Asymptotic Random Matrix Applications in Communication: *MIMO and More*

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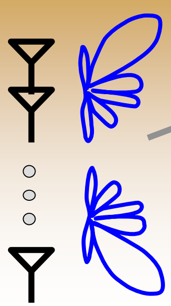

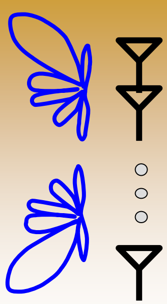

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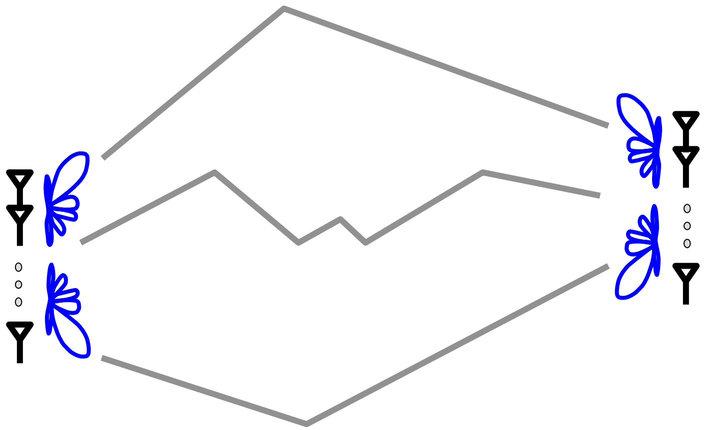
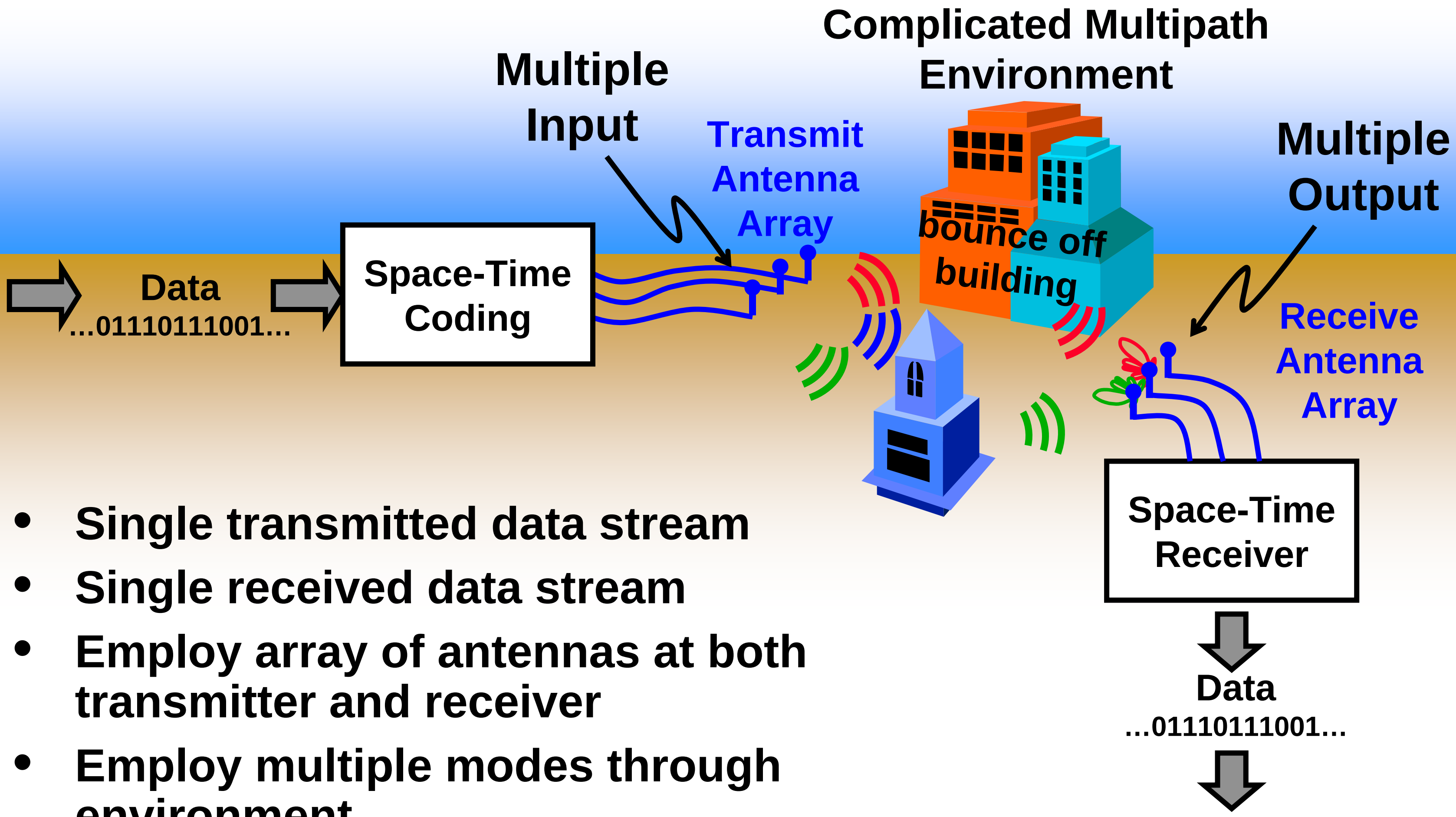
Topics

- **Multiple-Input Multiple-Output (MIMO)**
  - Introduction
  - Asymptotic results
- **Survey of Communication Examples**
  - Code-Division Multiple Access (CDMA)
  - Receiver performance in rich interference networks
- **Wish List**



MIMO Communication
Multiple-Input Multiple-Output



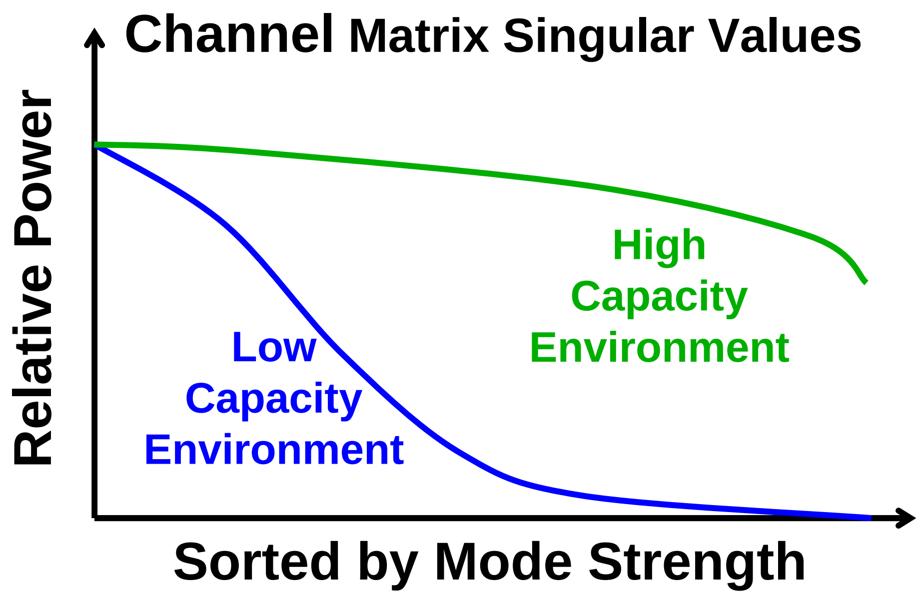
- Single transmitted data stream
- Single received data stream
- Employ array of antennas at both transmitter and receiver
- Employ multiple modes through environment

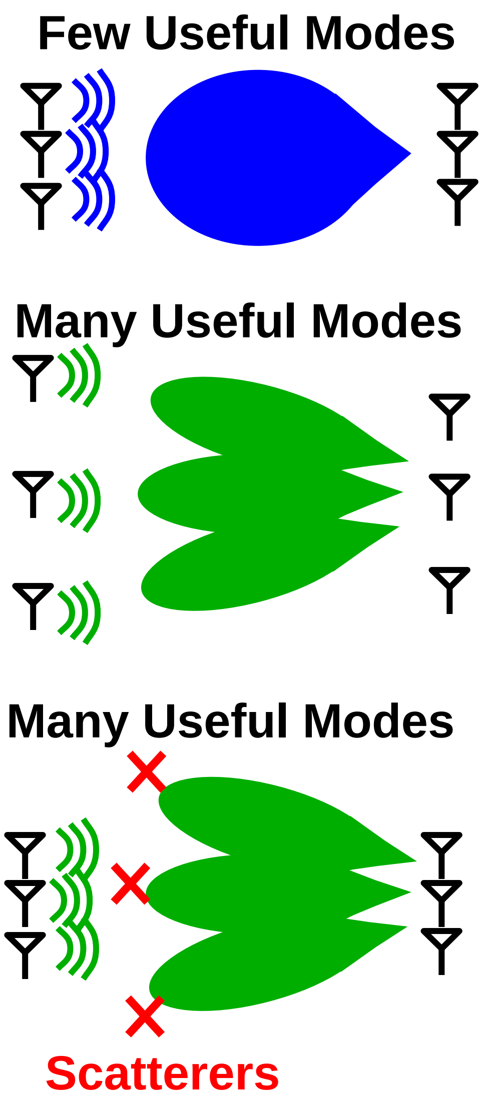
The Channel Matrix

- Channel matrix, $\mathbf{H}$, contains complex attenuation between each transmit and receive antenna

$$\vec{z}(t) = \mathbf{H}\,\vec{x}(t) + \vec{n}(t)$$

- Large channel matrix singular values are useful





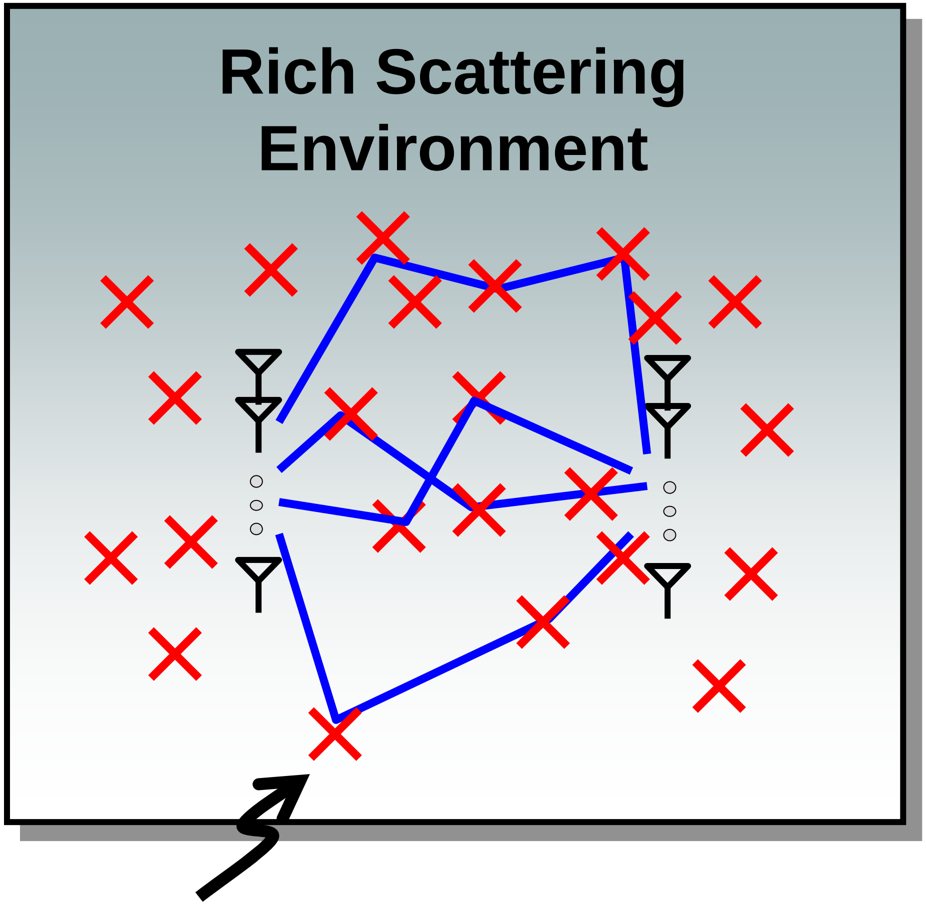
Rich Scattering Channel

- **Very complicated scattering environment**
- **Independent Rayleigh fading for each transmit-receive antenna pair**
- **Entries in $\mathbf{G}$ matrix independently sampled from unit-norm complex Gaussian distribution**

$$\mathbf{H} = a\,\mathbf{G}$$

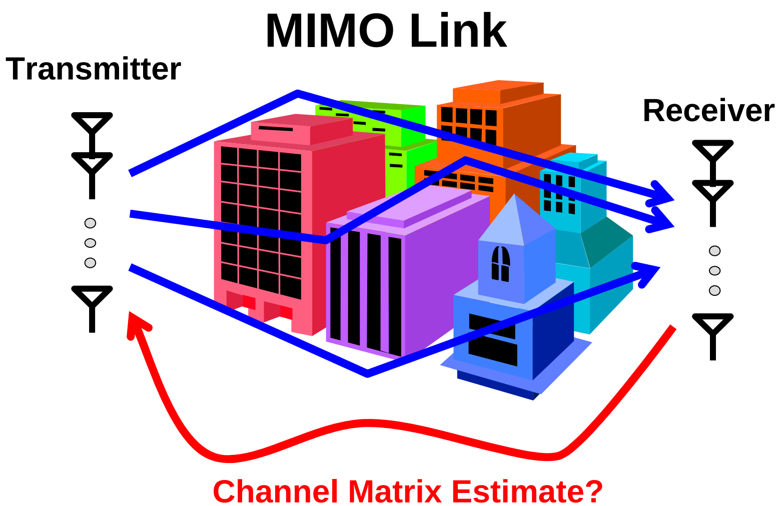
RMS antenna pair attenuation

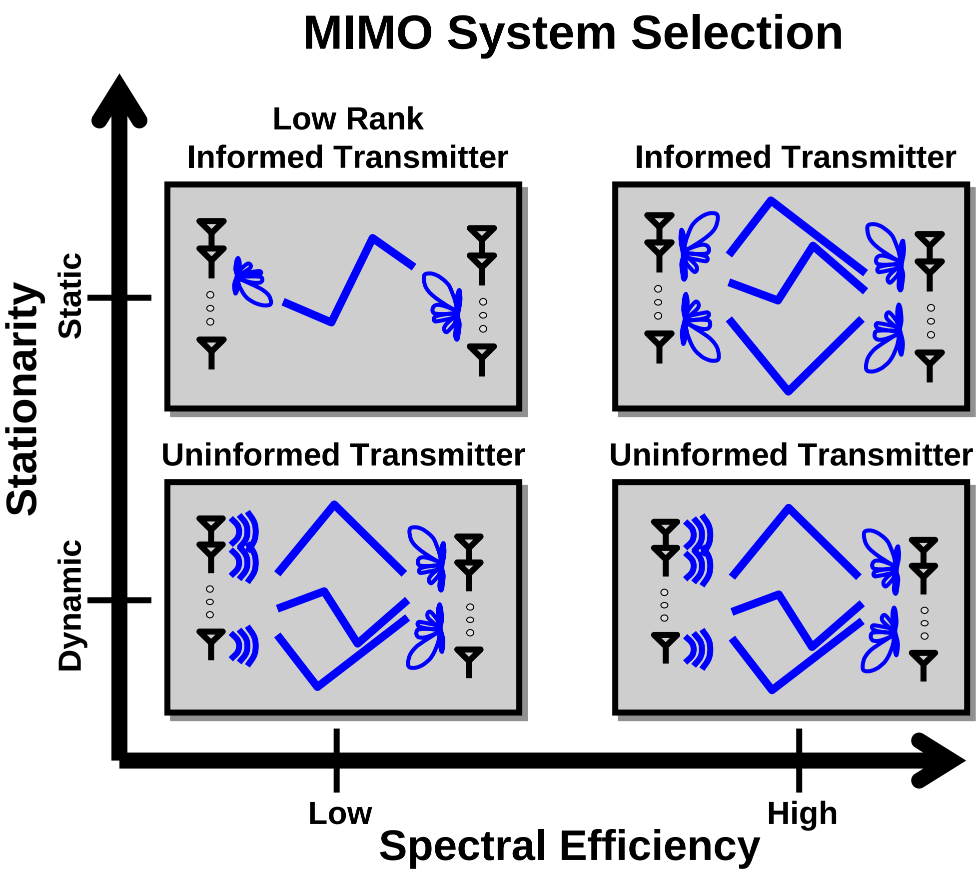
$$\mathbf{G} = \begin{pmatrix} g_{11} & g_{12} & & \\ g_{21} & g_{22} & & \\ & & \ddots & \\ & & & g_{mn} \end{pmatrix}$$



Scatterer

MIMO Taxonomy





- Stationarity required to pass channel estimate to transmitter
- Capacity formulation drives choice of transmitter rank
  - Low Spec. Eff. $\Rightarrow$ low rank TX
  - High Spec. Eff. $\Rightarrow$ high rank TX

Information Theoretic Capacity

Signal Model

$$\vec{z} = \mathbf{H}\vec{x} + \vec{n}$$

Mutual Information

$$\begin{aligned}\mathcal{I}(\vec{z}, \vec{x}|\mathbf{H}) &= h(\vec{z}|\mathbf{H}) - h(\vec{z}|\vec{x}, \mathbf{H}) \\ &= h(\vec{z}|\mathbf{H}) - h(\mathbf{H}\vec{x} + \vec{n}|\vec{x}, \mathbf{H}) \\ &= h(\vec{z}|\mathbf{H}) - h(\vec{n}) \,,\end{aligned}$$

Receive-Signal Entropy

$$\begin{aligned}h(\vec{z}|\mathbf{H}) &= \log_2 \left|\pi\, e \left\langle \vec{z}\vec{z}^\dagger \right\rangle\right| \\ &= \log_2 \left|\pi\, e\, \sigma_n^2 \left(\mathbf{I}_{n_R} + \mathbf{H} \left\langle \vec{x}\vec{x}^\dagger \right\rangle \mathbf{H}^\dagger\right)\right|\end{aligned}$$

Noise-Like Entropy

$$\begin{aligned}h(\vec{n}) &= \log_2 \left|\pi\, e \left\langle \vec{n}\vec{n}^\dagger \right\rangle\right| \\ &= \log_2 \left|\pi\, e\, \sigma_n^2 \mathbf{I}_{n_R}\right|\end{aligned}$$

In Interference Environment

$$h(\vec{z}|\mathbf{H}) \leq \log_2 \left\{\pi e \left|\sigma_n^2 \mathbf{I} + \sigma_n^2 \mathbf{R} + \mathbf{H} \left\langle \vec{x}\vec{x}^\dagger \right\rangle \mathbf{H}^\dagger\right|\right\}$$

$$h(\vec{z}|\vec{x}, \mathbf{H}) \leq \log_2 \left\{\pi e \left|\sigma_n^2 \mathbf{I} + \overbrace{\sigma_n^2 \mathbf{R}}^{\text{Interference Covariance Matrix}}\right|\right\}$$

Informed Transmitter Capacity

$$C_{IT} = \max_{\mathbf{P}} \log_2 \left|\mathbf{I}_{n_R} + \tilde{\mathbf{H}}\, \mathbf{P}\, \tilde{\mathbf{H}}^\dagger\right|$$

$\tilde{\mathbf{H}} \equiv (\mathbf{I} + \mathbf{R})^{-1/2}\, \mathbf{H}$ **; Whitened Channel Matrix**

Uninformed Transmitter Capacity

$$C_{UT} = \log_2 \left|\mathbf{I}_{n_R} + \frac{P_o}{n_T} \tilde{\mathbf{H}}\tilde{\mathbf{H}}^\dagger\right|$$

Informed Transmitter MIMO
Water Filling

- **Model**

$$\vec{z}(t) = \mathbf{H}\,\vec{x}(t) + \vec{n}(t)$$

- **Modes in channel matrix live in singular values**

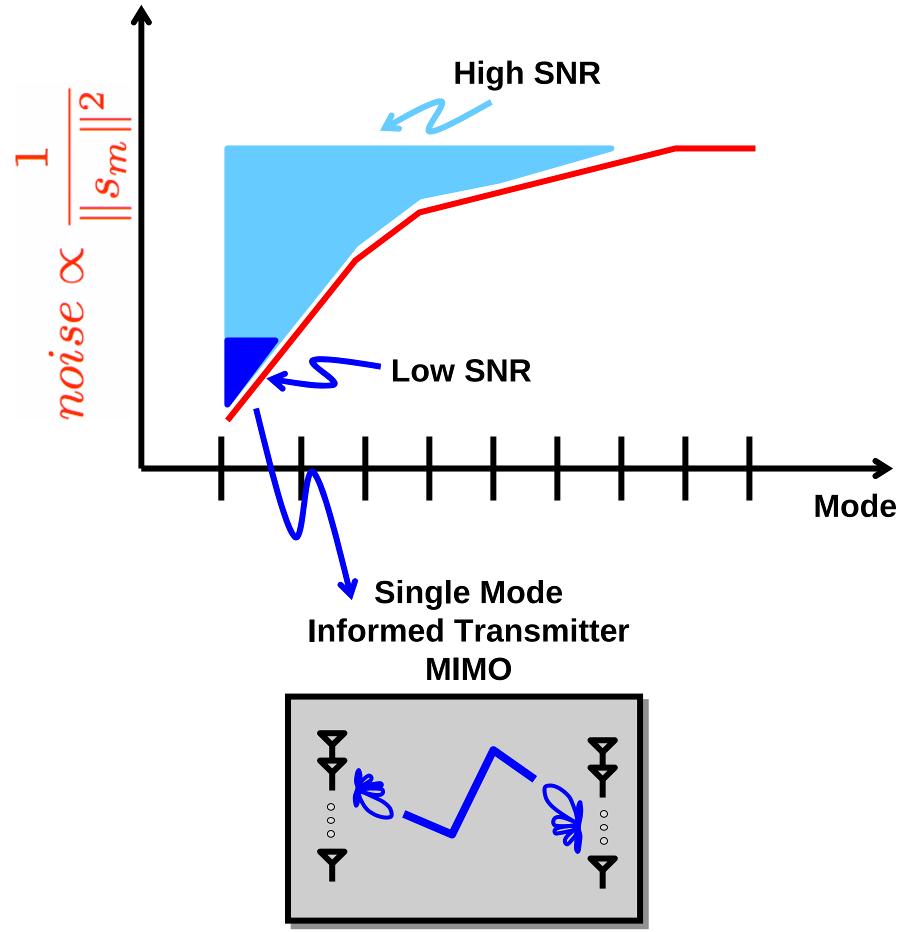
$$\mathbf{H} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^{\dagger}$$

Receive Beamformers

Transmit Beamformers

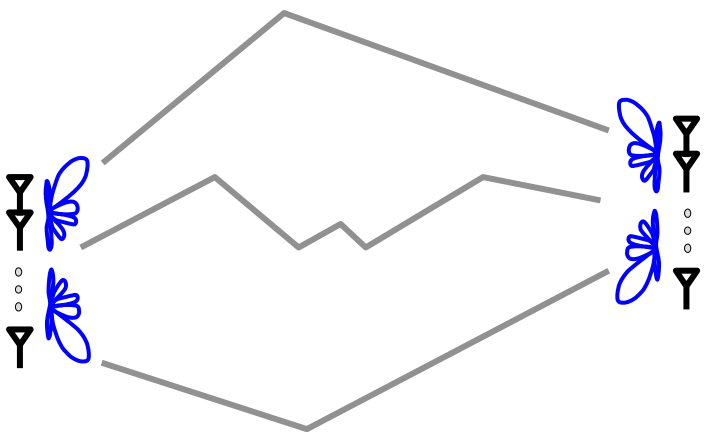
- **Number of employed modes depends on SNR**
  - Higher SNR $\Rightarrow$ Many modes
  - Lowest SNR $\Rightarrow$ Single mode

Water-Filling Analogy



Topics

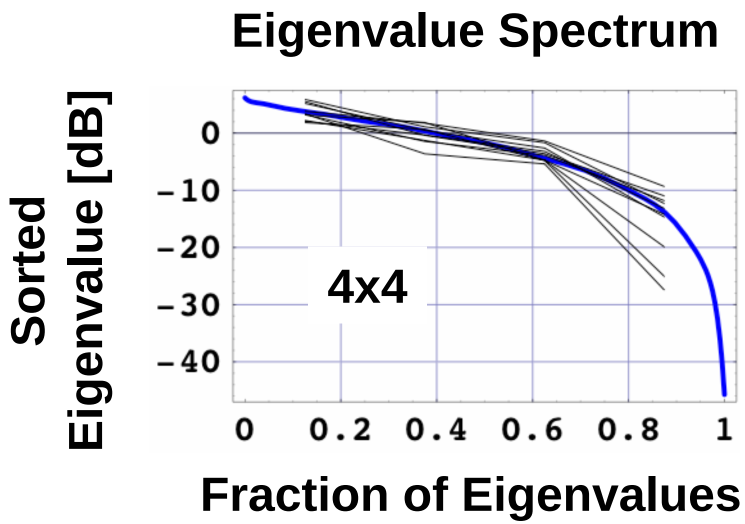
- **Multiple-Input Multiple-Output (MIMO)**
  - Introduction
  - **Asymptotic results**
- Survey of Communication Examples
  - Code-Division Multiple Access (CDMA)
  - Receiver performance in rich interference networks
- Wish List

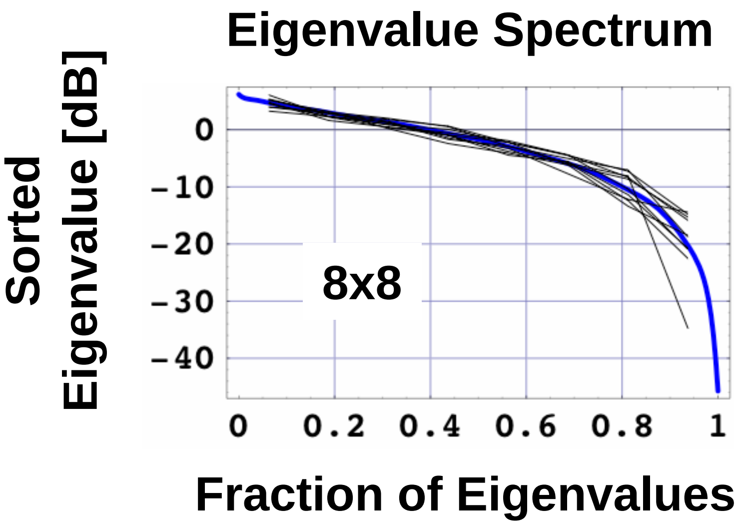


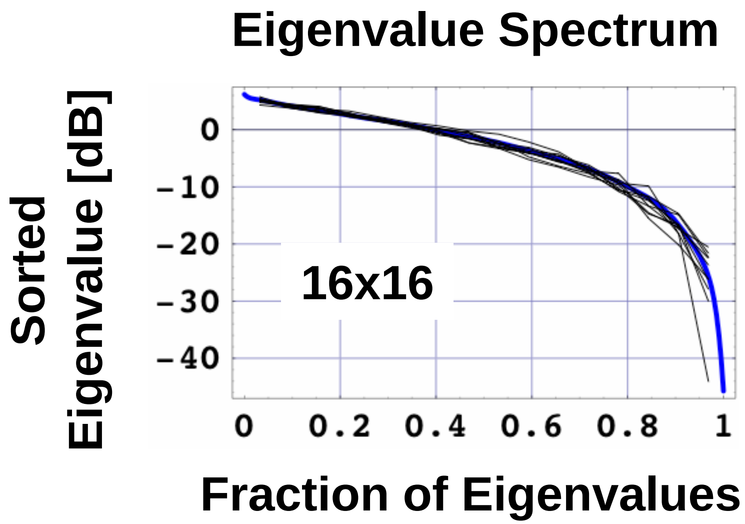
Convergence to the Continuum

How Big is Big?

- **Eigenvalue probability density of $\mathbf{HH}^{\dagger} = a^2\mathbf{GG}^{\dagger}$**
- **Ensemble of finite NxN eigenvalue spectra**
- **Converges quickly**







Continuous Uninformed Transmitter Capacity Approximation

- **Replace discrete matrix with continuous approximation**
- **Shape of random finite matrix eigenvalue distributions converge quickly**
- **Use number of receivers normalized capacity**

Total Transmit Power *(noise-normalized)*

Channel Matrix

$$C_{UT} = \log_2 \left| \mathbf{I} + \frac{P_o}{n_{Tx}} \mathbf{H}\,\mathbf{H}^\dagger \right|$$

$$= \log_2 \left| \mathbf{I} + a^2 P_o \frac{\mathbf{G}\mathbf{G}^\dagger}{n_{Tx}} \right|$$

$$= \sum_{m=1}^{n_{Rx}} \log_2 \left| \mathbf{I} + a^2 P_o \; \mu_m \right|$$

continuous approximation

$$\approx n_{Rx} \, \Phi\left(a^2 P_o ; r\right)$$

$$\Phi(x\,;\,r) \equiv \int_0^\infty d\mu \;\; \underbrace{f_r(\mu)}_{} \;\; \log_2(1 + \mu\, x)$$

Eigenvalue Density

D. W. Bliss, K. W. Forsythe, and A. F. Yegulalp, "MIMO communication capacity using infinite dimension random matrix eigenvalue distributions," Asilomar Conference, Nov., 2001

Continuous Informed Transmitter Capacity Approximation

$$C_{IT} = \max_{tr\{\mathbf{P}\}=P_o} \log_2 \left| \mathbf{I} + \mathbf{H}\mathbf{P}\mathbf{H}^\dagger \right|$$

$$= \log_2 \left| \frac{P_o + \frac{1}{a^2 M} tr\left\{\Lambda^{-1}\right\}}{L} a^2 M \underbrace{\Lambda}_{\text{Employed Eigenvalues}} \right|$$

continuous approximation

$$\approx g\, M \log_2 \left[\frac{a^2 P_o + \int_{\mu_{cut}}^{\infty} d\mu' \; f_1(\mu') \frac{1}{\mu'}}{g} \right] + M \int_{\underbrace{\mu_{cut}}_{\text{Smallest Eigenvalue Employed}}}^{\infty} d\mu \; f_1(\mu) \log_2(\mu)$$

$$\underbrace{g}_{\text{Fraction of Eigenvalues Employed}} = \frac{L}{M} \approx \int_{\mu_{cut}}^{\infty} d\mu \; f_1(\mu) \quad , \quad \underbrace{\mu_{cut}}_{\text{Implicitly Defined}} = \frac{\int_{\mu_{cut}}^{\infty} d\mu' \; f_1(\mu')}{a^2 P_o + \int_{\mu_{cut}}^{\infty} d\mu \; f_1(\mu) \frac{1}{\mu}}$$

- **For this example assume that**

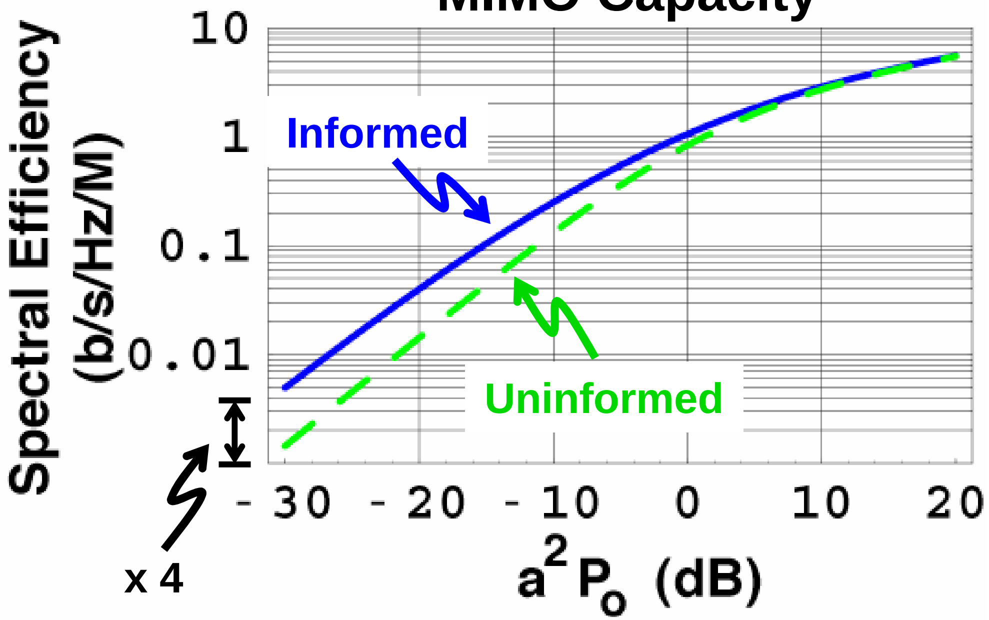
  $M = n_{Rx} = n_{Tx} \quad , \quad r = 1$

- **Employ top L modes of $\mathbf{H}\mathbf{H}^\dagger$ contained in Λ**

Large Random Matrix MIMO Capacity
Informed vs. Uninformed Transmitter

- **Rich scattering environment**
- **M x M channel**
- **capacity using asymptotic eigenvalue distribution**
- **Low-SNR informed capacity higher**

Antenna Number Normalized MIMO Capacity



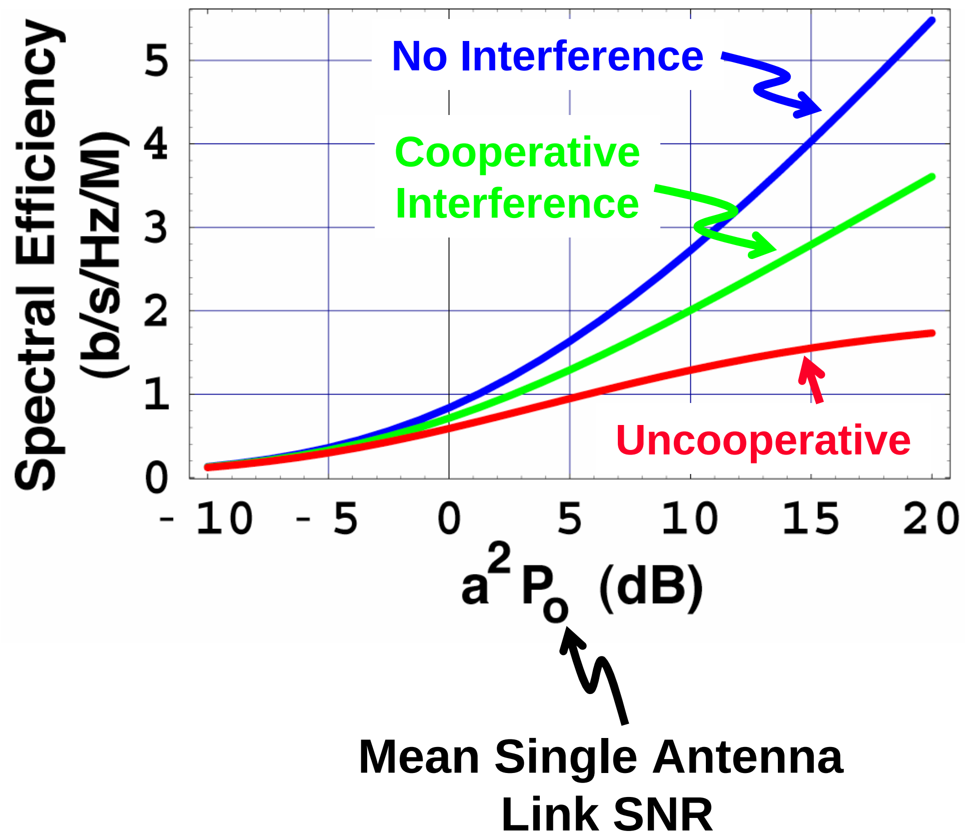
Mean Single Antenna Link SNR

MIMO Capacity In Interference

Large Dimension Limit (UT)

- Capacity in presence of interference
- Equivalent MIMO Interferer
- Interference type
  - Cooperative
  - Uncooperative
- Cooperative interference can be demodulated
- Uninformed transmitter
- Square $\mathbf{H}$

Antenna Number Normalized MIMO Capacity



Topics

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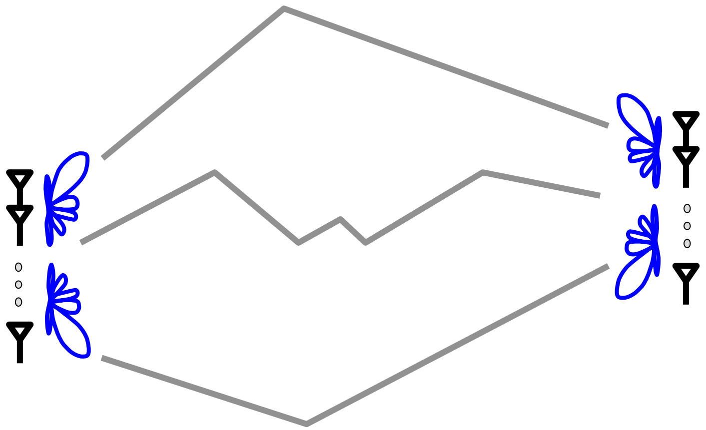
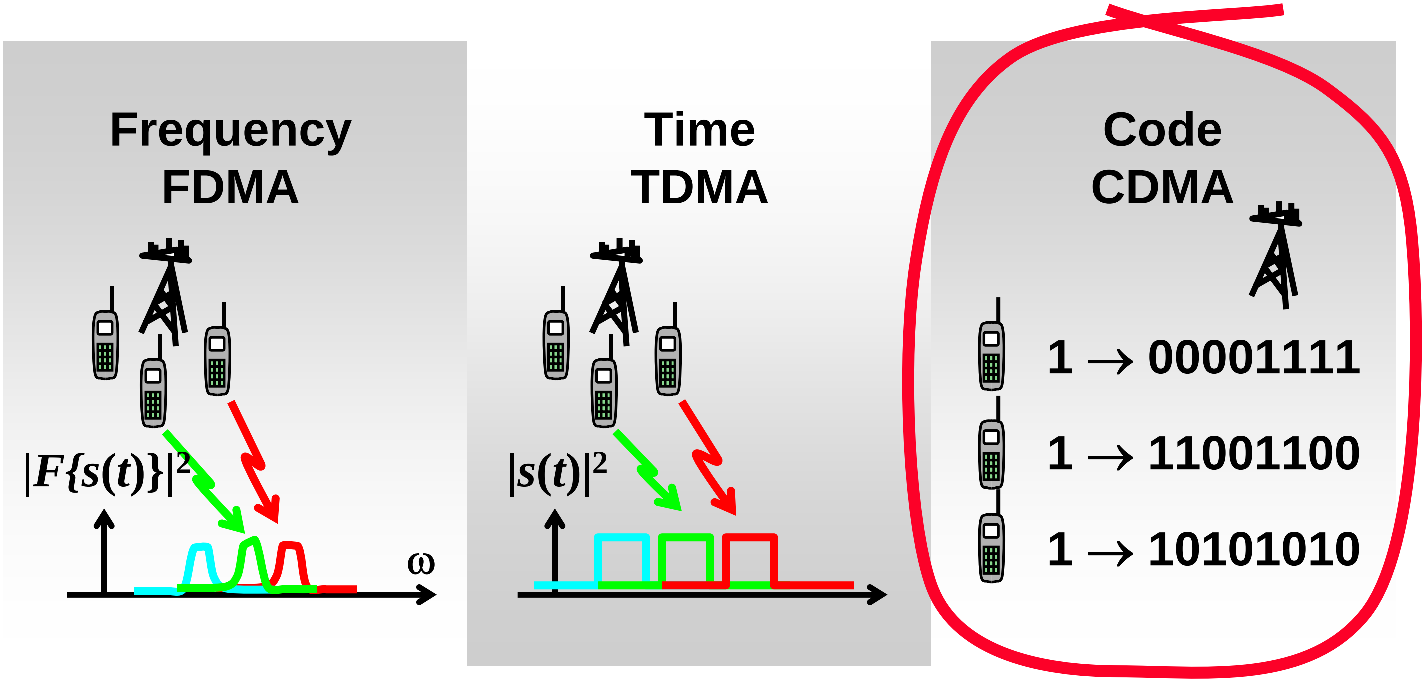
  - Asymptotic results
- **Survey of Communication Examples**
  - **Code-Division Multiple Access (CDMA)**
  - **Receiver performance in rich interference networks**
- Wish List



Multiple Access

- Multiple users trying to talk to the base station at once
- Separate users
  - Frequency (frequency division multiple access)
  - Time (time division multiple access)
  - Code (code division multiple access)

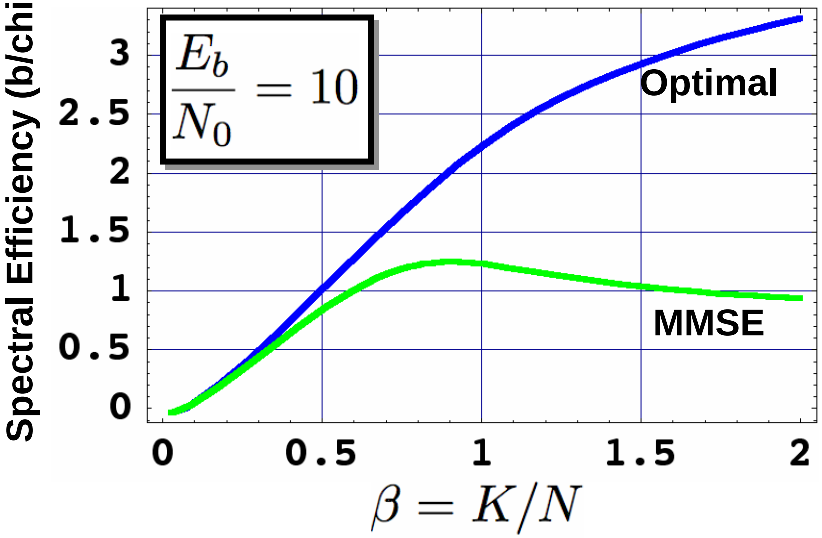


CDMA Performance Bounds

- **Verdu's multiuser detection bounds**
- **Asymptotic in number of users and spreading**
- **Eigenvalue distribution in code cross-correlation**

Users	$K \to \infty$
Spreading	$N \to \infty$
Loading	$\beta = K/N \to \text{finite}$
SNR	$\rho = \dfrac{C}{\beta}\dfrac{E_b}{N_0/2}$

Spectral Efficiency Bounds



$$C^{mmse} = \frac{\beta}{2} log_2 \left(1 + \rho - \frac{1}{4} f(\rho, \beta)\right)$$

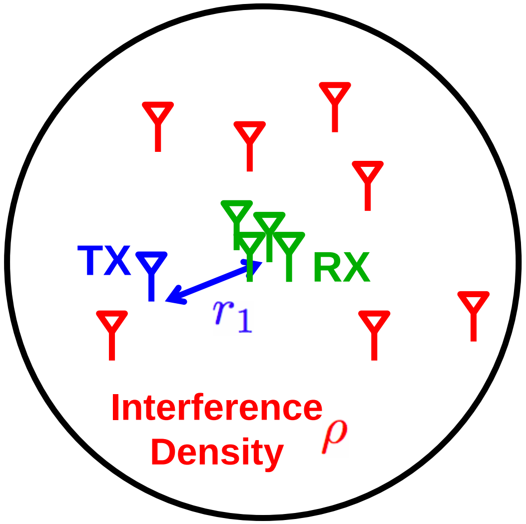
$$C^{opt} = \frac{\beta}{2} log_2 \left(1 + \rho - \frac{1}{4} f(\rho, \beta)\right) + \frac{1}{2}\log_2\left(1 + \rho\beta - \frac{1}{4} f(\rho, \beta)\right) - \frac{\log_2 e}{8\rho} f(\rho, \beta)$$

$$f(x, y) \equiv \left(\sqrt{x(1+\sqrt{y})^2 + 1} - \sqrt{x(1-\sqrt{y})^2 + 1}\right)^2$$

Verdu and Shamai, "Spectral Efficiency of CDMA with Random Spreading," IEEE Trans. On IT, March, 1999

Multiple-Antenna Performance Bounds

- **Random distribution of interferers on a plane**
- **Spatial interference mitigation**
- **Propagation**
  - **Power law attenuation,** $P \propto r^{-\alpha}$
  - **Rayleigh fading**
- **High interference, low noise regime**



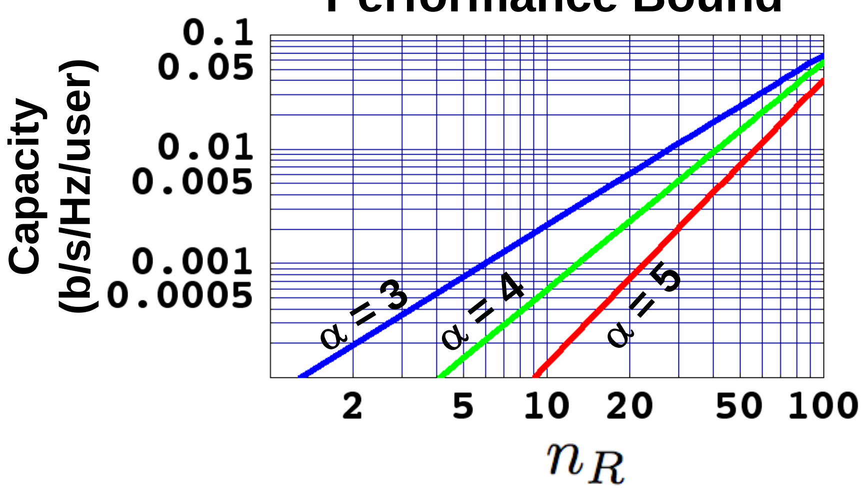
Number of Receive Antennas

$$C \approx \log_2\left(1 + \left[\frac{n_R}{r_1^2\,\rho}\,\frac{\alpha\sin(\frac{2\pi}{\alpha})}{2\pi^2}\right]^{\alpha/2}\right)$$

Link Length

Interference Density

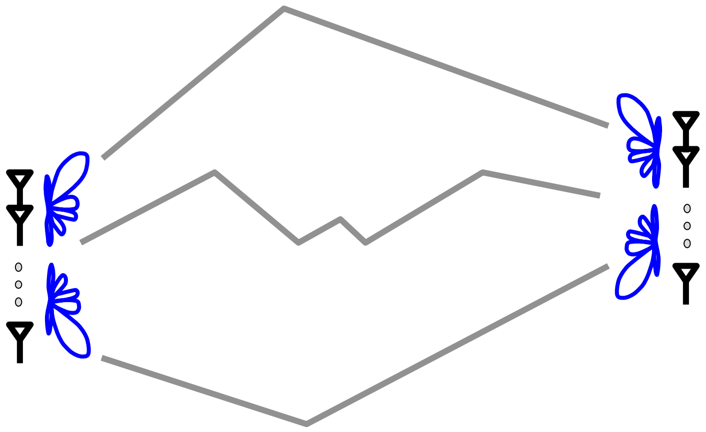
Performance Bound



S. Govindasamy, et al, "The performance of linear multiple-antenna receivers with interferers distributed on a plain," IEEE Workshop on SWAWC, June, 2005

Topics

- Multiple-Input Multiple-Output (MIMO)
  - Introduction
  - Asymptotic results
- Survey of Communication Examples
  - Code-Division Multiple Access (CDMA)
  - Ultra-Wideband (UWB)
  - Receiver performance in rich interference networks
- **Wish List**



Wish List

What I would like for the holidays

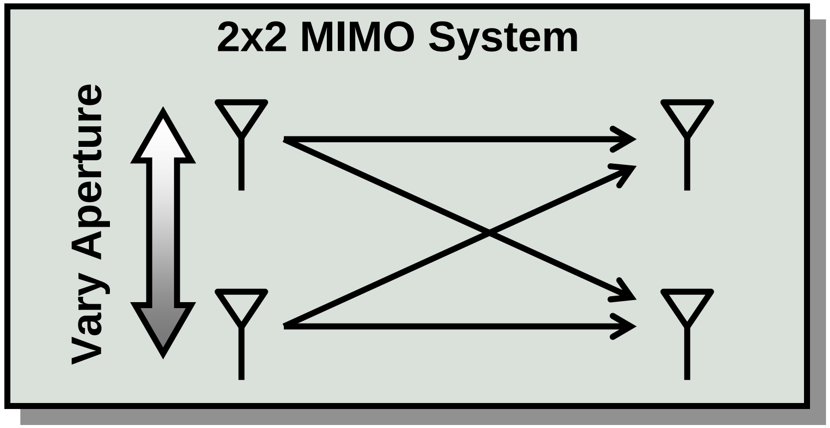
- **Densities for functions of random complex matrices**
  - **Finite or infinite**

$$f(\mathbf{X}\mathbf{X}^\dagger, \mathbf{Y}\mathbf{Y}^\dagger, \cdots)$$

- **Ratios, Products, Sums, Differences, Det's, Traces**
  - **Signal in the mean**
  - **Colored matrices**
- **Make it easy for me…**

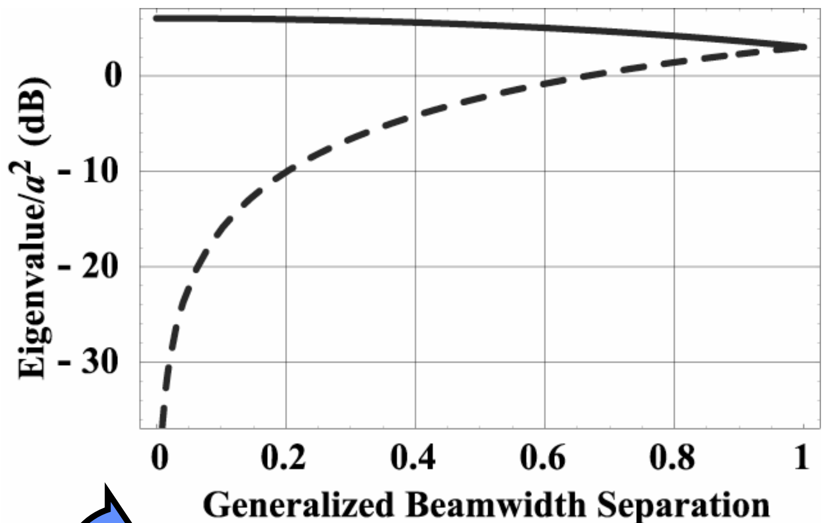
Backup Slides

The Channel Matrix
A Toy Model



- Toy MIMO channel model
  - 2x2
  - line of sight
- Resolving individual antennas increases eigenvalue
- MIMO systems in real environments employ scatterers to increase effective aperture

Eigenvalues of Channel Matrix



$$b = \frac{2}{\pi} \arccos \frac{\left\| \vec{h}_1^{\dagger} \vec{h}_2 \right\|}{\left\| \vec{h}_1 \right\| \left\| \vec{h}_2 \right\|}$$

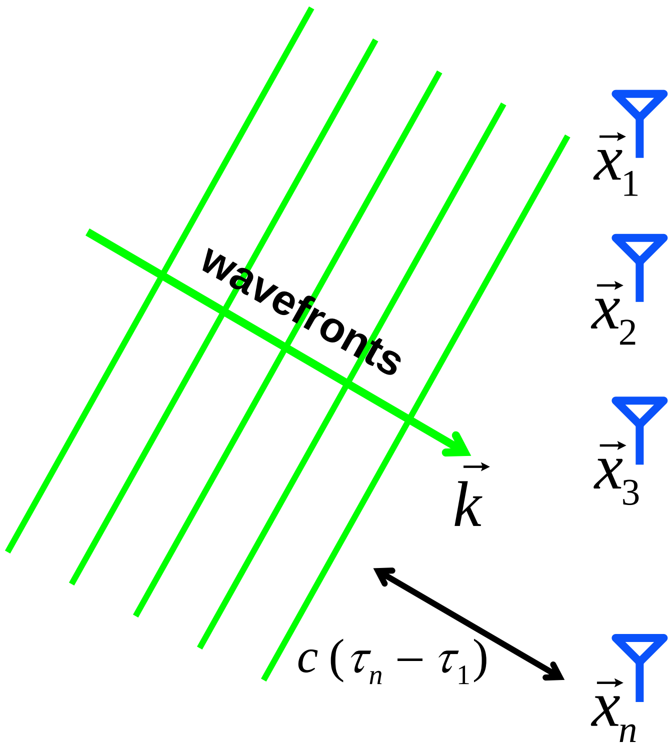
Channel Matrix, $\mathbf{H} = \left(\vec{h}_1 \;\; \vec{h}_2 \right)$

$$= 2\, a \left(\vec{v}_1 \;\; \vec{v}_2 \right)$$

Unit norm steering vector

Beamforming

- **Phase multiple antenna elements to control direction of transmitted or received power**
- **Narrowband model - can't resolve antennas,** $B \ll \dfrac{c}{\|\vec{x}_n - \vec{x}_1\|}$



$$\to s(t-\tau_1) \approx a\,u(t)\,e^{i\omega t - i\vec{k}\cdot\vec{x}_1}$$

$$\to s(t-\tau_2) \approx a\,u(t)\,e^{i\omega t - i\vec{k}\cdot\vec{x}_2}$$

$$\to s(t-\tau_3) \approx a\,u(t)\,e^{i\omega t - i\vec{k}\cdot\vec{x}_3}$$

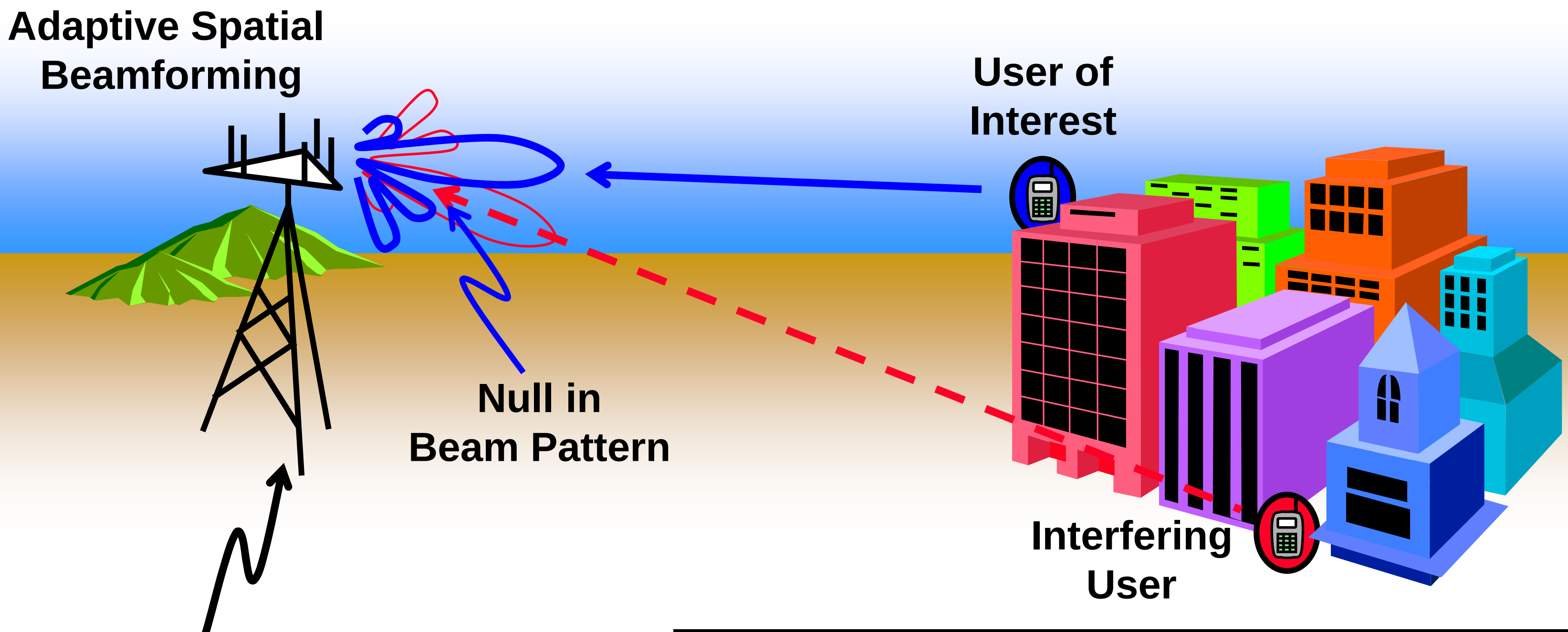
$$\vdots$$

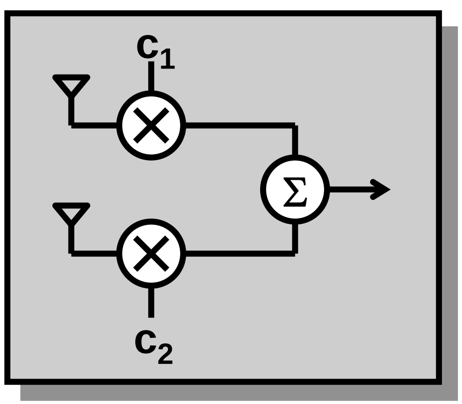
$$\to s(t-\tau_n) \approx a\,u(t)\,e^{i\omega t - i\vec{k}\cdot\vec{x}_n}$$

$$= a\,u(t)\,e^{i\omega t}\,\sqrt{n}\;\underbrace{\boxed{\vec{v}(\vec{k};\vec{x}_1,\cdots,\vec{x}_n)}}_{\text{Steering Vector}}$$

Multi-Antenna Processing

Simple Example





How it works:

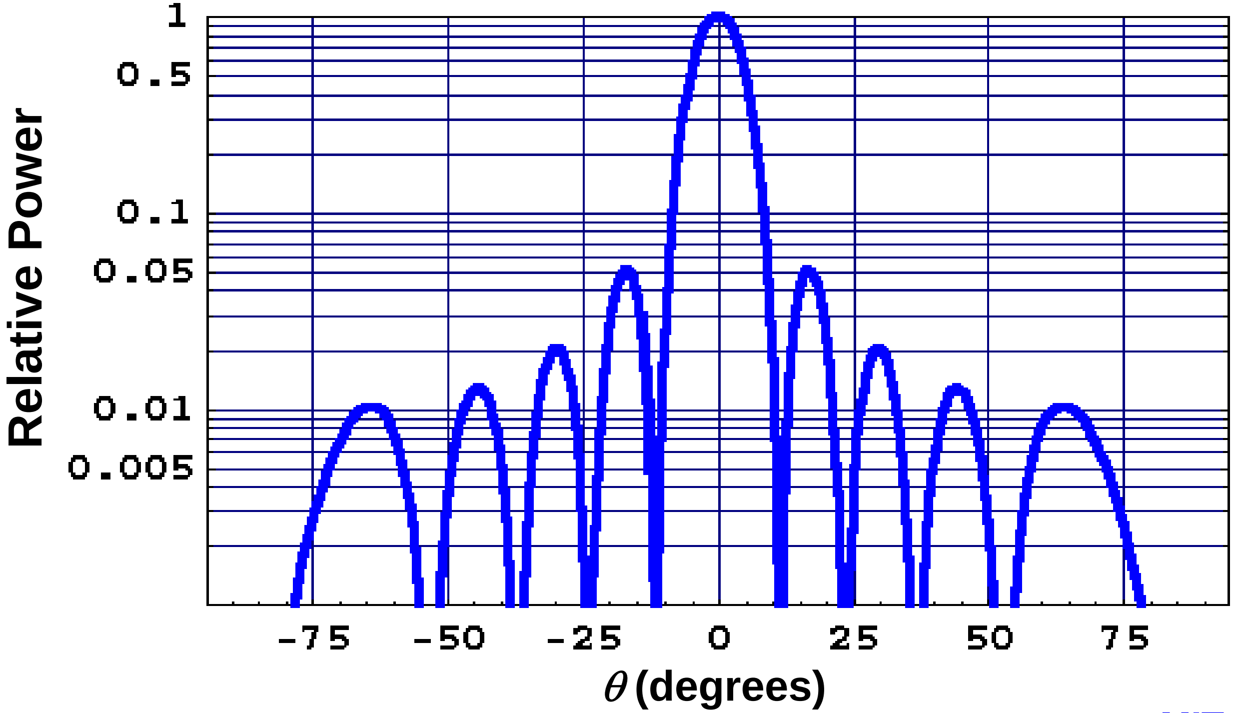
- Adapt antenna array beam pattern
- Distinct beam pattern for each user
- Done simultaneously for all users
- Spatial equivalent to adaptive filtering in frequency domain

Beamforming Pattern
(Receive or Transmit)

- **Phase multiple antenna elements to control direction of transmitted or received power**
- **Narrowband model**
- ***n* = 10**

Steering Vector

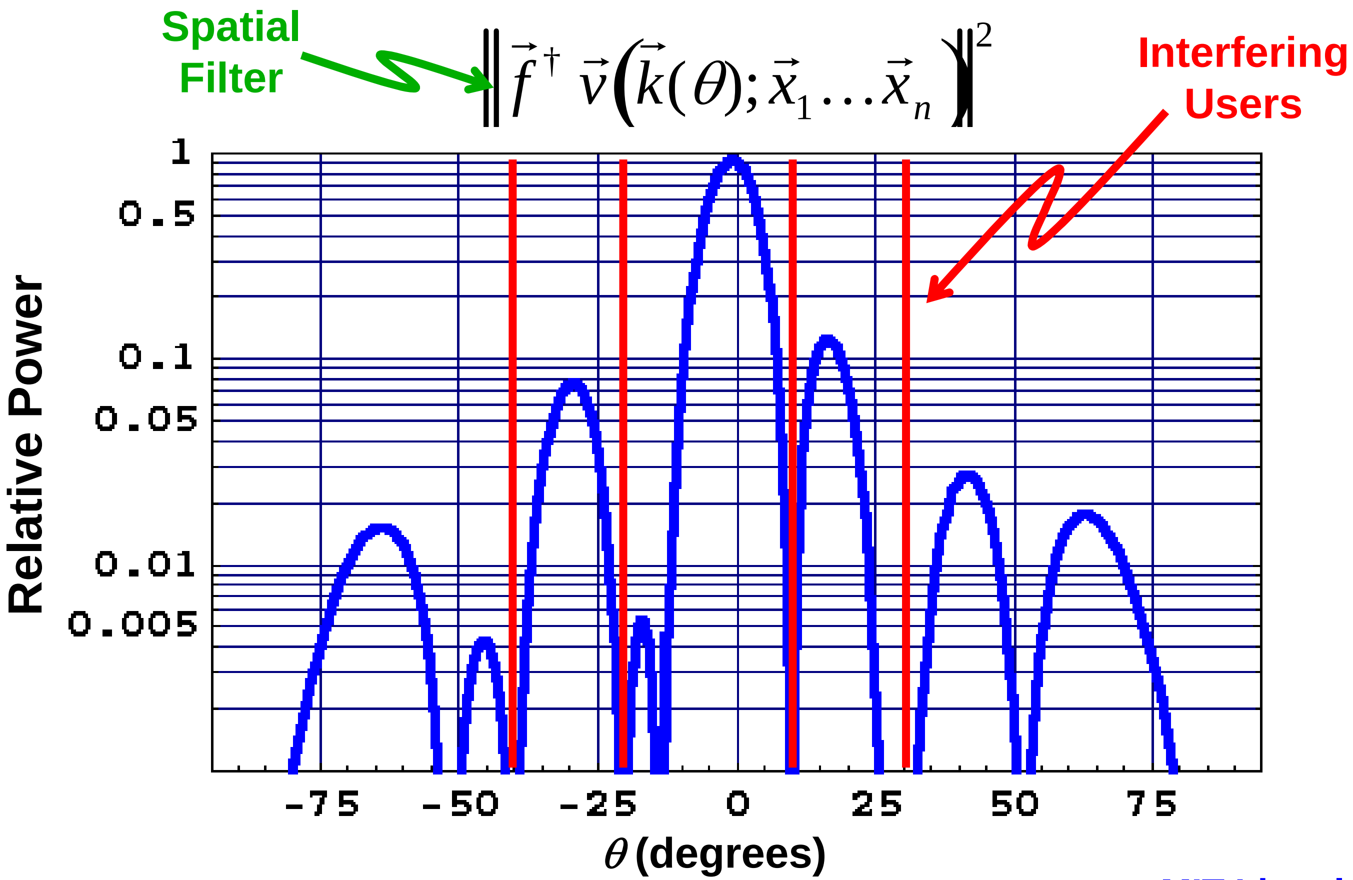
$$\left\| \vec{v}^{\dagger}\left(\vec{k}(\theta = 0); \vec{x}_1 \ldots \vec{x}_n\right) \vec{v}\left(\vec{k}(\theta); \vec{x}_1 \ldots \vec{x}_n\right) \right\|^2$$



Minimize Mean Square Error Example

- 10 antenna array
- 4 interfering users (10,-20,30,-40 degrees)
- Signal of interest at 0 degrees

$$\left\| \vec{f}^{\dagger} \, \vec{v}\left(\vec{k}(\theta); \vec{x}_1 \ldots \vec{x}_n\right) \right\|^2$$



Minimize Mean Square Error Example
Moving Interferer

- **10 antenna array**
- **1 interfering users**
- **Signal of interest at 0 degrees**

Interfering User

$$\left\| \vec{f}^{\,\dagger}\, \vec{v}\left(\vec{k}(\theta); \vec{x}_1 \ldots \vec{x}_n\right) \right\|^2$$

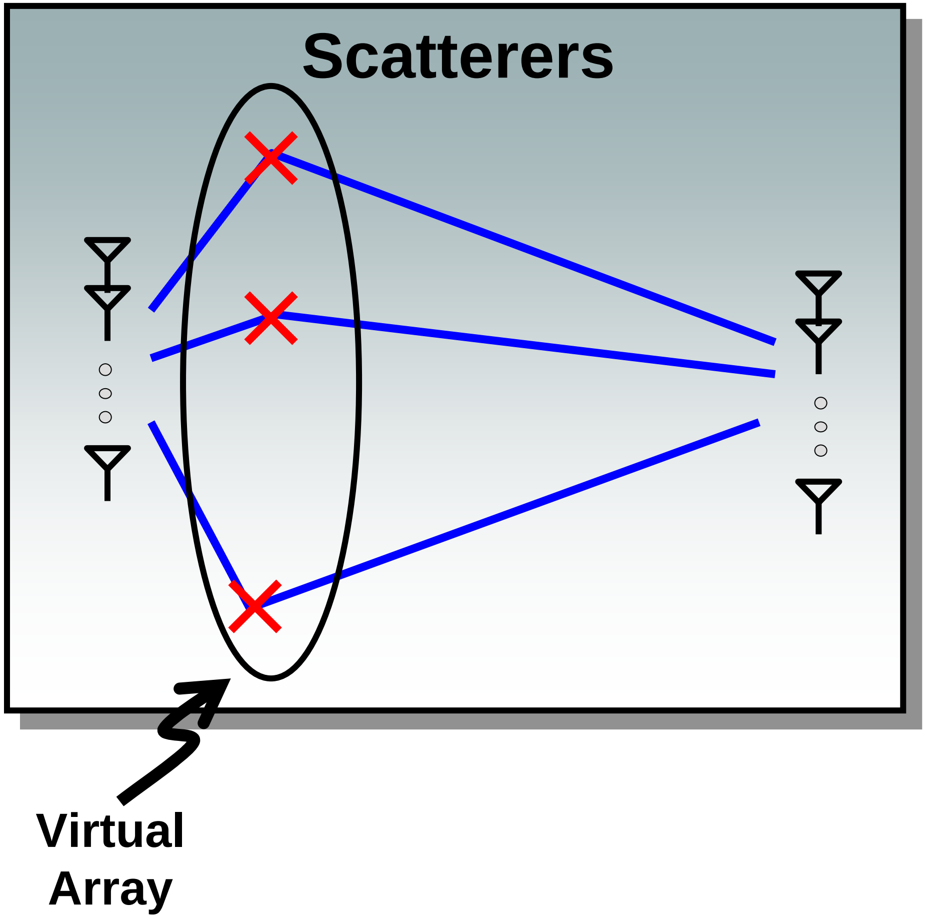
Relative Power

QuickTime™ and a Animation decompressor are needed to see this picture.

θ (degrees)

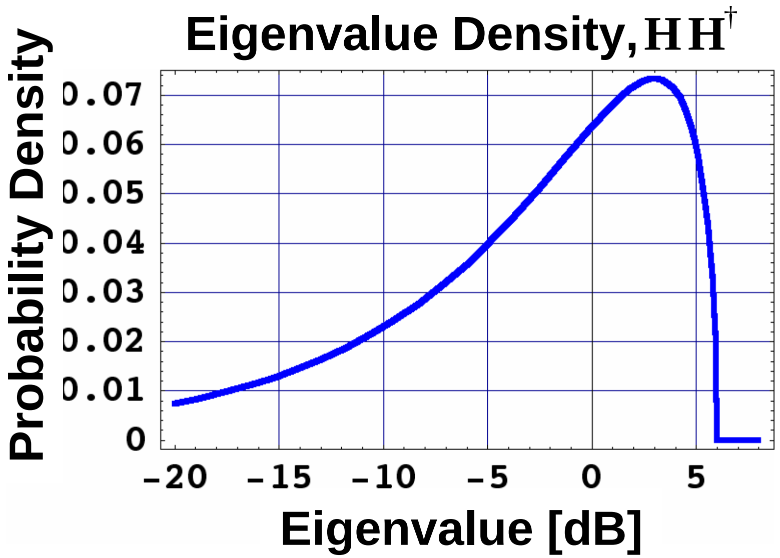
Channel Complexity

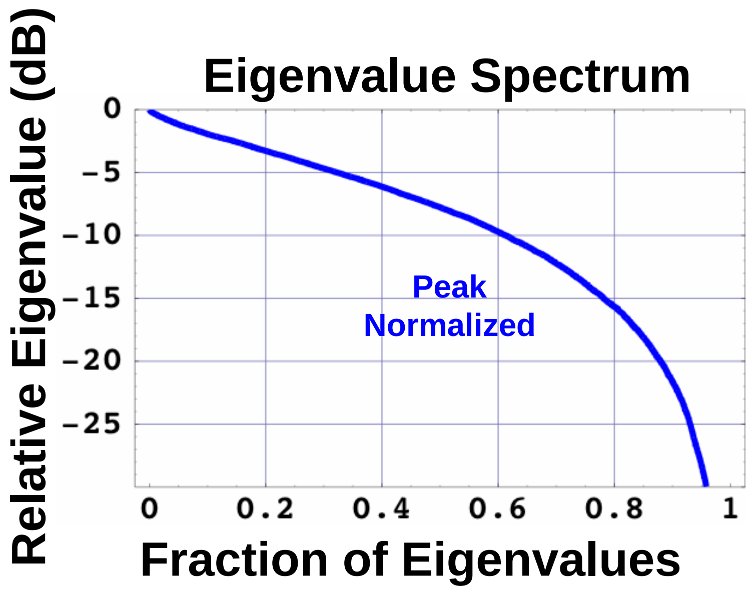
- Scatterers act as virtual elements in antenna array
- Channel matrix complexity increases with effective virtual aperture
- Virtual scattering array resolves antenna of other antenna array



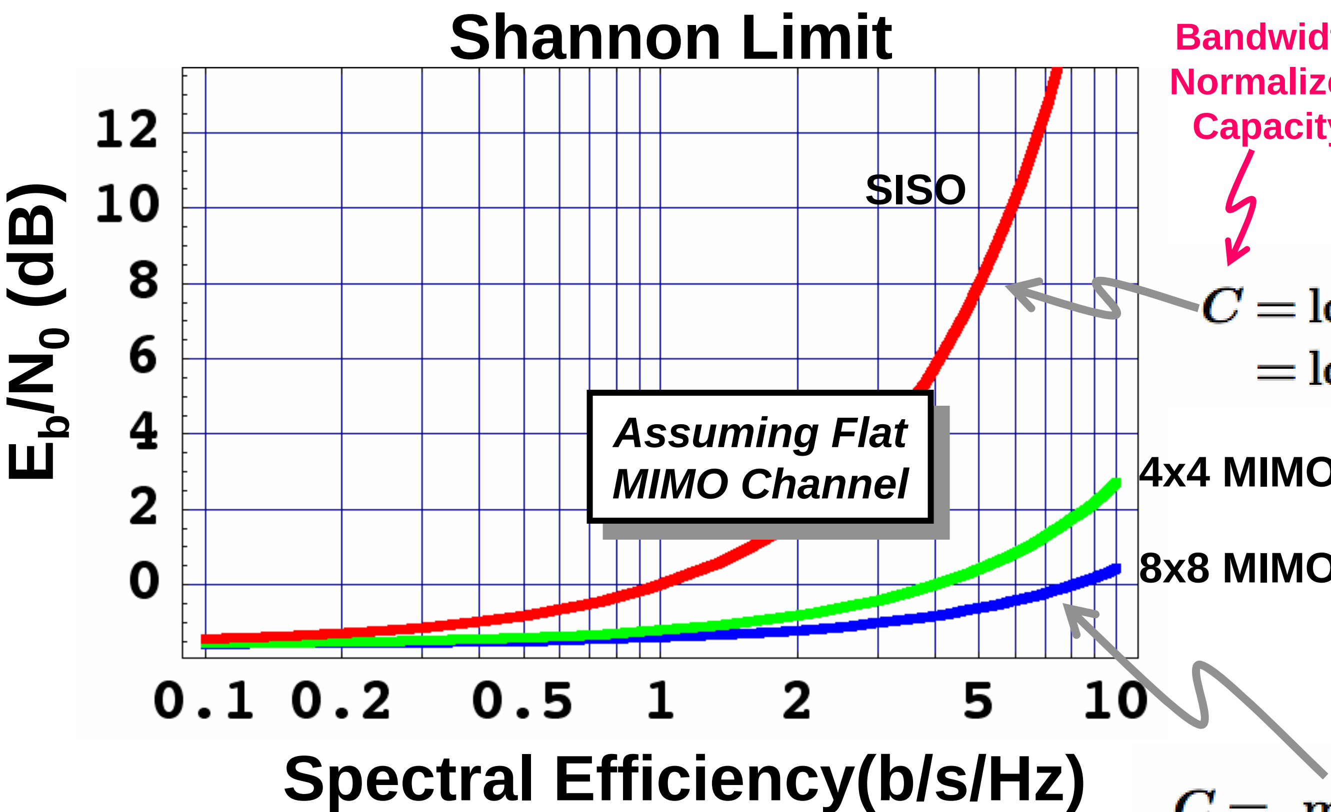
The Rich Scattering Channel
Large Dimension Limit

- **Eigenvalue probability density of $\mathbf{HH}^\dagger = a^2\mathbf{GG}^\dagger$**
- **Entries in $\mathbf{G}$ are drawn from unit norm complex Gaussian** $(1/n_{Tx})\ \mathbf{GG}^\dagger$
- **Eigenvalues probability density of**
  $$f_{r=1}(x)\,dx = \frac{\sqrt{x\,(1-x)}}{\pi\ x}\,dx$$
- **PDF dependent upon,** $r = n_{Rx} / n_{Tx}$





MIMO Capacity Bound



Bandwidth Normalized Capacity

Channel Loss

Transmit Power (noise normalized)

$$C = \log_2(1 + \|a\|^2 P_o) = \log_2(1 + SNR)$$

Transmit Power Matrix

Channel Matrix

$$C = \max_{\mathrm{tr}\,\mathrm{P} = P_o} \log_2 \left| \mathrm{I} + \mathrm{HPH}^\dagger \right|$$

Determinant

- MIMO bound follows different theoretical limit
- Divide total energy amongst transmitters avoiding compressive regime of SISO Shannon limit

Uninformed Transmitter Redux

Re-Express bound for uncooperative interference

$$\tilde{C}_{UT,Un} = \log_2 \left| \mathbf{I} + \frac{P_o}{n_{Tx}} \tilde{\mathbf{H}}\tilde{\mathbf{H}}^\dagger \right|$$

$$= \log_2 \left| \mathbf{R} + \frac{P_o}{n_{Tx}} \mathbf{H}\,\mathbf{H}^\dagger \right| - \log_2 |\mathbf{R}|$$

$$= \log_2 \left| \mathbf{I} + \frac{P_{\text{int}}}{n_{\text{int}}} \mathbf{J}\,\mathbf{J}^\dagger + \frac{P_o}{n_{Tx}} \mathbf{H}\mathbf{H}^\dagger \right| - \log_2 \left| \mathbf{I} + \frac{P_{\text{int}}}{n_{\text{int}}} \mathbf{J}\,\mathbf{J}^\dagger \right|$$

Interference plus noise can written in the form

$$\mathbf{R} = \mathbf{I} + \frac{P_{\text{int}}}{n_{\text{int}}} \mathbf{J}\,\mathbf{J}^\dagger = \mathbf{I} + \frac{a_{\text{int}}^2 P_{\text{int}}}{n_{\text{int}}} \Gamma\,\Gamma^\dagger$$

Assume same equal average receive power per antenna

$$\frac{a^2 P_o}{n_{Tx}} = \frac{a_{\text{int}}^2 P_{\text{int}}}{n_{\text{int}}}$$

$$\tilde{C}_{UT,Un,Eq} = \log_2 \left| \mathbf{I} + \frac{a^2 P_o}{n_{Tx}} \mathbf{A}\,\mathbf{A}^\dagger \right| - \log_2 \left| \mathbf{I} + \frac{a^2 P_o}{n_{Tx}} \Gamma\,\Gamma^\dagger \right| \quad ; \quad \mathbf{A} = \left(\mathbf{G} \;\; \Gamma \right)$$

Uncooperative External Interference

Continuous Uninformed Transmitter

- **Continuous form can be expressed as the difference of two UT bounds**
- **Assumptions**
  - **Interference has same channel statistics**
  - **Interference has same power per transmit antenna**

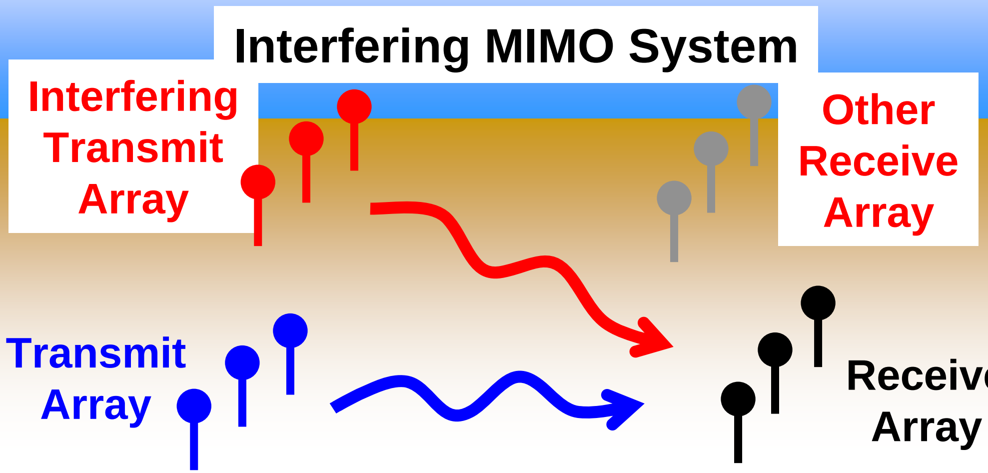
$$\tilde{C}_{UT,Un,Eq} = \log_2 \left| \mathbf{I} + \frac{a^2 P_o}{n_{Tx}} \mathbf{A}\mathbf{A}^\dagger \right| - \log_2 \left| \mathbf{I} + \frac{a^2 P_o}{n_{Tx}} \Gamma\Gamma^\dagger \right|$$

Continuous approximation

$$\approx n_{Rx} \left\{ \Phi\left(a^2 P_o \frac{n_{Tx} + n_{\text{int}}}{n_{Tx}} ; \frac{n_{Rx}}{n_{Tx} + n_{\text{int}}} \right) - \Phi\left(a^2 P_o \frac{n_{\text{int}}}{n_{Tx}} ; \frac{n_{Rx}}{n_{\text{int}}} \right) \right\}$$

Cooperative Interference
Other MIMO Systems

- Cooperative interference affects performance less
- Receiver jointly demodulates signals from both interfering MIMO system and system of interest
  - Discards information from interfering MIMO system



Cooperative External Interference

Continuous Uninformed Transmitter

- **Interfering and intended MIMO systems allowed to coordinate**
- **Assumptions**
  - **Interference has same channel statistics**
  - **Interference has same power per transmit antenna**

$$\tilde{C}_{UT,Co} = \underbrace{\gamma}_{\text{Fraction of data associated with system of interest}} \log_2 \left| \mathbf{I} + \frac{P_o}{n_{Tx}} \mathbf{H}\,\mathbf{H}^\dagger + \frac{P_{\text{int}}}{n_{\text{int}}} \underbrace{\mathbf{J}}_{\text{Interfering Channel Matrix}} \mathbf{J}^\dagger \right|$$

$$\tilde{C}_{UT,Co,Eq} = \frac{n_{Tx}}{n_{Tx} + n_{\text{int}}} \log_2 \left| \mathbf{I} + \frac{a^2 P_o}{n_{Tx}} (\mathbf{G} \quad \Gamma)(\mathbf{G} \quad \Gamma)^\dagger \right| \quad ; \quad \frac{a^2 P_o}{n_{Tx}} = \frac{a_{\text{int}}^2 P_{\text{int}}}{n_{\text{int}}}$$

Continuous approximation

$$\approx \frac{n_{Rx}\; n_{Tx}}{n_{Tx} + n_{\text{int}}} \; \Phi\left(a^2 P_o \frac{n_{Tx} + n_{\text{int}}}{n_{Tx}} \; ; \; \frac{n_{Rx}}{n_{Tx} + n_{\text{int}}} \right)$$