Two random matrix central limit theorems

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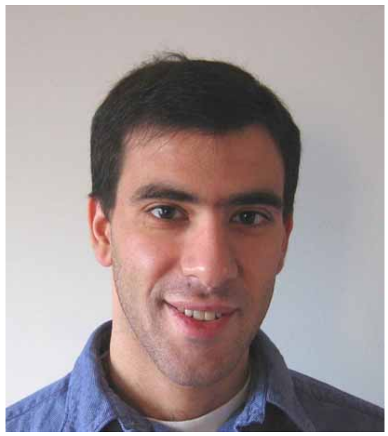


References:

[1] *A GUE Central Limit Theorem and Universality of Directed Last and First Passage Site Percolation*, **Int. Math. Res. Not., no.6:325–338, 2005.**

[2] *Random Matrix Central Limit Theorems for Non-Intersecting Random Walks*, **preprint**

Available at my website or arXive

Two probabilistic models

- Non-intersecting random walks (Dyson's Brownian motions, random Hexagon tiling)

- Last passage percolation

Universal fluctuations as in the random matrix theory at least in certain asymptotic regimes

Universality theorems in random matrix theory

A. Unitary invariant ensembles

The set of $N \times N$ Hermitian matrices H with measure $e^{-N \operatorname{tr} V(H)}$

Take $N \to \infty$.

Limiting density of eigenvalues depend on V.

But <u>local</u> statsitics are independent of V: given by sine kernel (bulk) or Airy kernel (edge)

[Pastur-Shcherbina], [Bleher-Its], [Deift-Kricherbau McLaughlin-Venakides-Zhou], [Deift-Gioev]

$$\mathbb{P}\Big((\xi_{\max}(N)-a)bN^{2/3} \le x\Big) \to F_{TW}(x) = \det(1-\mathcal{A}|_{(x,\infty)})$$

$$\mathbb{P}\Big(\textbf{no eigenvalue in } (x_0 - \frac{cx}{N}, x_0 + \frac{cx}{N})\Big) \to \det(1-\mathcal{S}|_{(-x,x)}$$

$$\mathcal{A}(u,v) = \frac{\mathrm{Ai}(u)\,\mathrm{Ai}'(v) - \mathrm{Ai}'(u)\,\mathrm{Ai}(v)}{u-v}$$

$$\mathcal{S}(u,v) = \frac{\sin(\pi(u-v))}{\pi(u-v)}$$

B. Orthogonal and Symplectic ensembles

[Deift-Gioev]

C. Wigner matrices

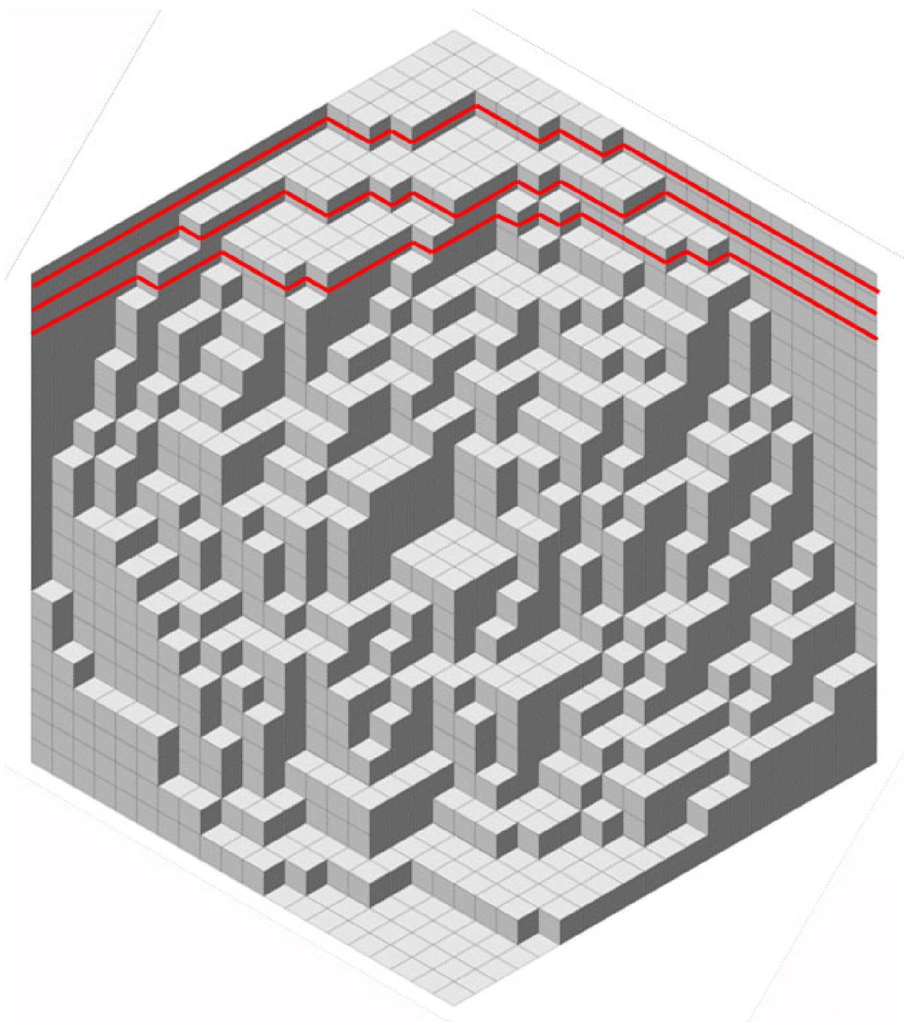
Hermitian matrices with independent and 'identically distributed' entries

Limiting distribution of the largest eigenvalue is universal [Soshnikov]

Non-intersecting random walks

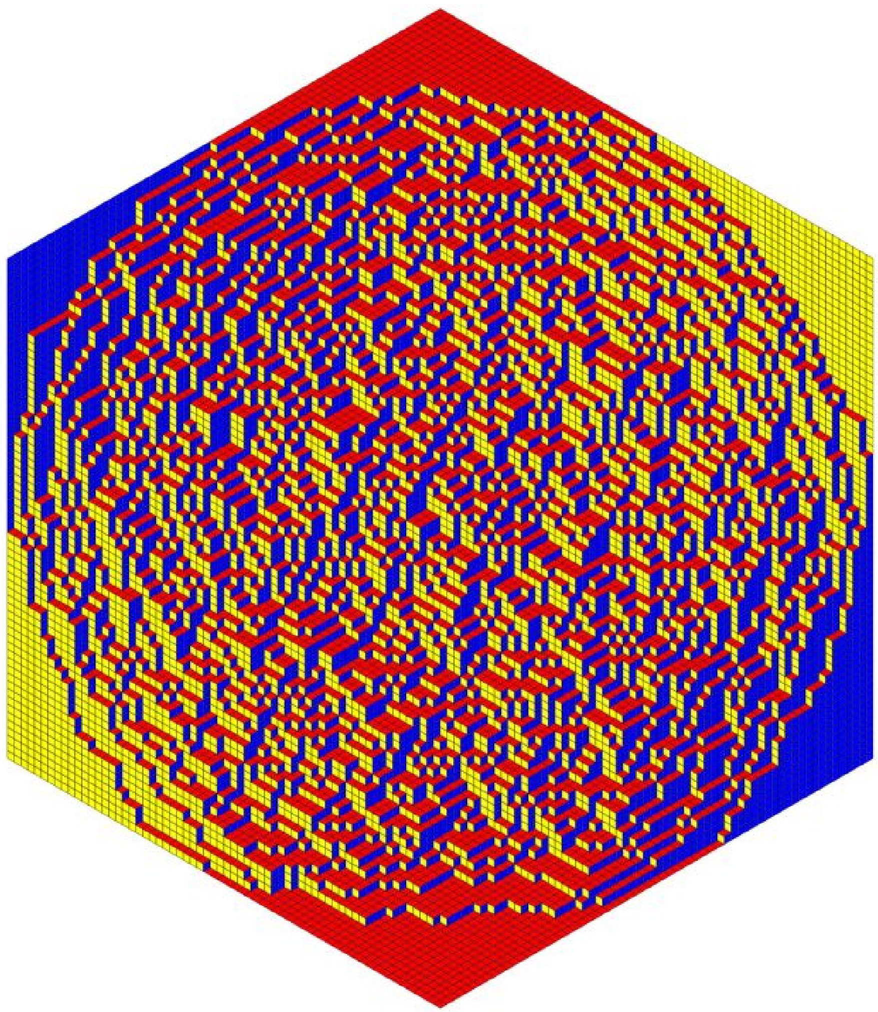
Example 1. Bernoulli walks

Picture by Propp, size=20



= random rhombus tiling of Hexagon

Propp, size=64



n= **number of walks**

T= **time steps**

For $n = O(T) \to \infty$**, limiting distributions are given by those of GUE**

[Baik-Kriecherbauer-McLaughlin-Miller] (based on the algebraic work of [Johansson])

Example 2. Dyson process

n **Brownian bridge processes** $B_t = (B_t^{(1)}, \ldots, B_t^{(n)})$ **for** $t \in [0, 2]$ **such that** $B_0 = B_2 = 0$**, conditioned not to intersect (i.e.** $B_t^{(1)} > \cdots > B_t^{(n)}$**).**

Density of B_t **at time** $t = 1$ **is**

$$c_n \cdot \prod_{1 \le j < k \le n} |b_j - b_k|^2 \prod_{j=1}^{n} e^{-b_j^2}$$

Eigenvalue density of $n \times n$ **GUE!**

Proof: Karlin-McGregor argument

$$\begin{aligned} &\mathbb{P}(a_1 \to b_1, a_2 \to b_2, \textbf{no intersect}) \\ &= \mathbb{P}(a_1 \to b_1, a_2 \to b_2) - \mathbb{P}(a_1 \to b_2, a_2 \to b_1) \\ &= \mathbb{P}(a_1 \to b_1)\mathbb{P}(a_2 \to b_2) - \mathbb{P}(a_1 \to b_2)\mathbb{P}(a_2 \to b_1) \\ &= \det\left[\mathbb{P}(a_j \to b_k)\right]_{1 \le j,k \le n} \end{aligned}$$

Other examples.

Longest increasing subsequences, random Young diagram [Karlin]

Random domino tiling of Aztec diamond [Johansson]

A model for bus system in Cuernevaca, Mexico [Baik-Borodin-Deift-Suidan]

General non-intersecting random walks

Fix n, take $T \to \infty$: random walks $\approx$ Brownian motions.

If we take $T \to \infty$ first, and then $n \to \infty$, we obtain the random matrix limits.

Can we take $n, T \to \infty$ simultaneously?

Brownian approximation v.s. non-intersect conditioning

[Baik-Suidan] For general random variables with finite moment-generating function, yes when '$T \geq O(n^{8n^3})$'.

- **For a few special cases, yes even when $T = O(n)$**

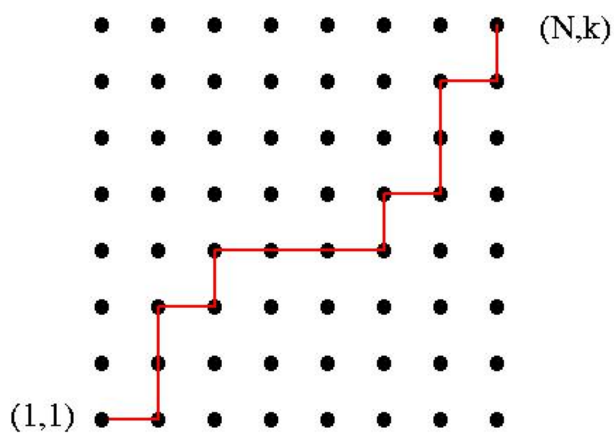
- **True for general random variables? Open question.**

Last passage percolation

i.i.d. random variables $X(i,j)$ at each site $(i,j) \in \mathbb{Z}^2$.

Admissible paths: $\Pi(N,k)$= the set of 'up/right' paths from $(1,1)$ to (N,k)

Total $\binom{N+k-2}{N-1}$ paths



Last passage time:

$$L(N,k) := \max_{\pi \in \Pi(N,k)} \left\{ \sum_{(i,j)\in\pi} X(i,j) \right\}$$

Random growth model, queues in tandem, interacting particle systems

1. Expectation

(1) $k = o(N)$, $\lim_{N,k\to\infty} \frac{\mathbb{E}L(N,k)-\mu N}{\sqrt{Nk}} = 2\sigma$ **[Seppaäläiner**

(2) $k = O(N)$, $\lim_{n\to\infty} \frac{\mathbb{E}L([xn],[yn])}{n} = a(x,y)$ **Formula of** $a(x,y)$ **is not known for general r.v.**

Exponential: $a(x,y) = (\sqrt{x} + \sqrt{y})^2$ **[Rost]**

Geometric: $a(x,y) = \frac{q(x+y)+2\sqrt{qxy}}{1-q}$ **[Johansson]**

2. Limiting distribution

Exponential, Geometric [Johansson]

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L([xn],[yn]) - a(x,y)n}{b(x,y)n^{1/3}} \le s\right) = F_{TW}(s).$$

Any general central limit theorem? Yes for thin models.

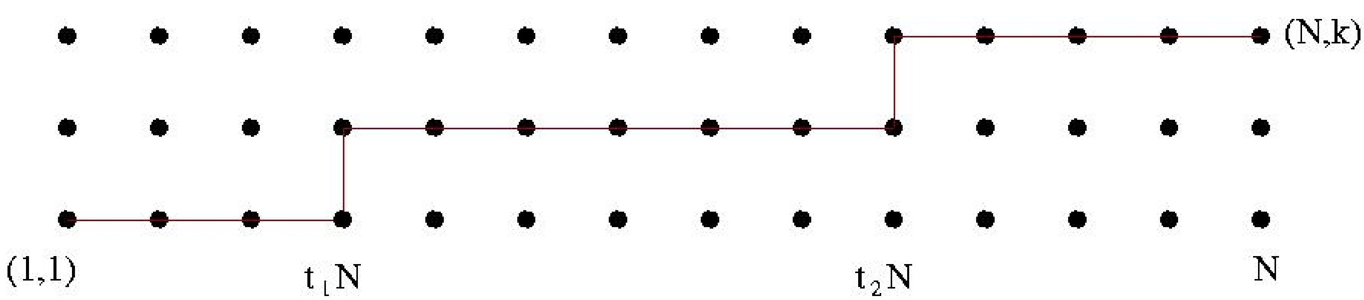
[Baik-Suidan] [Bodineau-Martin] [Suidan]

Suppose $\mathbb{E}(X_{11}) = \mu$, $Var(X_{11}) = \sigma^2$ **and** $\mathbb{E}|X_{11}|^4 < \infty$.

When $k = o(N^\alpha)$ **and** $\alpha < \frac{3}{14}$,

$$\lim_{N,k\to\infty} \mathbb{P}\left(\frac{L(N,k) - \mu(N+k-1) - 2\sigma\sqrt{Nk}}{\sigma k^{-1/6} N^{1/2}} \le s \right) = F(s).$$

If all the moments are finite, $\alpha < \frac{3}{7}$.



Proof

$N \to \infty$: each level $\approx$ Brownian motion

k fixed, $N \to \infty$ [Glynn + Whitt 1991]

$$\frac{L(N,k) - \mu N}{\sigma\sqrt{N}} \Rightarrow \widehat{D}_k(1)$$

$$\widehat{D}_k(1) := \sup_{0=t_0<t_1<\cdots<t_{N-1}<t_N:=1} \sum_{j=1}^{k} \Big(B_j(t_j) - B_j(t_{j-1})\Big)$$

What is this functional ?

'Solvable' case (exponential) [Baryshnikov 2001], [Gravner+Tracy+Widom]: $\widehat{D}_k(1)$= largest eigenvalue of $k \times k$ random Hermitian matrix

Random matrix theory: $k \to \infty$
$(\widehat{D}_k(1) - 2\sqrt{k})k^{1/6} \Rightarrow \chi$

"$k \to \infty, N \to \infty$" = "$N, k \to \infty$"?

Need to couple BM's: Skorohod embedding, KMT (Komlós-Major-Tusnády) approximation