Random Lévy Matrices

Not everything is Gaussian !!!

M.A. Nowak, J. Jurkiewicz (Jagellonian University)

G. Papp (Eötvös Loránd University)

I. Zahed (Stoony Brook, SUNY)

Not everything is Gaussian !!!

M.A. Nowak, J. Jurkiewicz (Jagellonian University)

G. Papp (Eötvös Loránd University)

I. Zahed (Stoony Brook, SUNY)

Outline:

- stable distributions (classical probability)
- large random matrices (symmetric real)
- Gaussian domain of attraction (GOE, Wigner)
- Lévy's domain of attraction
  - free Lévy random variables
  - Wigner-Lévy ensemble
- summary

Laws of addition in classical probability

Let x_1, x_2 be independent random variables with pdfs: $p_1(x), p_2(x)$

What is pdf for: $x_{1+2} = x_1 + x_2$?

$p_{1+2}(x) = \int \mathrm{d}x_1 p_1(x_1) p_2(x - x_1)$

Laws of addition in classical probability

Let x_1, x_2 be independent random variables with pdfs: $p_1(x), p_2(x)$

What is pdf for: $x_{1+2} = x_1 + x_2$?

$$p_{1+2}(x) = \int \mathrm{d}x_1 p_1(x_1) p_2(x - x_1)$$

Characteristic function:

$$\phi(k) = \int \mathrm{d}x \; p(x) e^{ikx}$$

$$\phi_{1+2}(k) = \phi_1(k) \cdot \phi_2(k)$$

Laws of addition in classical probability

Let x_1, x_2 be independent random variables with pdfs: $p_1(x), p_2(x)$

What is pdf for: $x_{1+2} = x_1 + x_2$?

$$p_{1+2}(x) = \int \mathrm{d}x_1 p_1(x_1) p_2(x - x_1)$$

Characteristic function:

$$\phi(k) = \int \mathrm{d}x \; p(x) e^{ikx}$$

$$\phi_{1+2}(k) = \phi_1(k) \cdot \phi_2(k)$$

Cumulants' generating function:

$$c(k) = \log(\phi(k))$$

$$c_{1+2}(k) = c_1(k) + c_2(k)$$

Normal distribution:

$$x \sim N(0, \sigma^2) \ , \quad \phi(k) = \exp -\tfrac{1}{2}\sigma^2 k^2 \ , \quad c(k) = -\frac{\sigma^2 k^2}{2}$$

Normal distribution:

$$x \sim N(0, \sigma^2) \ , \quad \phi(k) = \exp -\tfrac{1}{2}\sigma^2 k^2 \ , \quad c(k) = -\frac{\sigma^2 k^2}{2}$$

Sum of independent normal variables

$$x_1 \sim N(0, \sigma_1^2) \ , \ x_2 \sim N(0, \sigma_2^2) \implies x_{1+2} \sim N(0, \sigma_1^2 + \sigma_2^2)$$

Normal distribution:

$$x \sim N(0, \sigma^2) \ , \quad \phi(k) = \exp -\tfrac{1}{2}\sigma^2 k^2 \ , \quad c(k) = -\frac{\sigma^2 k^2}{2}$$

Sum of independent normal variables

$$x_1 \sim N(0, \sigma_1^2) \ , \ x_2 \sim N(0, \sigma_2^2) \implies x_{1+2} \sim N(0, \sigma_1^2 + \sigma_2^2)$$

Sum of iid normal variables

$$\sigma_1^2 = \sigma_2^2 \equiv \sigma^2$$

$$\frac{x_{1+2}}{\sqrt{2}} \sim N(0, \sigma^2)$$

Stability

x_j iid $N(0, \sigma^2)$

$X = \frac{x_1 + \cdots + x_n}{\sqrt{n}} \sim N(0, \sigma^2)$ (fixed point)

Stable laws are important !!

Stability

x_j iid $N(0,\sigma^2)$

$X = \frac{x_1+\cdots+x_n}{\sqrt{n}} \sim N(0,\sigma^2)$ (fixed point)

Stable laws are important !!

Let x_j be iid and $E(x)=0, \mathrm{var}(x)=\sigma^2$

$X = \frac{x_1+\cdots+x_n}{\sqrt{n}} \stackrel{n\to\infty}{\Longrightarrow} C(k) = nc\left(\frac{k}{\sqrt{n}}\right) \to -\frac{\sigma^2 k^2}{2}$

$X \sim N(0,\sigma^2)$ (Gaussian domain of attraction)

Universality !!

Lévy domain of attraction

Heavy tails in pdf $p(x) \sim \frac{A}{|x|^{1+\mu}}$ for $x \to \pm\infty$

Lévy stable law for addition of iid: $X = \frac{x_1 + \cdots + x_n}{n^{1/\mu}}$

$$L_\mu(x) = \frac{1}{2\pi} \int dk \hat{L}(k) e^{ikx}$$

$$c(k) = \log \hat{L}(k) = -\gamma^\mu |k|^\mu \ , \mu \in (0, 2]$$

$$c_{1+2}(k) = c_1(k) + c_2(k) \implies \gamma^\mu_{1+2} = \gamma^\mu_1 + \gamma^\mu_2$$

Lévy domain of attraction

Heavy tails in pdf $p(x) \sim \frac{A}{|x|^{1+\mu}}$ for $x \to \pm\infty$

Lévy stable law for addition of iid: $X = \frac{x_1 + \cdots + x_n}{n^{1/\mu}}$

$$L_\mu(x) = \frac{1}{2\pi} \int dk \hat{L}(k) e^{ikx}$$

$$c(k) = \log \hat{L}(k) = -\gamma^\mu |k|^\mu \;, \mu \in (0, 2]$$

$$c_{1+2}(k) = c_1(k) + c_2(k) \implies \gamma_{1+2}^\mu = \gamma_1^\mu + \gamma_2^\mu$$

Assymetry:

$p(x) \sim \frac{A_\pm}{|x|^{1+\mu}}$ for $x \to \pm\infty$

$$\beta = \frac{A_+ - A_-}{A_+ + A_-} \quad \beta \in [-1, 1]$$

$$L_\mu^{\gamma,\beta}(x) = \frac{1}{2\pi} \int dk \hat{L}(k) e^{ikx}$$

$$\log \hat{L}(k) = -\gamma^\mu |k|^\mu (1 + i\beta \,\mathrm{sgn}(k)\, \tan(\pi\mu/2))$$

Summary 1/3

- Law of addition in classical probability $c_{1+2}(k) = c_1(k) + c_2(k)$
- Importance of stable distributions: domains of attractions
- Classification of stable distributions: $L_\mu^{\gamma,\,\beta}(x)$

Random Matrices

$N \times N$ real symmetric matrices H, $N \to \infty$

GOE

Measure: $DH \exp\left(-\frac{N}{2\sigma^2}\mathrm{tr}H^2\right)$ where $DH = \prod_i dH_{ii} \prod_{i>j} dH_{ij}$

Orthogonal invariance: $H \to OHO^\tau$

Diagonal decomposition: $H = O\,\mathrm{diag}[\lambda_1, \ldots, \lambda_N]\,O^\tau$

Spectral density function (sdf) for $N \to \infty$:

$\rho(\lambda) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - \lambda^2}$

Product measure

GOE: $H_{ij} \sim N\left(0, \frac{\sigma^2}{2N}(1 + \delta_{ij})\right)$

Slightly modified ensemble: $H_{ij} \sim N\left(0, \frac{\sigma^2}{2N}\right)$

Product measure

GOE: $H_{ij} \sim N\left(0, \frac{\sigma^2}{2N}(1+\delta_{ij})\right)$

Slightly modified ensemble: $H_{ij} \sim N\left(0, \frac{\sigma^2}{2N}\right)$

Wigner ensemble

H_{ij} are iid:

Measure: $\prod_{i \geq j} dH_{ij} p(H_{ij}),\qquad E(H_{ij}) = 0\ ,\ \mathrm{var}(H_{ij}) = \frac{\sigma^2}{2N}$

sdf for $N \longrightarrow \infty$

$\rho(\lambda) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - \lambda^2}$

Summary 2/3

Two classes of matrices in the Gaussian domain of attraction:

- $H = O\,\mathrm{diag}[\lambda_1, \ldots, \lambda_N]\,O^\tau$,
- H_{ij} iid: $E(H_{ij}) = 0$, $\mathrm{var}(H_{ij}) = \frac{\sigma^2}{2N}$

Summary 2/3

Two classes of matrices in the Gaussian domain of attraction:

- $H = O \, \mathrm{diag}[\lambda_1, \ldots, \lambda_N] \, O^\tau$
- H_{ij} iid: $E(H_{ij}) = 0 \ , \ \mathrm{var}(H_{ij}) = \frac{\sigma^2}{2N}$

Remark:

$\mathcal{H} = \frac{H_1 + \cdots + H_n}{\sqrt{n}}$

The two classes have the same limit !!

Lévy matrices

Class 1: Orthogonally invariant (heavy-tailed) matrices:

$H = O\,\mathrm{diag}[\lambda_1, \ldots, \lambda_N]\,O^\tau$, sdf $\rho(\lambda) \sim \frac{1}{\lambda^{\mu+1}}$ $\quad 0 < \mu < 2$

Class 2: Wigner-Lévy

H_{ij} iid , pdf $p(H_{ij}) \sim \frac{1}{H_{ij}^{\mu+1}}$ $\quad 0 < \mu < 2$

$$\mathcal{H} = \frac{H_1 + \cdots + H_n}{n^{1/\mu}}$$

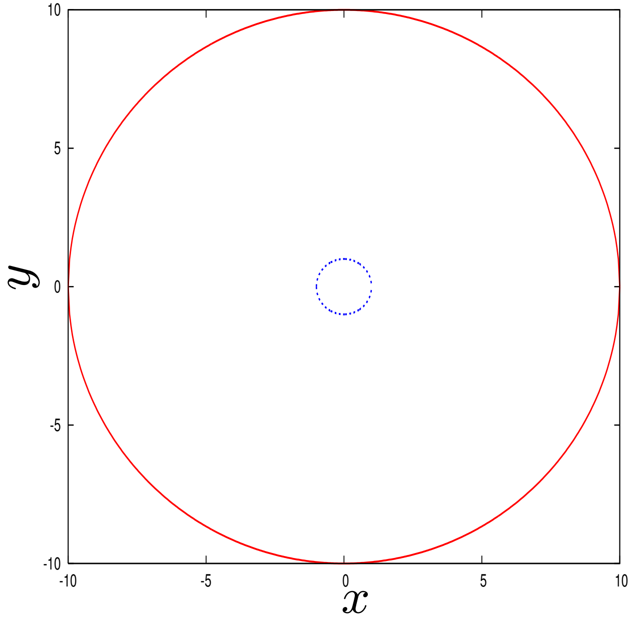
Class 1: $\mathcal{H}$ free random matrices (ortogonally invariant) with stable sdf

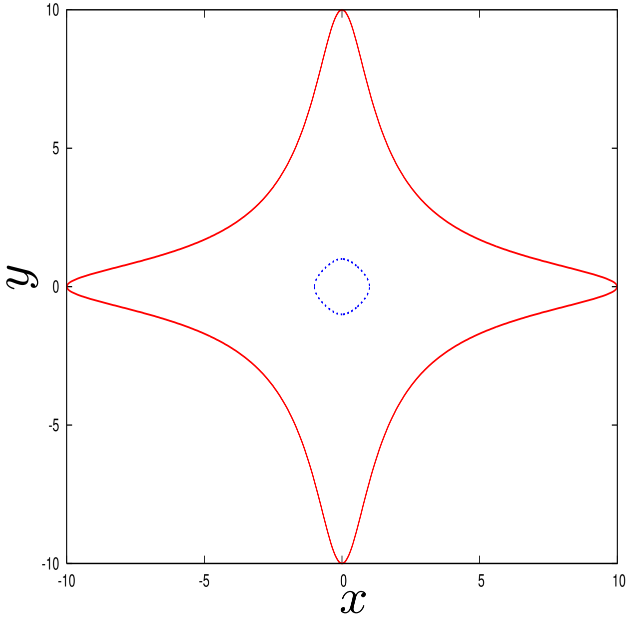
Class 2: $\mathcal{H}$ pdf $p(H_{ij}) \to L_\mu^{\gamma,\beta}(H_{ij})$

Simple example for 2d-random vectors:

pdfs for (x, y)

$$p_1(x, y) \sim e^{-x^2} e^{-y^2} \qquad p_2(x, y) \sim \frac{1}{1+x^2} \frac{1}{1+y^2}$$





$$p(x, y) = \text{const}$$

Class 1 = free random variables

Free probability (Voiculescu)

Assymptotic freeness of large matrices from class 1 (Speicher)

Classical probability

$p(x)$ - pdf

$\phi(k) = \int \mathrm{d}x p(x) e^{+ikx}$

$c(k) = \log \phi(k)$

$c_{1+2}(k) = c_1(k) + c_2(k)$

$\phi(k) = \exp c(k)$

$p(x) = \frac{1}{2\pi} \int \mathrm{d}k \phi(k) e^{-ikx}$

Free probability

$\rho(\lambda)$ - sdf

$G(z) = \int \frac{\rho(\lambda)\mathrm{d}\lambda}{\lambda - z}$

$G(R(z) + \frac{1}{z}) = z$

$R_{1+2}(z) = R_1(z) + R_2(z)$

$R(G(z)) + \frac{1}{G(z)} = z$

$\rho(\lambda) = -\frac{1}{\pi} \mathrm{Im}\, G(\lambda + i0^+)$

Free lunch

Bercovici-Pata bijection between classical and free random variable

Stable laws in free probability (Bercovici-Voiculescu)

$$R(z) = 0$$

$$R(z) = bz^{\mu-1}$$

$$b = \begin{cases} \gamma^{\mu}\; e^{i(\frac{\mu}{2}-1)(1+\beta)\pi} & \text{for} \quad 1 < \mu \le 2 \\ \gamma^{\mu}\; e^{i[\pi+\frac{\mu}{2}(1+\beta)\pi]} & \text{for} \quad 0 < \mu < 1 \end{cases} .$$

$$R(z) = -i\gamma(1+\beta) - \frac{2\beta\gamma}{\pi}\ln \gamma z \qquad \text{for} \;\; \mu = 1$$

Two examples

$R(G(z)) + \frac{1}{G(z)} = z$

Cauchy: $\mu = 1, \beta = 0 \;\longrightarrow\; R(z) = -i\gamma$

$-i\gamma + \frac{1}{G(z)} = z$

$G(z) = \frac{1}{z+i\gamma} \;\longrightarrow\; \rho(\lambda) = \frac{1}{\pi}\frac{\gamma}{\lambda^2+\gamma^2}$

Gauss: $\mu = 2 \;\longrightarrow\; R(z) = \gamma^2 z$

$\gamma^2 G(z) + \frac{1}{G(z)} = z$

$G(z) = \frac{z-\sqrt{z^2-4\gamma^2}}{2\gamma^2} \;\longrightarrow\; \rho(\lambda) = \frac{1}{2\pi\gamma^2}\sqrt{4\gamma^2-\lambda^2}$

analytically solvable for $\mu = 1/4, 1/3, 1/2, 2/3, 3/4, 4/3, 3/2$

Class 2 - Wigner-Lévy

Bouchaud-Cizeau, (tour de force !!)

ZB, M.A.Nowak, J.Jurkiewicz, G. Papp, I. Zahed

Class 2 - Wigner-Lévy

Bouchaud-Cizeau, (tour de force !!)

ZB, M.A.Nowak, J.Jurkiewicz, G. Papp, I. Zahed

Solution for $H_{ij} \sim L_{\mu}^{1,\beta}(H_{ij})$

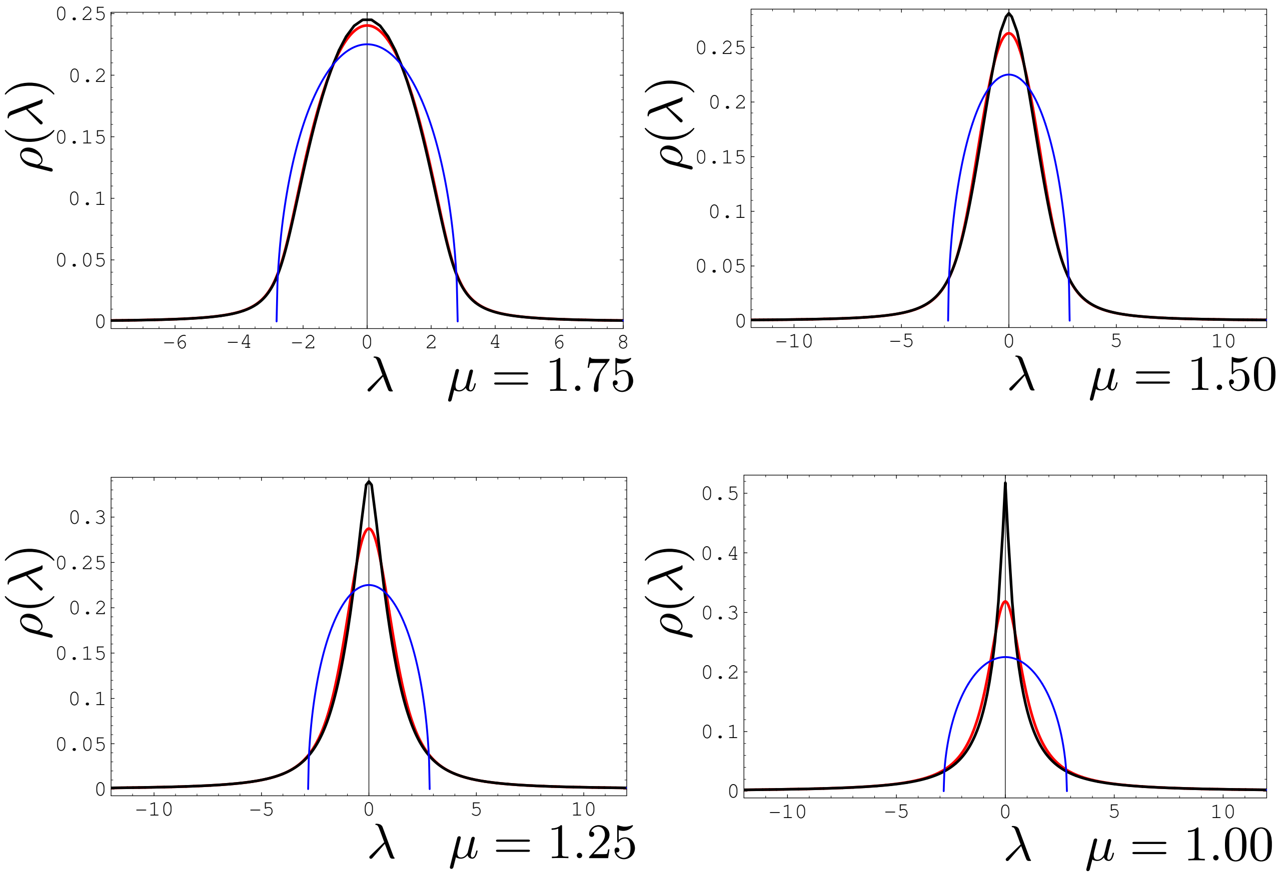
$$\gamma^{\mu/2}(z) = \int_{-\infty}^{+\infty} dx|x|^{-\mu/2} L_{\mu/2}^{\gamma(z),\beta(z)}(z-x)$$

$$\beta(z) = \frac{\int_{-\infty}^{+\infty} dx \ \mathrm{sign}(x)|x|^{-\mu/2} L_{\mu/2}^{\gamma(z),\beta(z)}(z-x)}{\int_{-\infty}^{+\infty} dx|x|^{-\mu/2} L_{\mu/2}^{\gamma(z),\beta(z)}(z-x)}$$

$$\rho(\lambda) = L_{\mu/2}^{\gamma(\lambda),\beta(\lambda)}(\lambda)$$

Numerical comparison:

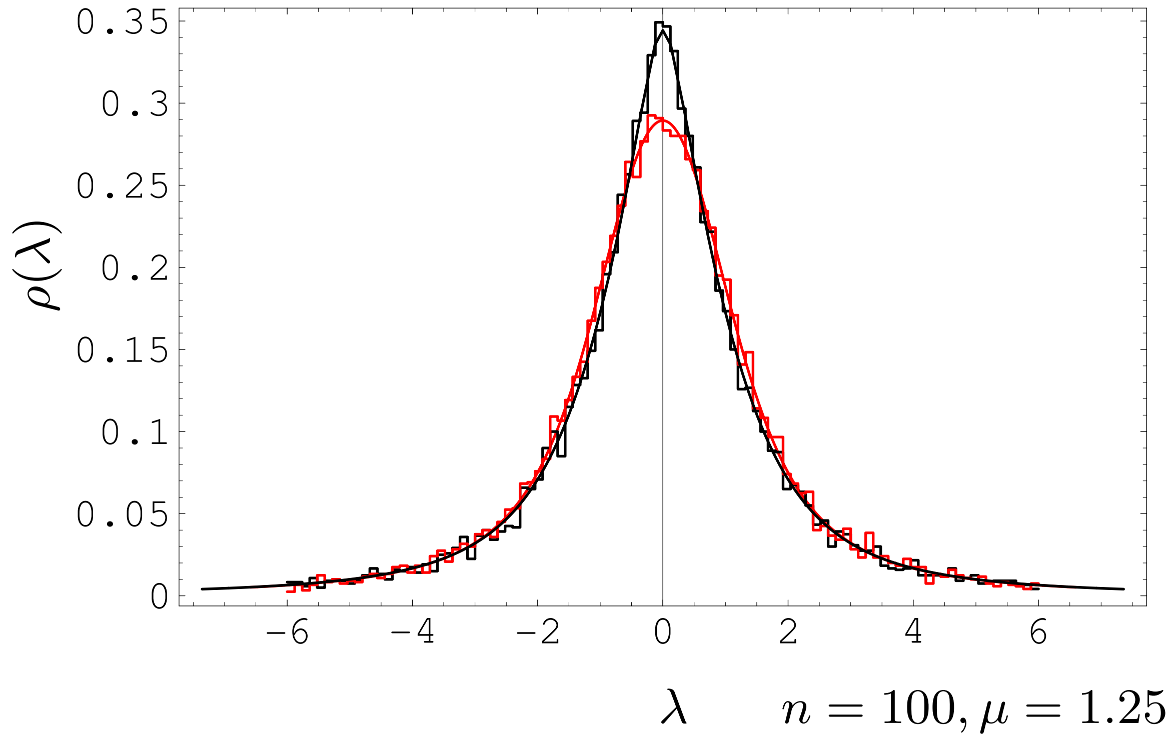
methods: J. P. Nolan + our group



From Wigner-Lévy to free random Lévy

H_i random Wigner-Lévy O_i random orthogonal matrices

$$\mathcal{H} = \frac{1}{n^{1/\mu}} \sum_{i=1}^{n} O_i H_i O_i^{\tau}$$



$n = 100, \mu = 1.25$

Summary 3/3

- Wigner-Lévy and free Lévy ensembles
- spectral denisty can be calculated in both cases
- CLT for randomly rotated Wigner-Lévy give free Lévy

Summary 3/3

- Wigner-Lévy and free Lévy ensembles
- spectral denisty can be calculated in both cases
- CLT for randomly rotated Wigner-Lévy give free Lévy
- Lévy spacing: Wigner surmise for small λ and Poissonian for large (Bouchaud-Cizeau); largest eigenvalue (Soshnikov);
- *Random Levy Matrices Revisited* ZB, J.Jurkiewicz, M.A. Nowak, G. Papp, I. Zahed, **cond-mat/0602087**
- *Applying Free Random Variables . . .* ZB, A. Jarosz, J.Jurkiewicz, M.A. Nowak, G. Papp, I. Zahed, **physics/0603024**
- Challenge: finite size effects for Wigner/Wishart ensembles with pdf$(x) \sim x^{-1-\mu}$ for $\mu > 2$