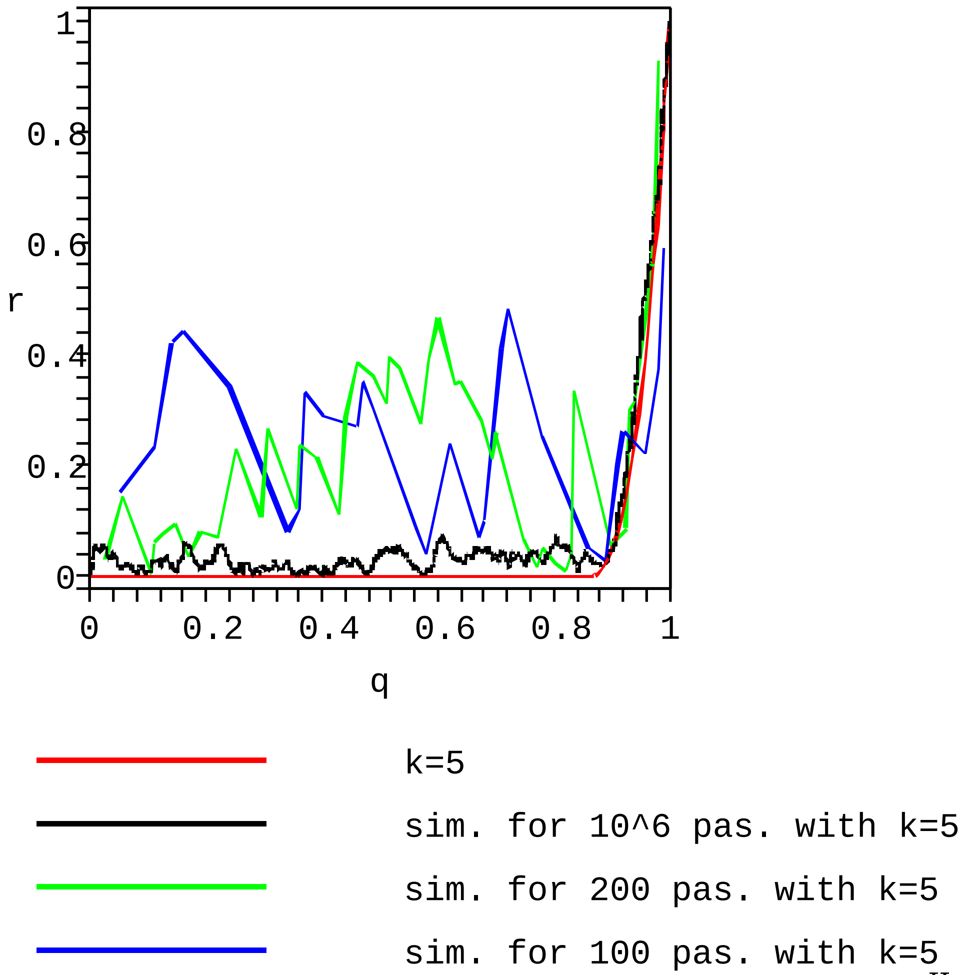


Figure 1: Maximal length curves for several values of k.



When $k \leq ln(2)$ the error term is (after proper normalization) distributed between the $F_2 = GUE$ and $F_4 = GSE$ Tracy-Widom distributions

When $k > ln(2)$ the error term is no longer $n^{1/6}$, but sits somewhere between Tracy-Widom and Gaussian behavior, because it doesnt "explore" much.



Here is how to import EPS art

Figure 2: Comparison of the maximal curve for $k = 5.0$ with simulation based maximal chains.

index	policy	$k=4$	V-B results
1	F_1	1.00	1.00
2	F_2	1.21	1.12
3	F_3	1.40	1.16
4	F_{3,σ_0}	1.42	1.21
5	F_4	1.56	1.27
6	F_{4,σ_1}	1.47	1.24
7	F_{4,σ_2}	1.47	1.24
8	F_6	1.86	1.37
9	F_{6,σ_3}	1.60	1.27
10	F_{6,σ_4}	1.52	1.25
11	F_{6,σ_5}	1.52	1.32
12	F_{10}	2.34	1.61
13	F_{10,σ_6}	2.01	1.40
14	F_{10,σ_7}	1.58	1.20
15	$F_{2,2}$	1.10	1.11
16	$F_{2,2,\sigma_8}$	1.10	1.01
17	$F_{2,3}$	1.18	1.13
18	$F_{2,3,\sigma_9}$	1.25	1.20
19	$F_{2,4}$	1.28	1.15
20	$F_{2,4,\sigma_{10}}$	1.09	1.07
21	$F_{2,4,\sigma_{11}}$	1.28	1.18
22	$F_{2,6}$	1.43	1.23
23	$F_{2,6,\sigma_{12}}$	0.90	1.00
24	$F_{2,6,\sigma_{13}}$	1.30	1.06
25	$F_{2,6,\sigma_{14}}$	0.90	1.04
26	$F_{2,10}$	1.73	1.44
27	$F_{2,10,\sigma_{15}}$	0.70	1.10
28	$F_{2,10,\sigma_{16}}$	1.68	1.09

Table 1: Comparison results

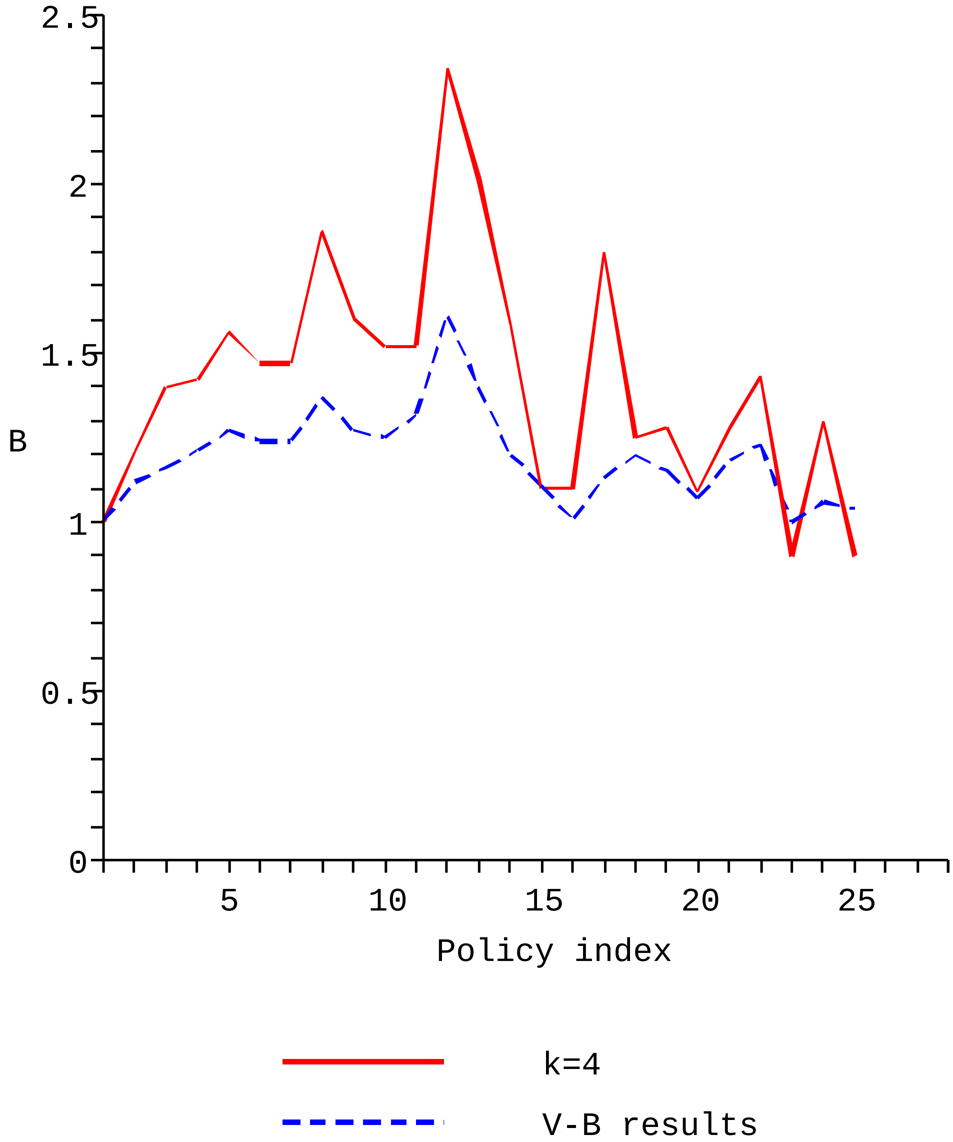


Figure 3: Boarding time (B) according to the model and the V-B simulations.

0	2, 3, 1
1	4, 2, 3, 1
2	4, 1, 3, 2
3	6, 4, 2, 5, 3, 1
4	6, 3, 5, 2, 4, 1
5	6, 2, 5, 1, 4, 3
6	10, 8, 6, 4, 2, 9, 7, 5, 3, 1
7	10, 5, 9, 4, 8, 3, 7, 2, 6, 1
8	4, 1, 2, 3
9	6, 4, 5, 3, 1, 2
10	8, 6, 7, 5, 4, 2, 3, 1
11	8, 3, 6, 1, 4, 7, 2, 5
12	12, 10, 8, 11, 9, 7, 6, 4, 2, 5, 3, 1
13	12, 9, 11, 8, 10, 7, 6, 3, 5, 2, 4, 1
14	12, 4, 8, 5, 9, 1, 11, 3, 7, 6, 10, 2

Table 2: Permutations