On some detection problems in radar array processing involving eigenvalues of Wishart and related random matrices

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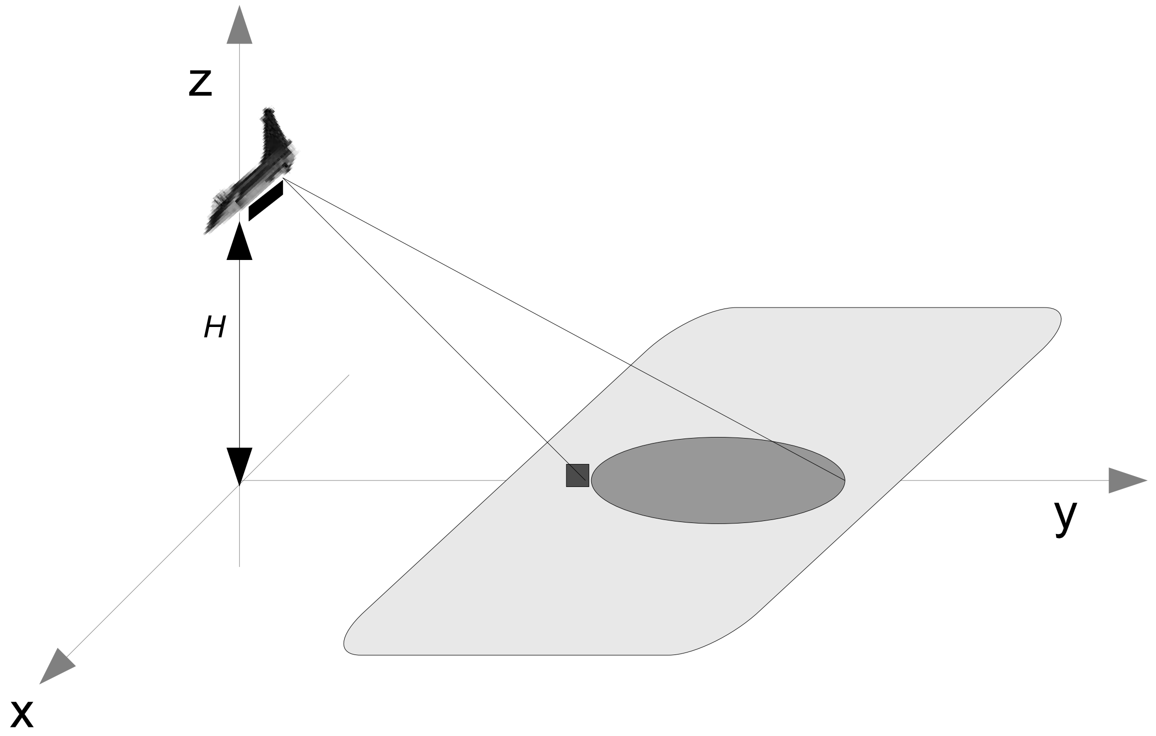
Outline

- Framework : radar detection using an array of sensors.
- Generalized Likelihood Ratio Test for detecting a signal whose signature is subject to uncertainties.
- Analysis of the detectors (which involve the ratio of the largest eigenvalue of a Wishart-type matrix to its trace).
- Concluding remarks.

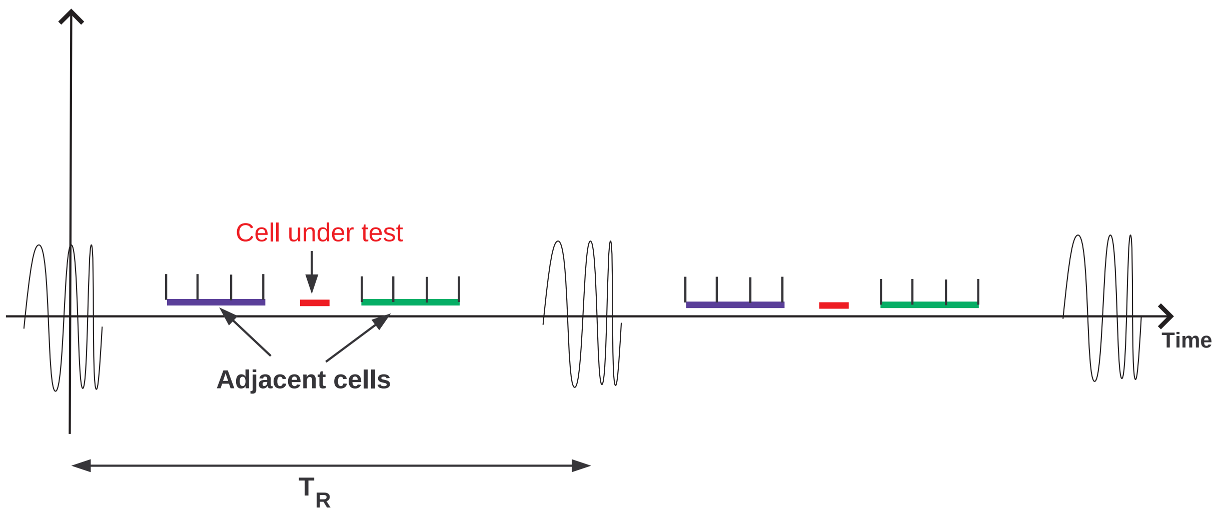
Radar detection

- A fundamental task of any radar is to detect a target against a background of noise.
- Consider an airborne radar aimed at detecting a moving target on the ground.



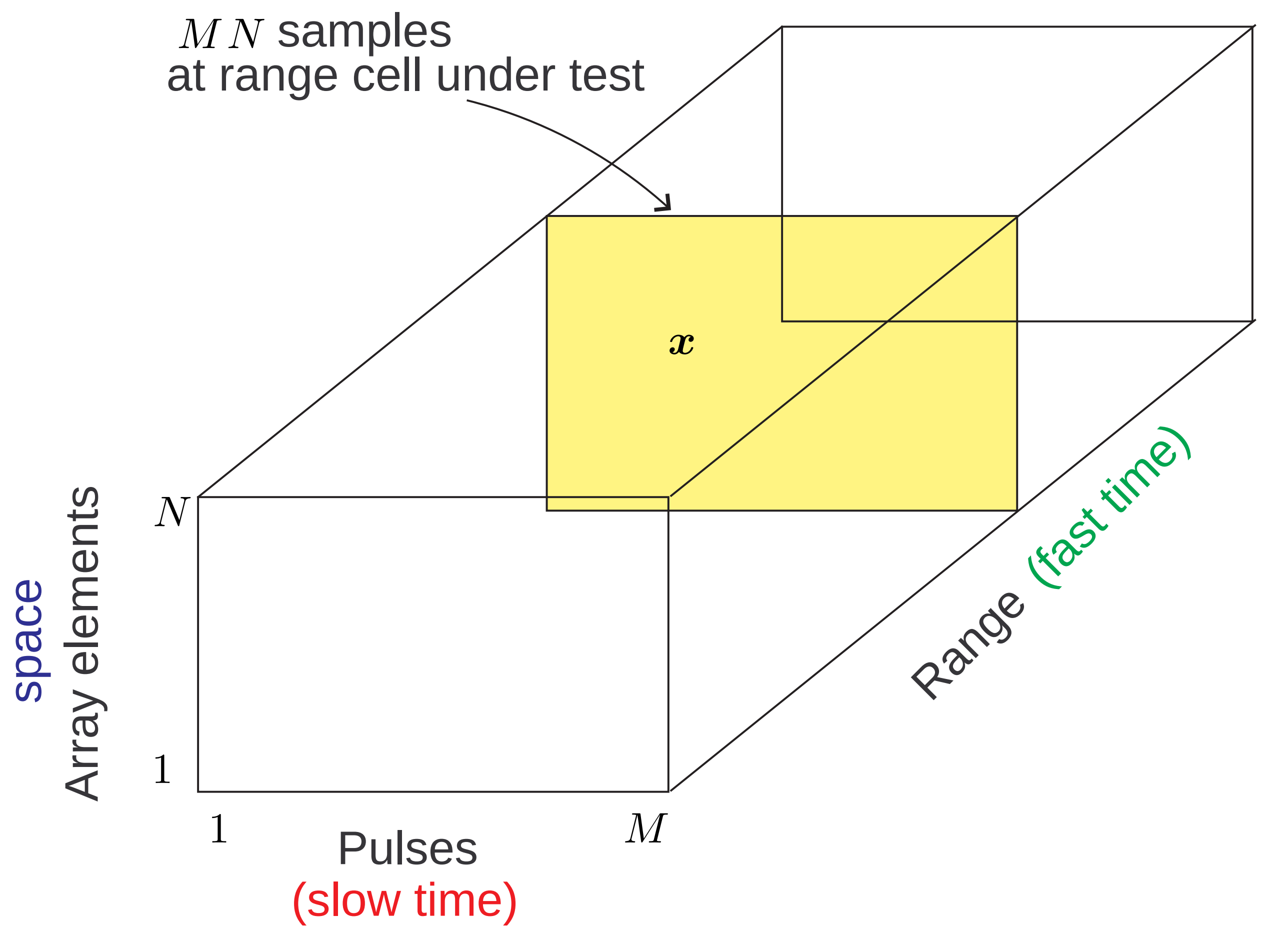
Radar detection

- The radar send a series of pulses and receives the various echoes (clutter + target).
- Within each pulse, the received signal is decomposed in range cells which correspond to the echoes at a given distance of the radar.



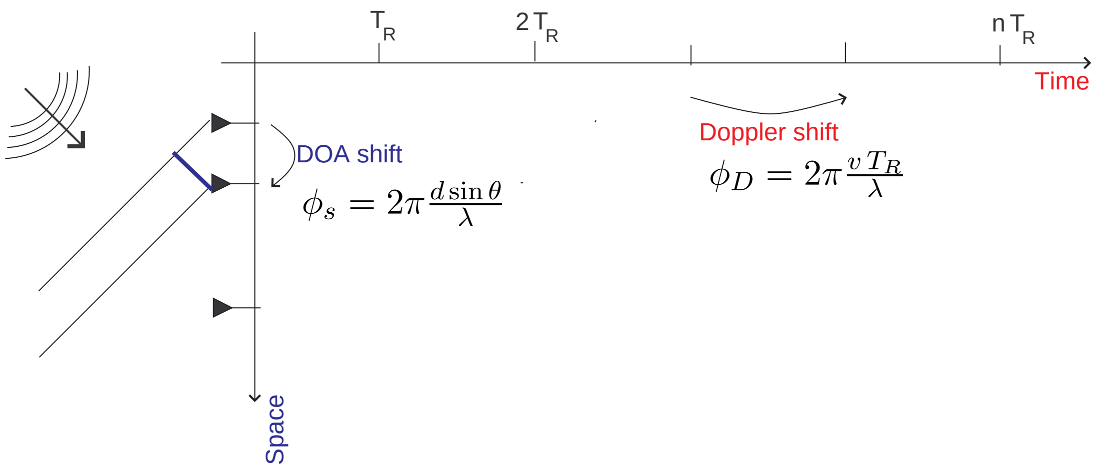
Radar detection

- If an arrays of sensors is used in reception, we have the following datacube organization of the data:



Radar detection

- If a target is present at given direction of arrival (DOA) and velocity, the emitted waveform will undergo
  - a phase shift from pulse to pulse (Doppler effect);
  - a phase shift from antenna to antenna (propagation).

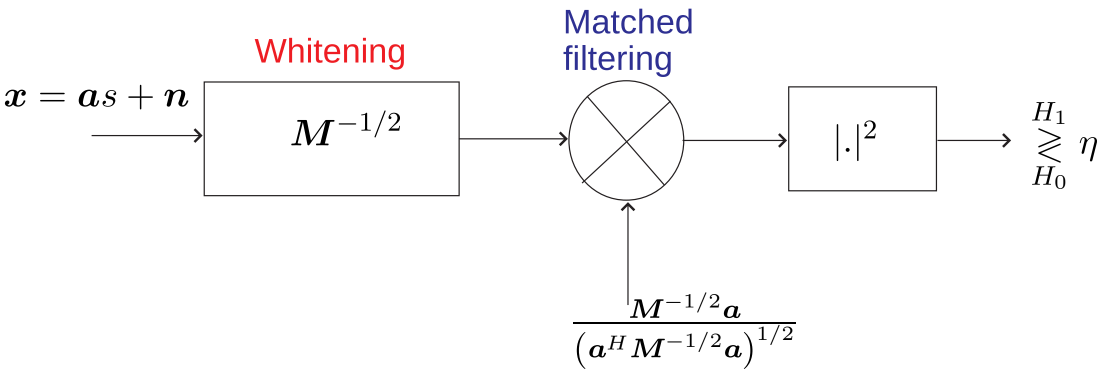


Radar detection

- The space-time signature (steering vector) of a target is a known function of its DOA and velocity.
- The clutter is a random (for the sake of mathematical tractability Gaussian) space-time process whose statistical properties are function of the nature of the ground, the geometry of the array.
- The basic problem is to detect, in a range cell under test (CUT), the presence of a target with known space-time signature, in the presence of noise.

Radar detection

- If the noise covariance matrix $\boldsymbol{M}$ is known the optimal detector has the following form:



- When $\boldsymbol{M}$ is unknown, it can be inferred from range cells adjacent to the CUT, assuming that the noise in the adjacent cells shares the same statistical properties as the noise in the CUT.

The (usual) detection problem

- The problem consists of deciding between the two hypotheses

$$H_0 : \begin{cases} \boldsymbol{x}(t) = \boldsymbol{n}_p(t);\ t = 1, \cdots, N_p & \text{primary data} \\ \boldsymbol{y}(t) = \boldsymbol{n}_s(t);\ t = 1, \cdots, N_s & \text{secondary data} \end{cases}$$

$$H_1 : \begin{cases} \boldsymbol{x}(t) = \boldsymbol{a} s^*(t) + \boldsymbol{n}_p(t);\ t = 1, \cdots, N_p & (\boldsymbol{a} : \text{steering vector}) \\ \boldsymbol{y}(t) = \boldsymbol{n}_s(t);\ t = 1, \cdots, N_s \end{cases}$$

- We let $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}(1) & \cdots & \boldsymbol{x}(N_p) \end{bmatrix}$ and $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{y}(1) & \cdots & \boldsymbol{y}(N_s) \end{bmatrix}$ be the primary and secondary data array.

The detection problem considered herein

- We consider a situation where there exit uncertainties about the actual steering vector, due e.g. to uncalibrated arrays, pointing errors, wavefront distortions.
- To account for these uncertainties, we assume that $\boldsymbol{a}$ belongs to a known p-dimensional subspace spanned by the columns of the $m \times p$ matrix $\boldsymbol{H}$.
- The noise vectors $\boldsymbol{n}_p(t)$ and $\boldsymbol{n}_s(t)$ are proper zero-mean independent and Gaussian distributed with

$$\boxed{\mathcal{E}\left\{\boldsymbol{n}_s(t)\boldsymbol{n}_s^H(t)\right\} = \boldsymbol{M};\ \mathcal{E}\left\{\boldsymbol{n}_p(t)\boldsymbol{n}_p^H(t)\right\} = \alpha\boldsymbol{M}}$$

The detection problem

- $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent random matrices with distribution

$$\begin{cases} \boldsymbol{X} \sim \tilde{\mathcal{N}}_{m,N_p}\left(\mu \boldsymbol{H}\boldsymbol{\theta}\boldsymbol{s}^H, \alpha \boldsymbol{M}, \boldsymbol{I}_{N_p}\right) \\ \boldsymbol{Y} \sim \tilde{\mathcal{N}}_{m,N_s}\left(\boldsymbol{0}, \boldsymbol{M}, \boldsymbol{I}_{N_s}\right) \end{cases}$$

  with $\mu = 0$ under H_0 and $\mu = 1$ under H_1.
- We consider two cases : $\boldsymbol{M}$ known and $\boldsymbol{M}$ unknown.
- In either case, $\boldsymbol{\theta}$, $\boldsymbol{s}$ and α are unknowns.

GLRT - M known

- The probability density function (p.d.f.) of $\boldsymbol{X}$ is given by

$$f(\boldsymbol{X};\alpha,\boldsymbol{M},\boldsymbol{\theta},\boldsymbol{s}|H_k)=\frac{e^{-\mathrm{Tr}\left\{\alpha^{-1}\left(\boldsymbol{X}-\mu\boldsymbol{H}\boldsymbol{\theta}\boldsymbol{s}^H\right)^H\boldsymbol{M}^{-1}\left(\boldsymbol{X}-\mu\boldsymbol{H}\boldsymbol{\theta}\boldsymbol{s}^H\right)\right\}}}{\pi^{mN_p}\alpha^{mN_p}\left|\boldsymbol{M}\right|^{N_p}}$$

- The generalized likelihood ratio test (GLRT) is given by

$$\frac{\max_{\alpha,\boldsymbol{M},\boldsymbol{\theta},\boldsymbol{s}} f(\boldsymbol{X};\alpha,\boldsymbol{M},\boldsymbol{\theta},\boldsymbol{s}|H_1)}{\max_{\alpha,\boldsymbol{M}} f(\boldsymbol{X};\alpha,\boldsymbol{M}|H_0)}\overset{H_1}{\underset{H_0}{\gtrless}}\eta$$

GLRT - M known

- Maximizing over the unknowns yields

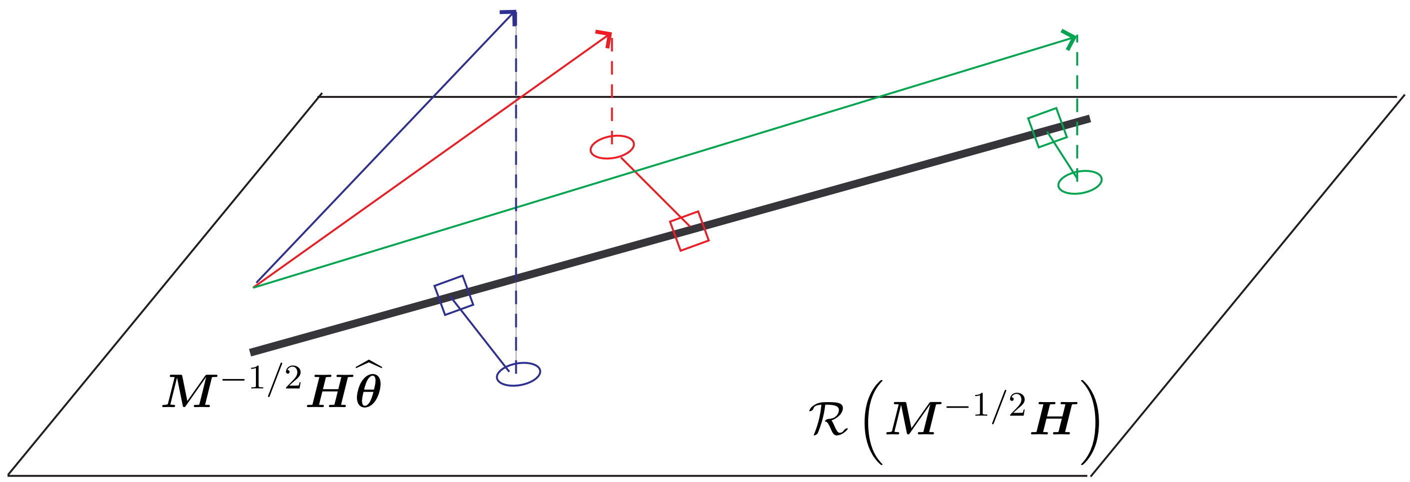
$$\frac{\lambda_{\max}\left\{\left(\boldsymbol{H}^H\boldsymbol{M}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^H\boldsymbol{M}^{-1}\boldsymbol{X}\boldsymbol{X}^H\boldsymbol{M}^{-1}\boldsymbol{H}\right\}}{\mathrm{Tr}\left\{\boldsymbol{M}^{-1}\boldsymbol{X}\boldsymbol{X}^H\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Leftrightarrow \frac{\lambda_{\max}\left\{\boldsymbol{P}_{\boldsymbol{M}^{-1/2}\boldsymbol{H}}\boldsymbol{M}^{-1/2}\boldsymbol{X}\boldsymbol{X}^H\boldsymbol{M}^{-1/2}\right\}}{\mathrm{Tr}\left\{\boldsymbol{M}^{-1/2}\boldsymbol{X}\boldsymbol{X}^H\boldsymbol{M}^{-1/2}\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- The detector performs whitening, looks for a preferred direction in the whitened subspace $\mathcal{R}\left(\boldsymbol{M}^{-1/2}\boldsymbol{H}\right)$ and compares the energy along this direction with the total energy.

GLRT - M known

- In contrast to a matched subspace detector, the estimated signal is constrained to lie on a line in $\mathcal{R}\left(\boldsymbol{M}^{-1/2}\boldsymbol{H}\right)$:



- The matched direction detector involves the ratio of the largest eigenvalue of a Wishart-type matrix to the trace of a Wishart-type matrix.

Adaptive detection - M unknown

- When $\boldsymbol{M}$ is unknown, the exact GLRT is difficult to obtain.
- A simpler (and widely used) method consists of replacing $\boldsymbol{M}$ by its estimate obtained from secondary data, viz $\widehat{\boldsymbol{M}} = N_s^{-1}\boldsymbol{S}$, with $\boldsymbol{S} = \boldsymbol{Y}\boldsymbol{Y}^H$.
- This yields the adaptive matched direction detector

$$\boxed{\frac{\lambda_{\text{max}}\left\{\left(\boldsymbol{H}^H\boldsymbol{S}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^H\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{X}^H\boldsymbol{S}^{-1}\boldsymbol{H}\right\}}{\operatorname{Tr}\left\{\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{X}^H\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \zeta}$$

Detector's analysis

- From a radar point of view, it is important
  - to have constant false alarm rate (CFAR) detectors (so that the threshold can be set independently of the nuisance parameters);
  - to obtain the distribution of the test statistic under H_0 in order to avoid extensive simulations.

  $\Rightarrow$ need to derive the p.d.f. of the test statistic under H_0.
- A bonus is to have this p.d.f. under H_1 also, so as to obtain analytically the probability of detection P_d (usually more complicated).

Matched Direction Detector's analysis

- Let $M^{-1/2}H = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ and let

$$\tilde{X} = \alpha^{-1/2} Q^H M^{-1/2} X = \begin{array}{c} \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} \end{array} \begin{array}{c} p|N_p \\ m-p|N_p \end{array}$$
$$\sim \tilde{\mathcal{N}}_{m,N_p}\left(\mu\alpha^{-1/2} \begin{bmatrix} \tilde{\theta} \\ 0 \end{bmatrix} s^H, I_p, I_{N_p} \right)$$

These transformations perform whitening and rotation on the first p components.

MDD's analysis

- The test statistic can be rewritten as

$$g = \frac{\lambda_{\text{max}}\left\{\tilde{\boldsymbol{X}}_1\tilde{\boldsymbol{X}}_1^H\right\}}{\operatorname{Tr}\left\{\tilde{\boldsymbol{X}}_1\tilde{\boldsymbol{X}}_1^H\right\} + \operatorname{Tr}\left\{\tilde{\boldsymbol{X}}_2\tilde{\boldsymbol{X}}_2^H\right\}}$$

$$= \frac{\lambda_1/t_1}{1 + t_2/t_1} = \frac{a}{1+f} = ab$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_{\min(p,N_p)}$ are the eigenvalues of $\tilde{\boldsymbol{X}}_1\tilde{\boldsymbol{X}}_1^H$, $t_1 = \operatorname{Tr}\left\{\tilde{\boldsymbol{X}}_1\tilde{\boldsymbol{X}}_1^H\right\}$ and $t_2 = \operatorname{Tr}\left\{\tilde{\boldsymbol{X}}_2\tilde{\boldsymbol{X}}_2^H\right\}$.

- To obtain the p.d.f. of $g = ab$ under H_0, we'll show that a and b are independent, and derive the respective p.d.f. of a and b.

MDD's analysis under H_0

- Since t_1 and t_2 are independent chi-squared distributed r.v., $b = 1/(1+f)$ has a beta distribution:

$$f_B(b) = \frac{\Gamma(mN_p)}{\Gamma(pN_p)\Gamma((m-p)N_p)} b^{pN_p-1}(1-b)^{(m-p)N_p-1}; \quad 0 \leq b \leq 1$$

- Let $z_k = t_1^{-1}\lambda_k$ for $k = 1, \cdots, p$ ($p \leq N_p$) denote the normalized eigenvalues and let us define

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \cdots & \lambda_p \end{bmatrix}^T; \quad \boldsymbol{\Lambda} = \mathrm{diag}\left(\boldsymbol{\lambda}\right)$$
$$\boldsymbol{z} = \begin{bmatrix} z_1 & \cdots & z_{p-1} \end{bmatrix}^T; \quad \tilde{\boldsymbol{z}} = \begin{bmatrix} \boldsymbol{z} & z_p \end{bmatrix}^T$$

nota: $z_p = 1 - \sum_{k=1}^{p-1} z_k$.

MDD's analysis under H_0

- The p.d.f of $\boldsymbol{\lambda}$ is [James64]

$$\boxed{f(\boldsymbol{\lambda}) = C\, e^{-\mathrm{Tr}\{\boldsymbol{\Lambda}\}} \left|\boldsymbol{\Lambda}\right|^{Np-p} \left|\boldsymbol{V}(\boldsymbol{\lambda})\right|^2}$$

  with $|\boldsymbol{V}(\boldsymbol{\lambda})| = \prod_{k<\ell}(\lambda_k - \lambda_\ell)$.

- Making the change of variables, the p.d.f. of $(t_1, \boldsymbol{z})$ is $f(t_1, \boldsymbol{z}) = f(t_1) f(\boldsymbol{z})$ with

$$\boxed{f(t_1) = \frac{t_1^{pN_p - 1} e^{-t_1}}{\Gamma(pN_p)}; \quad f(\boldsymbol{z}) = C\, \Gamma(pN_p) \left|\tilde{\boldsymbol{Z}}\right|^{Np-p} \left|\boldsymbol{V}(\tilde{\boldsymbol{z}})\right|^2}$$

  where $\tilde{\boldsymbol{Z}} = \mathrm{diag}\,(\tilde{\boldsymbol{z}})$.

- Therefore t_1 and $z_1 = \lambda_1 / t_1$ are independent.

MDD's analysis under H_0

- The p.d.f. of $a = z_1$ could in principle be obtained as

$$f(z_1) = C\Gamma(pN_p) \int_{z_2} \cdots \int_{z_{p-1}} \prod_{k=1}^{p} z_k^{N_p - p} \prod_{k<\ell} (z_k - z_\ell)^2 dz_2 \cdots dz_{p-1}$$

  where the integration is over the domain $0 \leq z_p < z_{p-1} < \ldots < z_1 \leq 1$ and $z_p + z_{p-1} + \ldots + z_1 = 1$.

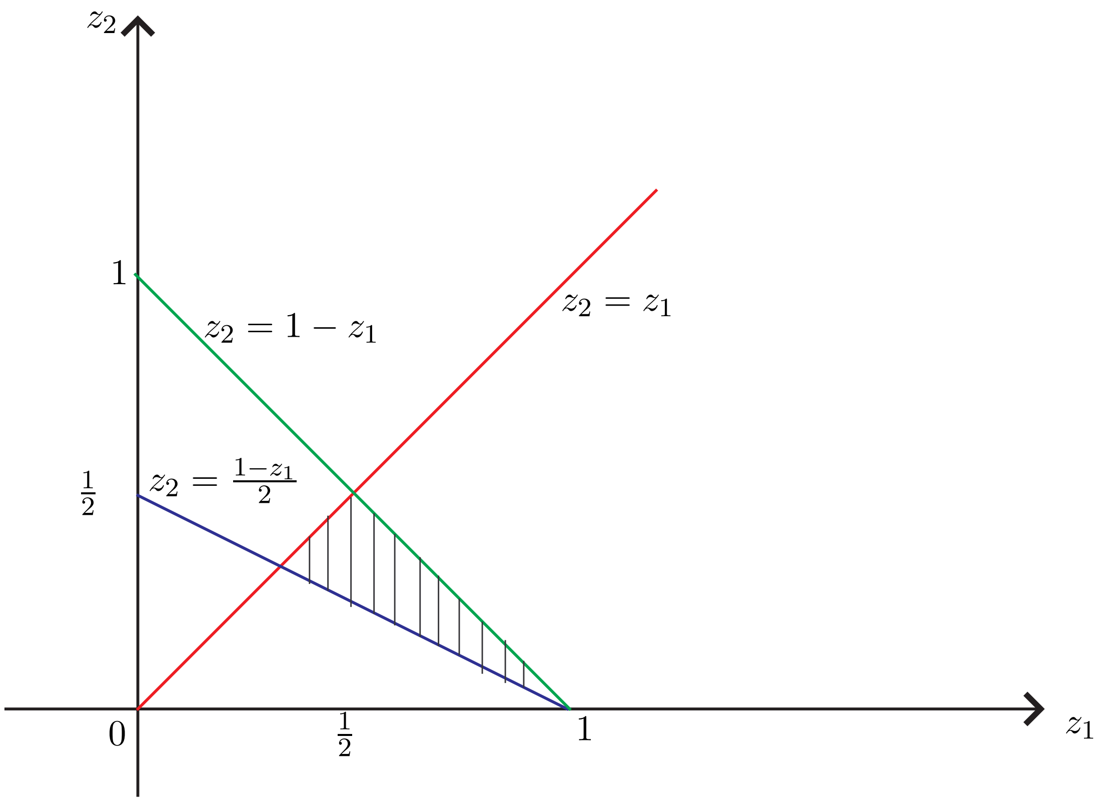
- For $p = 2$, there is no integral and

$$\boxed{f(z_1) = C\,\Gamma(2N_p) z_1^{N_p-2} (1 - z_1)^{N_p-2} (2z_1 - 1)^2; \quad \frac{1}{2} \leq z_1 \leq 1}$$

MDD's analysis under H_0

For $p = 3$

$$f(z_1) = C' \; z_1^{N_p-3} \int z_2^{N_p-3} (1 - z_1 - z_2)^{N_p-3} (z_1 - z_2)^2 (2z_1 - z_2 - 1)^2 (2z_2 - z_1 + 1)^2 \, dz_2$$



MDD's analysis under H_0

- Evaluating the integral yields for the pdf of $a = z_1$

$$
\begin{aligned}
f(a) = C_1 \ (1-a)^{2N_p+1} a^{N_p-3} \\
\times \Big\{ \left[(2N_p+1)(2N_p-1)h^2(a) - 6(2N_p+1)h(a) + 15 \right] B_{z(a)}(\frac{3}{2}, N_p - 2) \\
+ [1 - z(a)]^{N_p-2} z^{3/2}(a) \left[2(2N_p+3)z^2(a) - 10 \right] \Big\}; \qquad \frac{1}{3} \le a \le 1
\end{aligned}
$$

where

$$
h(a) = \left(\frac{3a-1}{1-a} \right)^2, \qquad z(a) = \min\left(1, h(a)\right)
$$

and $B_z(a,b)$ is the incomplete Beta function.

- For $p > 3$??

MDD's analysis under H_0

- Going back to our initial detector $g = ab$, its p.d.f is given (for $p = 2$) is given by

$$
\begin{aligned}
f_G(g) &= \int f_A(x) f_B\left(\frac{g}{x}\right)\frac{dx}{x} \\
&= C_2\, g^{2N_p-1}\int_{\max(g,\frac{1}{2})}^{1} x^{N_p-mN_p-1}(1-x)^{N_p-2}(2x-1)^2(x-g)^{(m-2)N_p-1}\, dx \\
&= \int f_A\left(\frac{g}{x}\right) f_B(x)\frac{dx}{x} \\
&= C_2\, g^{N_p-2}\int_{g}^{\min(1,2g)}(1-x)^{(m-2)N_p-1}(2g-x)^2(x-g)^{N_p-2}\, dx
\end{aligned}
$$

- The detector is CFAR as $f_G(g)$ does not depend on any nuisance parameter, viz α or $\boldsymbol{M}$.
- The threshold can be obtained by integrating $f_G(g)$.

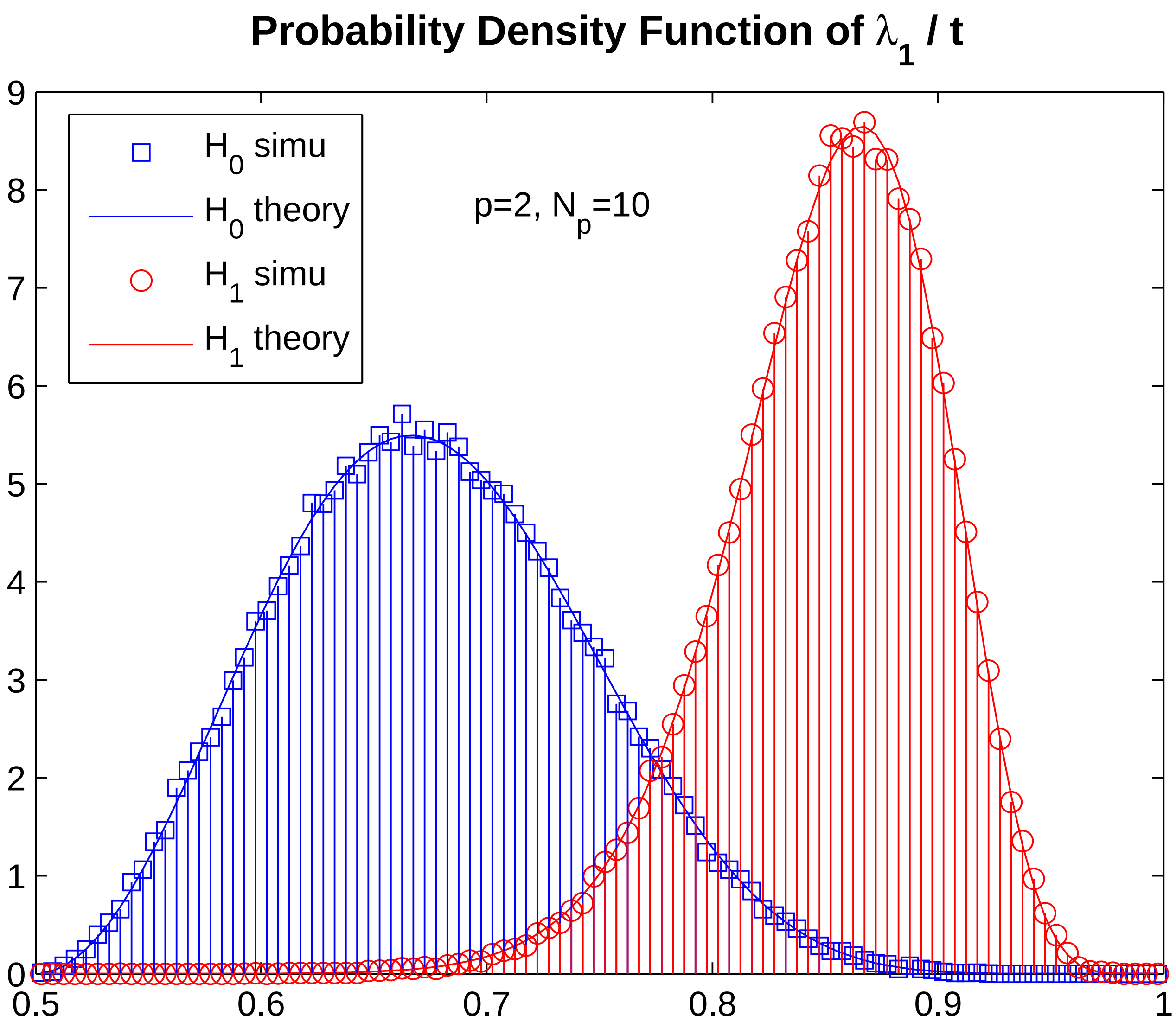
MDD's analysis under H_1

- Under H_1, notable complications:
  - t_1 and λ_1/t_1 are no longer independent;
  - the distribution of the eigenvalues is known when $\mathcal{E}\left\{\tilde{\boldsymbol{X}}\right\}$ is a full-rank matrix (and not a rank-one matrix as here).
- We were able to obtain the joint p.d.f. of $\begin{pmatrix} z_1 & \cdots & z_{p-1} \end{pmatrix}$. In the case $p = 2$, the p.d.f. of $a = z_1$ is given by

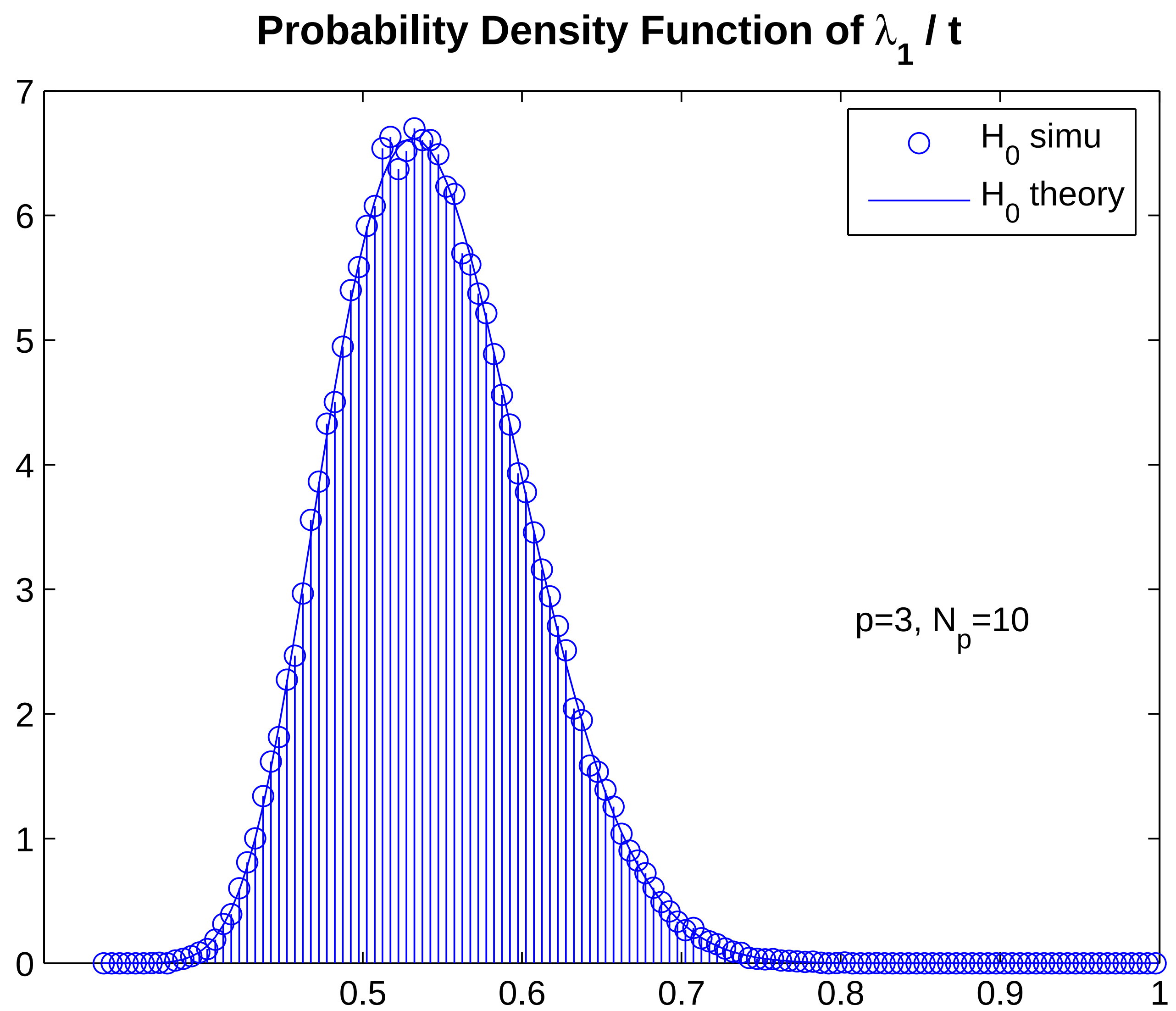
$$f_A(a) = \frac{\Gamma(2N_p - 1)}{\left[\Gamma(N_p - 1)\right]^2} \frac{e^{-\omega_1}}{\omega_1} a^{N_p - 2} (1-a)^{N_p - 2} (2a - 1) \times$$
$$\left[{}_1F_1\left(2N_p - 1; N_p - 1; a\omega_1\right) - {}_1F_1\left(2N_p - 1; N_p - 1; (1-a)\omega_1\right) \right]$$

with $\omega_1 = \alpha^{-1}(\boldsymbol{s}^H \boldsymbol{s})(\widetilde{\boldsymbol{\theta}}^H \widetilde{\boldsymbol{\theta}})$.

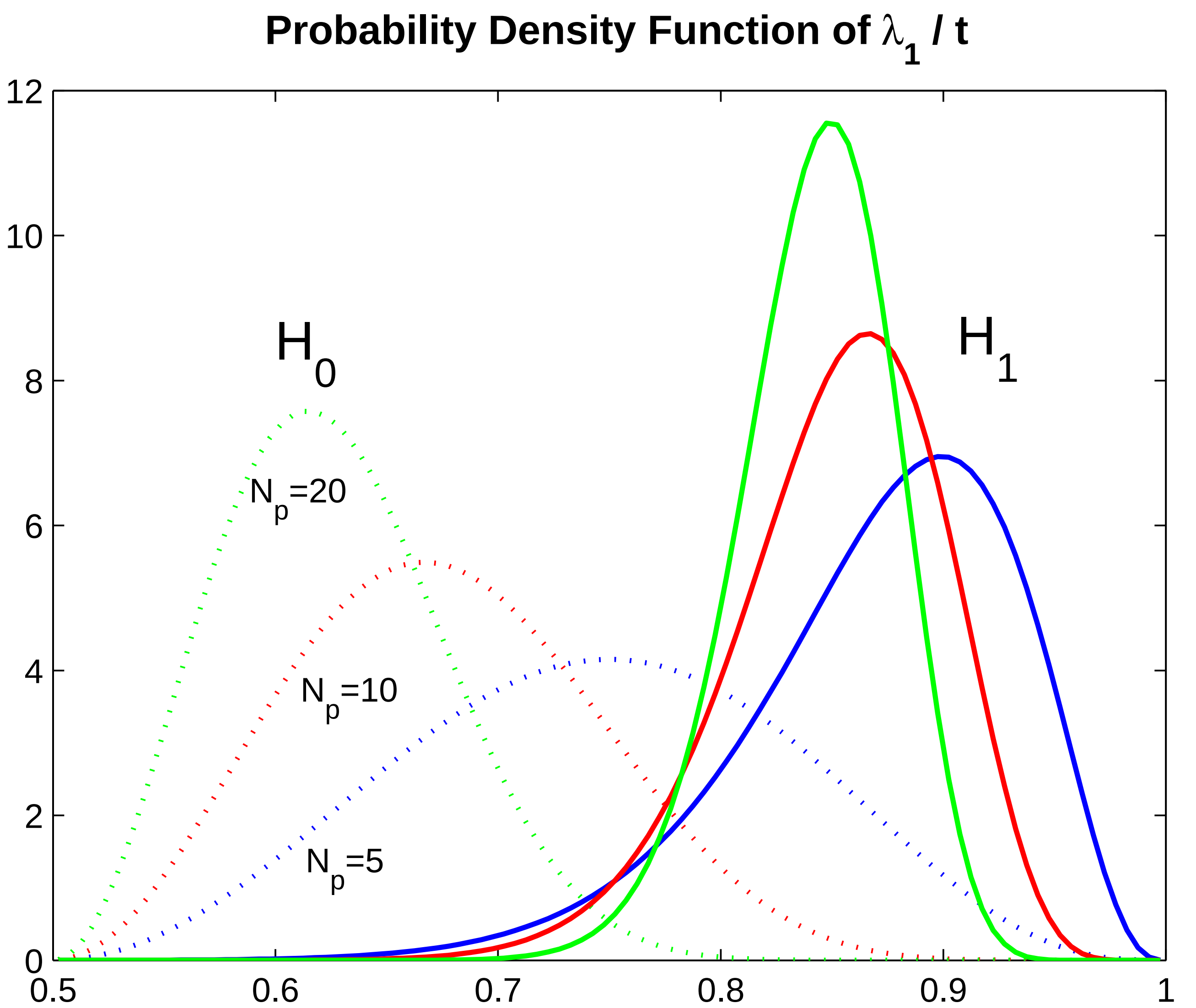
Illustrations



Illustrations



Illustrations



Synthesis

- So far, we were able to derive the p.d.f. of $a = \lambda_{\text{max}} \{\boldsymbol{S}\} / \text{Tr} \{\boldsymbol{S}\}$ when $\boldsymbol{S} \sim \tilde{\mathcal{W}}_p (N_p, \boldsymbol{\Sigma})$ is a Wishart matrix, in the case $p = 2$ and $p = 3$.
- Some open questions :
  - Can the p.d.f. of a be obtained for any p?
  - Is there a simpler (e.g. asymptotic in N_p) analysis?
- These p.d.f show that the detector has a constant-false alarm rate (CFAR). Hence, the threshold can be set theoretically without resorting to extensive simulations.
- Only for $p = 2$ were we able to derive the p.d.f. of a under H_1.

Adaptive MDD's analysis

- The AMDD is given by

$$\frac{\lambda_{\text{max}}\left\{\left(\boldsymbol{H}^H\boldsymbol{S}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^H\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{X}^H\boldsymbol{S}^{-1}\boldsymbol{H}\right\}}{\mathrm{Tr}\left\{\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{X}^H\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \zeta$$

- Similarly to the MDD, we define

$$\tilde{\boldsymbol{X}} = \alpha^{-1/2}\boldsymbol{Q}^H\boldsymbol{M}^{-1/2}\boldsymbol{X} \sim \tilde{\mathcal{N}}_{m,N_p}\left(\mu\alpha^{-1/2}\begin{bmatrix}\tilde{\boldsymbol{\theta}}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{s}^H, \boldsymbol{I}_m, \boldsymbol{I}_{N_p}\right)$$

$$\tilde{\boldsymbol{Y}} = \boldsymbol{Q}^H\boldsymbol{M}^{-1/2}\boldsymbol{Y} \sim \tilde{\mathcal{N}}_{m,N_s}\left(\boldsymbol{0}, \boldsymbol{I}_m, \boldsymbol{I}_{N_s}\right)$$

AMDD's analysis

The adaptive matched direction detector can be written as

$$\frac{\lambda_{\text{max}}\left\{\tilde{\boldsymbol{S}}_{1.2}^{-1}\boldsymbol{N}\boldsymbol{N}^H\right\}}{\text{Tr}\left\{\tilde{\boldsymbol{S}}_{1.2}^{-1}\boldsymbol{N}\boldsymbol{N}^H\right\}+\text{Tr}\left\{\boldsymbol{F}\right\}}$$

where $\boldsymbol{F}=\tilde{\boldsymbol{X}}_2^H\tilde{\boldsymbol{S}}_{22}^{-1}\tilde{\boldsymbol{X}}_2$ and

$$\tilde{\boldsymbol{S}}_{1.2}\sim\tilde{\mathcal{W}}_p\left(N_s-m+p,\boldsymbol{I}_p\right)$$

$$\boldsymbol{N}|\tilde{\boldsymbol{X}}_2,\tilde{\boldsymbol{S}}_{22}\sim\tilde{\mathcal{N}}_{p,N_p}\left(\mu\alpha^{-1/2}\tilde{\boldsymbol{\theta}}\boldsymbol{s}^H,\boldsymbol{I}_p,\boldsymbol{I}_{N_p}+\boldsymbol{F}\right)$$

$$f(\boldsymbol{F})=C_F\,\frac{\left|\boldsymbol{F}\right|^{m-p-N_p}}{\left|\boldsymbol{I}_{N_p}+\boldsymbol{F}\right|^{N_p+N_s}}$$

AMDD's analysis

- The AMDD is a CFAR detector; its p.d.f. under H_0 does not depend on α and $\boldsymbol{M}$: the threshold can be set independently of the noise statistics.
- The distribution of the eigenvalues of $\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{X}^H$ are known when $\boldsymbol{S} \sim \tilde{\mathcal{W}}_p\left(n, \boldsymbol{I}_p\right)$ and $\boldsymbol{X} \sim \tilde{\mathcal{N}}_{p,q}\left(\mathbf{0}, \boldsymbol{I}_p, \boldsymbol{I}_q\right)$. Here, the columns of $\boldsymbol{N}$ are not independent.
- Possible to use some recent results on correlated Wishart matrices?

Numerical examples

- We consider a scenario with $m = 10$, $N_p = 2$ and $N_s = 20$.
- $p = \dim \mathcal{R}(\boldsymbol{H}) = 2$ and the presumed steering vector $\overline{\boldsymbol{a}}$ belongs to $\mathcal{R}(\boldsymbol{H})$.
- We consider three different situations
  - $\boldsymbol{a} = \overline{\boldsymbol{a}} \in \mathcal{R}(H)$: no mismatch.
  - $\boldsymbol{a} \neq \overline{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{R}(H)$: we let $\theta = \langle \boldsymbol{M}^{-1/2}\boldsymbol{a}, \boldsymbol{M}^{-1/2}\overline{\boldsymbol{a}} \rangle$.
  - $\boldsymbol{a} \neq \overline{\boldsymbol{a}}$, $\boldsymbol{a} \notin \mathcal{R}(H)$: we let $\gamma = \langle \boldsymbol{M}^{-1/2}\boldsymbol{a}, \boldsymbol{M}^{-1/2}\boldsymbol{H} \rangle$.

Numerical examples

We compare the probability of detection of the three detectors

$$\frac{\lambda_{\max}\left\{\left(\boldsymbol{H}^H\boldsymbol{M}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^H\boldsymbol{M}^{-1}\boldsymbol{X}\boldsymbol{X}^H\boldsymbol{M}^{-1}\boldsymbol{H}\right\}}{\mathrm{Tr}\left\{\boldsymbol{M}^{-1}\boldsymbol{X}\boldsymbol{X}^H\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta_{mdd}$$

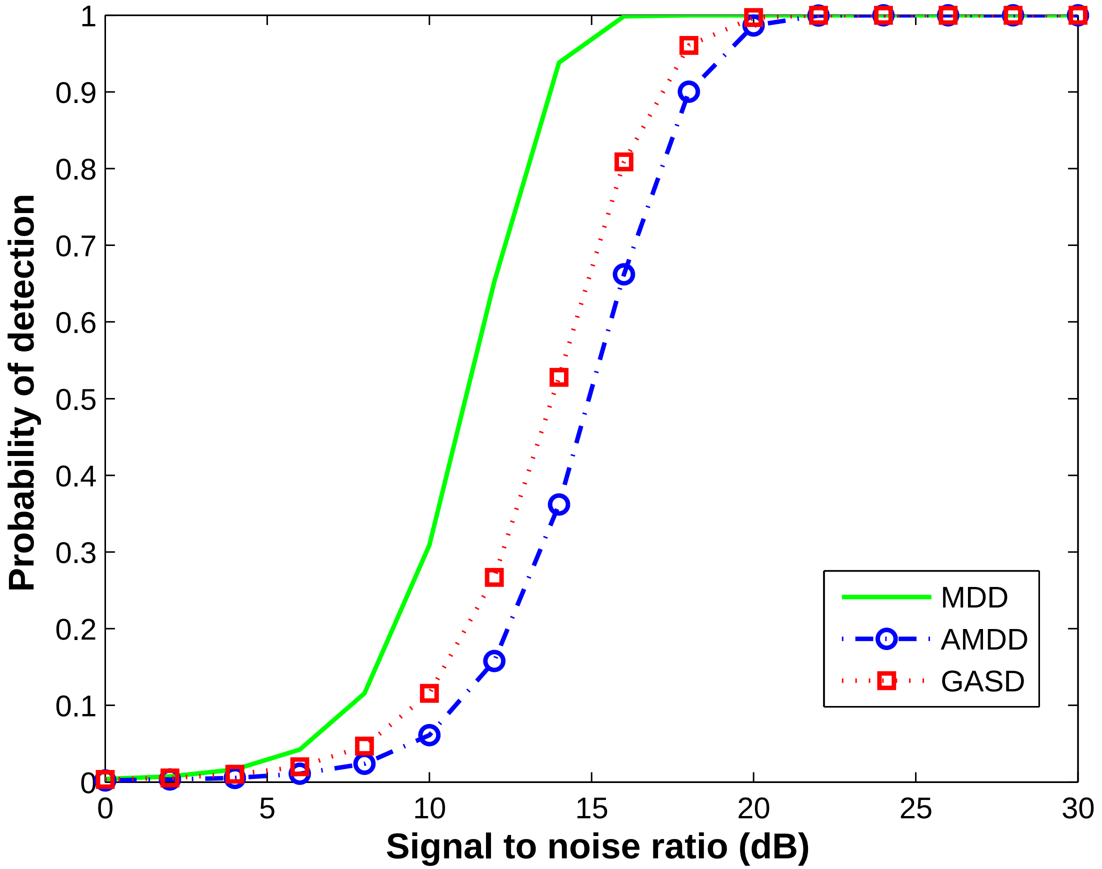
(MDD)

$$\frac{\lambda_{\max}\left\{\left(\boldsymbol{H}^H\boldsymbol{S}^{-1}\boldsymbol{H}\right)^{-1}\boldsymbol{H}^H\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{X}^H\boldsymbol{S}^{-1}\boldsymbol{H}\right\}}{\mathrm{Tr}\left\{\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{X}^H\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta_{amdd} \quad \text{(AMDD)}$$

$$\frac{\overline{\boldsymbol{a}}^H\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{X}^H\boldsymbol{S}^{-1}\overline{\boldsymbol{a}}}{\left(\overline{\boldsymbol{a}}^H\boldsymbol{S}^{-1}\overline{\boldsymbol{a}}\right)\mathrm{Tr}\left\{\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{X}^H\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta_{gasd} \quad \text{(GASD)}$$

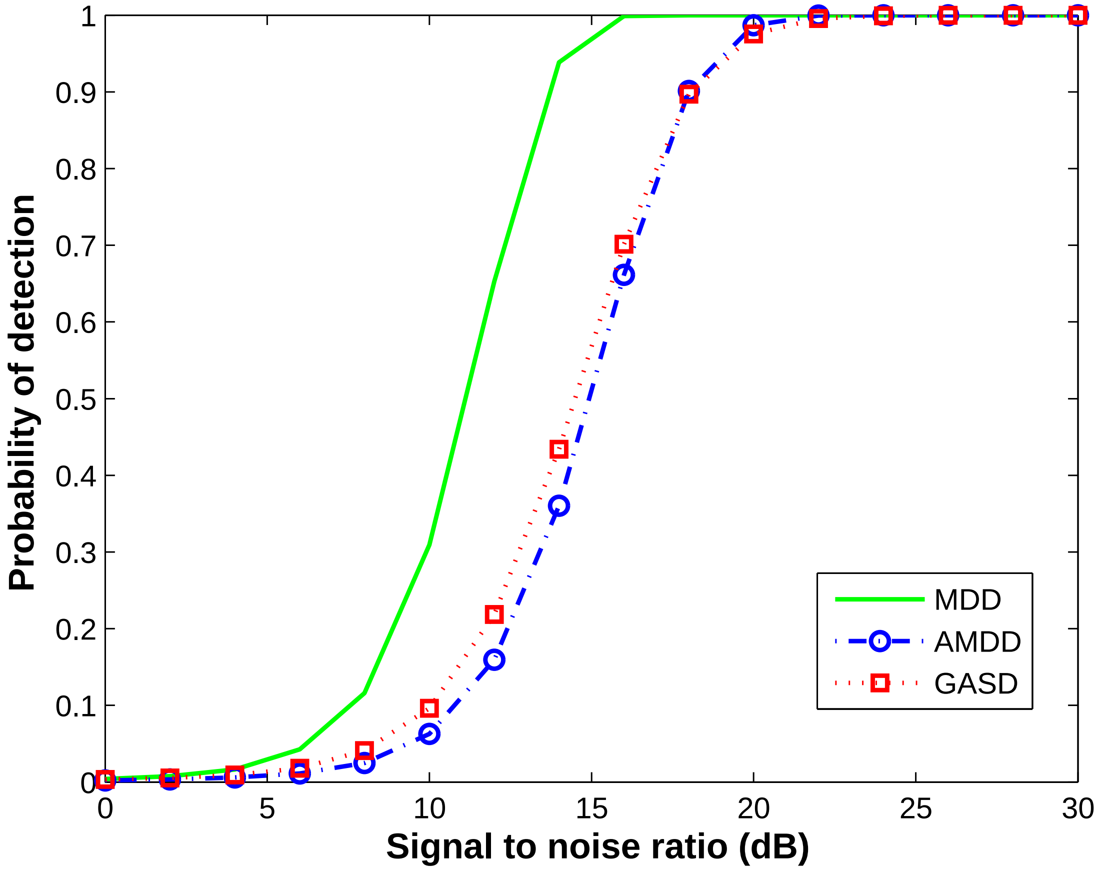
Nota: GASD = AMDD for $p = 1$.

Numerical examples



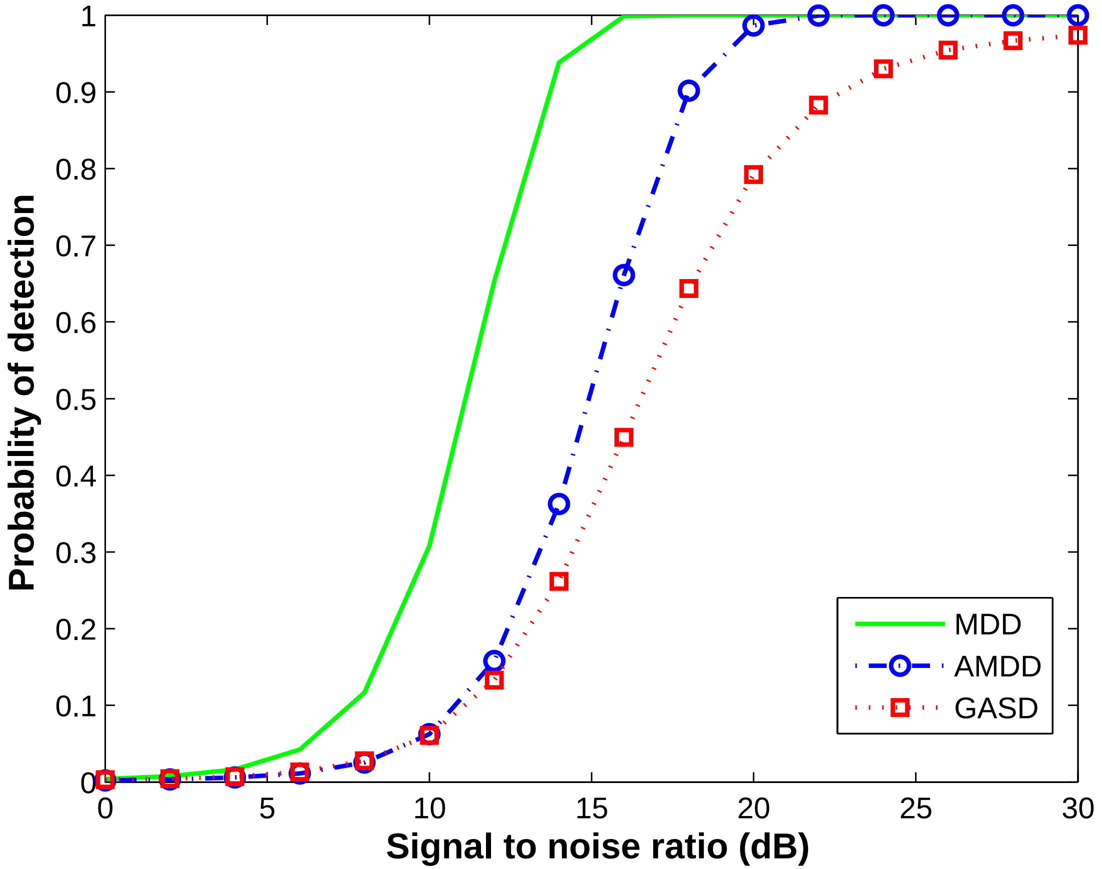
$\boldsymbol{a} = \overline{\boldsymbol{a}} \in \mathcal{R}(H)$.

Numerical examples



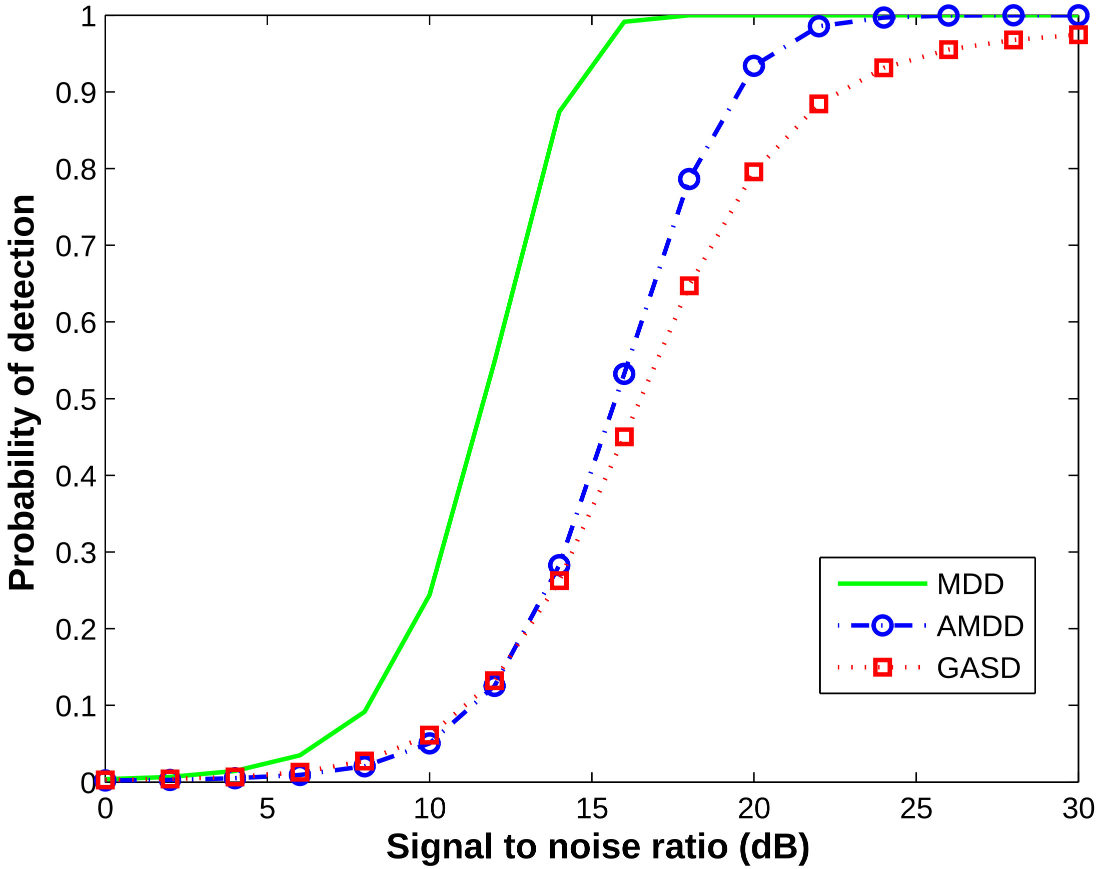
$\boldsymbol{a} \neq \overline{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{R}(H)$. $\cos\theta = 0.97$.

Numerical examples



$\boldsymbol{a} \neq \overline{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{R}(H)$. $\cos\theta = 0.90$.

Numerical examples



$\boldsymbol{a} \neq \overline{\boldsymbol{a}}$, $\boldsymbol{a} \notin \mathcal{R}(H)$. $\cos\theta = 0.9$ and $\cos\gamma = 0.97$.

Concluding remarks

- We addressed the problem of detecting a rank-one signal matrix, which belongs to a known subspace, in the presence of Gaussian noise with known or unknown covariance matrix.
- The detectors involve the ratio of the largest eigenvalue of a Wishart type matrix to its trace.
- When the size of the subspace is small ($p = 2, 3$), it is possible to obtain the distribution of such a statistic. For larger values, this is still an open problem.