Asymptotic Statistical Analysis on Special Manifolds
(Stiefel and Grassmann manifolds)

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[I] Stiefel manifold $V_{k,m}$ $(k \leq m)$

$\overset{\text{def}}{\equiv}$ { k - frames in R^m; k - frame = a set of k orthonormal vectors in R^m}

$$\therefore = \{ X(m \times k) \; ; \; X'X = I_k \} \; .$$

Dimension of $V_{k,m} = km - \frac{1}{2}k(k+1)$.

Ex: (i) $O(m) = V_{m,m}$: orthogonal group.

(ii) $V_{1,m}$: unit hypersphere in R^m.

($m=2$ circle, $m=3$ sphere)

Applications in Earth Sciences, Medicine, Astronomy, Meteorology, Biology.

$k=1$: A large literature exists for analysis of directional statistics.

$2 \le k \le m \le 3$: vector cardiogram; orbits of comets.

[II] Grassmann manifold $G_{k,m-k}(k \leq m)$

$\stackrel{\text{def}}{\equiv}$ { k - planes in R^m, i.e., k - dimensional hyperplanes containing the origin in R^m}

To each "k - plane" $\in G_{k,m-k}$, corresponds a unique "$m \times m$ orthogonal projection matrix P idempotent of rank k" $\in P_{k,m-k}$.

$\therefore G_{k,m-k} \equiv P_{k,m-k}$

We carry out statistical analysis on $P_{k,m-k}$.

Ex: $G_{1,m-1}$ ={axes or undirected lines through $\vec{0}$ }

Applications: in the signal processing of radar with m elements observing k targets. A rather new sample space.

We confine our discussion mainly on $V_{k,m}$ in the following.

[III] Population distributions on $V_{k,m}$

1. **Uniform distribution (normalized invariant measure)** $[dX]$

invariant under $X \to H_1 X H'_2$,

$$H_1 \in O(m), \quad H_2 \in O(k).$$

$[dX] = (dX) / \int_{V_{k,m}} (dX)$, where the differential form

$(dX) = \bigwedge_{j=1}^{m-k} \bigwedge_{i=1}^{k} \vec{x}'_{k+j}\, d\vec{x}_i \bigwedge_{i<j}^{k} \vec{x}'_j\, d\vec{x}_i$, for $X = (\overrightarrow{x_1} \cdots \overrightarrow{x_k})$

choosing $\vec{x}_{k+1} \cdots \vec{x}_m$ s.t. $(\vec{x}_1 \cdots \vec{x}_k\ \vec{x}_{k+1} \cdots \vec{x}_m) \in O(m)$,

$\int_{V_{k,m}} (dx) = 2^k \pi^{km/2} / \Gamma_k(m/2)$, with

$$\Gamma_k(a) = \int_{S>0} etr(-S) |S|^{a-(k+1)/2} (dS) = \pi^{k(k-1)/4} \prod_{i=1}^{k} \Gamma(a-(i-1)/2).$$

(multivariate Gamma function)

2. Matrix Langevin $L(m,k;F)$ distribution

pdf $f_x(X) = etr(F'X)/{}_0F_1(m/2; F'F/4)$

for F $m \times k$ matrix, w.r.t. $[dX]$.

$(\because X \underset{d}{\sim} N_{m,k}(M; I_m, \Sigma))$, with $F = M\Sigma^{-1} \mid X'X = I_k$).

${}_0F_1$: hypergeometric function with matrix argument.

Exponential type distribution.

Parametrization of $F = \Gamma\Lambda\Theta'$ (svd), where

$\Gamma \in \tilde{V}_{k,m}$, $\Theta \in O(k)$, and $\Lambda = diag(\lambda_1 \cdots \lambda_k)$,

(with first row positive) $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$.

$$\begin{cases} \text{Mode: } X_0 = \Gamma\Theta'(\in V_{k,m}) & (\because) etr(F'X) = etr(\Lambda\Gamma' X\Theta) \\ \text{Concentrations: } \Lambda & (\because) etr(F'X_0) = etr\Lambda. \end{cases}$$

Assume rank $F = k$, unless otherwise stated.

$F = 0$ (or $\Lambda = 0$): uniform distribution.

Historical background:

$k = 1$: directional disrtibution

$$\begin{cases} m = 2 & \text{von Mises distribution (1918)} \\ m = 3 & \text{Fisher distribution (1953)} \\ m \geq 2 & \text{Langevin (1905)} \end{cases}$$

generalization by Watson & Williams (1956).

Matrix-variate normal $N_{m,k}(M;\Sigma_1,\Sigma_2)$ distribution

$Z(m\times k) \underset{d}{\sim} N_{m,k}(0; I_m, I_k)$ if Z has the pdf

$$\varphi^{(m,k)}(Z) = (2\pi)^{-km/2}\exp[tr(-Z'Z/2)]$$

i.e., all elements of Z are i.i.d. $N(0,1)$.

Let $Y = \Sigma_1^{1/2} Z\,\Sigma_2^{1/2} + M$ [i.e., $Z = \Sigma_1^{-1/2}(Y-M)\Sigma_2^{-1/2}$]

for $M(m\times k)$, $\Sigma_1(m\times m) > 0$, $\Sigma_2(k\times k) > 0$,

then $Y(m\times k) \underset{d}{\sim} N_{m,k}(M;\Sigma_1,\Sigma_2)$.

Hypergeometric functions with symmetric matrix argument.

Starting with ${}_0F_0(S) = etr(S),$ for S $m \times m$ symmetric,

(Laplace transforms)

$$\frac{1}{\Gamma_m(a)} \int_{S>0} etr(-S)\,|S|^{a-(m+1)/2}\,{}_pF_q(a_1 \cdots a_p; b_1 \cdots b_q; YS)(dS)$$

$$= {}_{p+1}F_q(a_1 \cdots a_p a;\ b_1 \cdots b_q;\ Y).$$

$$ {}_pF_q(a_1 \cdots a_p;\ b_1 \cdots b_q;\ S)$$

$$= \sum_{l=0}^{\infty} \sum_{\lambda \vdash l} \frac{(a_1)_\lambda \cdots (a_p)_\lambda}{(b_1)_\lambda \cdots (b_q)_\lambda} \frac{C_\lambda(S)}{l!},$$

where $\lambda = (l_1, \ldots, l_m)$ ordered partition of l,

$$(l_1 \geq \cdots \geq l_m \geq 0, \ \sum_{i=1}^{m} l_i = l)$$

$$(a)_\lambda = \prod_{i=1}^{m} \left(a - \frac{i-1}{2} \right)_{l_i}, \text{ with}$$

$$(a)_l = a(a+1)\cdots(a+l-1), \ (a)_0 = 1,$$

$C_\lambda(S)$: zonal polynomial with $m \times m$

symmetric matrix S.

3. Matrix Bingham $B(m,k;B)$ distribution

pdf $f_x(X) = etr(X'BX) \; / \; {}_1F_1(k/2;\; m/2;\; B)$

for B $m \times m$ symmetric matrix.

$(\because X \underset{d}{\sim} N_{m,k}(0;\Sigma, I_k))$, with $B = -\frac{1}{2}\Sigma^{-1} \mid X'X = I_k)$.

Exponential type distribution.

Parametrization of $B = \Gamma\Lambda\Gamma'$ (sd), where

$\Gamma \in \tilde{O}(m)$, and $\Lambda = diag(\lambda_1 \cdots \lambda_m)$, $\lambda_1 \geq \cdots \geq \lambda_m$.

- Multi-mode $\Gamma_1 H_2$, for $\Gamma = (\Gamma_1 \Gamma_2)$ with $\Gamma_1 (m \times k)$ and any $H_2 \in O(k)$
- Concentrations: Λ.

4. A general form of distributions

pdf $f_x(X) = \dfrac{{}_pF_q(a_1 \cdots a_p;\ b_1 \cdots b_q;\ X'BX)}{{}_{p+1}F_{q+1}(a_1 \cdots a_p, k/2;\ b_1 \cdots b_q, m/2;\ B)}$

for $B : m \times m$ symmetric matrix.

Ex: ${}_0F_0\left(X'BX\right) = etr\left(X'BX\right)$,

${}_1F_0\left(b; X'BX\right) = | I_k - X'BX |^{-b}$.

5. Distributions with pdf of the form $f(P_\nu X)$,

where $\begin{cases} \nu : \text{a subspace of } R^m \text{ of dimension } q \\ P_\nu : \text{orthogonal projection matrix onto } \nu. \end{cases}$

Ex: $\begin{cases} L(m,k;F=\Gamma\Lambda\Theta'), \text{ with pdf } \propto etr(F'X) \\ = etr(\Theta\Lambda\Gamma'\Gamma\Gamma'X) = etr(F'P_\nu X), \\ B(m,k;B=\Gamma\Lambda\Gamma'), \text{ with pdf } \propto etr(X'BX) \\ = etr(X'\Gamma\Gamma'\Gamma\Lambda\Gamma'\Gamma\Gamma'X) = etr[(P_\nu X)'B(P_\nu X)], \end{cases}$

with $\begin{cases} P_\nu = \Gamma\Gamma' \\ \nu : \text{subspace spanned by the the columns of } \Gamma. \end{cases}$

6. A method to generate a new family of distributions

For an $m \times k$ random matrix Z,

$$Z = \underbrace{Z(Z'Z)^{-1/2}}_{H_Z} \cdot \underbrace{(Z'Z)^{1/2}}_{T_Z^{1/2}} = H_Z \cdot T_Z^{\ 1/2}$$

(polar decomposition of Z).

$H_Z \in V_{k,m}$: orientation of Z.

What is the distribution of H_Z?

using $(dZ) = \dfrac{\pi^{km/2}}{\Gamma_k(m/2)}[dH_Z](dT_Z)$,

where (dZ), (dT_Z): Lebesgue measures,

$[dH_Z]$: normalized invariant measure on $V_{k,m}$.

Ex: (i) When $Z \underset{d}{\sim} N_{m,k}\left(0; \Sigma(m \times m), I_k\right)$, then

$$f_{H_Z}(H_Z) = |\Sigma|^{-k/2} |H_Z' \Sigma^{-1} H_Z|^{-m/2}:$$

Matrix angular central Gaussian [MACG(Σ)] distribution.

(MACG(I_m) = uniform distribution)

(ii) More generally,

when $f_Z(Z) = |\Sigma|^{-k/2} g\left(Z'\Sigma^{-1}Z\right) = |\Sigma|^{-k/2} g\left(H'Z'\Sigma^{-1}ZH\right)$,

for $H \in O(k)$,

then $H_Z \underset{d}{\sim} MACG(\Sigma)$.

[IV] Inference for $F = \Gamma\Lambda\Theta'$ of Langevin distribution

Let $X_1, \cdots, X_n$ a random sample from $L(m,k;F)$.

$\overline{X} = \sum_{i=1}^{n} X_i / n,$ then the log-likelihood is

$$l(F; X_1, \cdots, X_n) = n\left[tr\left(F'\overline{X}\right) - \log {}_0F_1\left(m/2;\ \Lambda^2/4\right)\right]$$

$$= n\left[tr\left(\overline{H}_2'\Theta\Lambda\Gamma'\overline{H}_1\overline{X}_d\right) - \log {}_0F_1\left(m/2;\ \Lambda^2/4\right)\right],$$

where $\overline{X} = \overline{H}_1\overline{X}_d\overline{H}'_2$ (svd), with $\overline{H}_1 \in \tilde{V}_{k,m}$,
$\overline{H}_2 \in O(k)$, $\overline{X}_d = diag(x_1 \cdots x_k)$, $x_1 > \cdots > x_k > 0.$

M.l.e.'s $\hat{\Gamma} = \overline{H}_1$, $\hat{\Theta} = \overline{H}_2$, and $\hat{\Lambda} = diag\left(\hat{\lambda}_1 \cdots \hat{\lambda}_k\right)$

where $\dfrac{\partial \log {}_0F_1\left(m/2; \hat{\Lambda}^2/4\right)}{\partial \hat{\lambda}_i} = x_i, \quad i = 1, \cdots, k. \quad (*)$

[V] Asymptotic theorems in Inference and disrtibution theory

for $\left\{ \begin{array}{l} \text{large sample } n \\ \text{large or small concentrations } \Lambda \\ \qquad \text{(near uniformity)} \\ \text{high dimension } m \end{array} \right.$

[V.1] Large sample asymptotics

Testing

$$\begin{cases} H_0: \quad \Lambda = 0 \text{ (uniformity) against} \\ H_1: \quad F = n^{-1/2}F_0 \text{ (or } \Lambda = n^{-1/2}\Lambda_0), \end{cases}$$

$$Z = \sqrt{nm}\,\overline{X} \underset{d}{\dot{\approx}} \begin{cases} N_{m,k}(0; I_m, I_k) & \text{under } H_0 \\ N_{m,k}(m^{-1/2}F_0; I_m, I_k) & \text{Under } H_1 \text{, for large } n. \end{cases}$$

$$\therefore trZ'Z \underset{d}{\dot{\approx}} \begin{cases} x^2_{km} & \text{under } H_0 \\ x^2_{km;tr\Lambda_0^2/m} & \text{under } H_1\text{, for large } n; \end{cases}$$

asymptotically Rayleigh-style, likelihood ratio, Rao score, locally best invariant tests.

[V . 2] Small or large Λ asymptotics

Solve (*) by using the asymptotic expansion for the ${}_0F_1\left(m/2;\ \hat{\Lambda}^2/4\right)$ with $\hat{\Lambda}$ small or large:

$$\begin{cases} \hat{\lambda}_i = mx_i + O(\hat{\Lambda}^3), \qquad i = 1, \cdots, k, & \text{for small } \hat{\Lambda}. \\ 1 - \dfrac{m-k}{2\hat{\lambda}_i} - \dfrac{1}{2} \sum_{j=1, j\neq i}^{k} \dfrac{1}{\hat{\lambda}_i + \hat{\lambda}_j} + O\left(\hat{\Lambda}^{-2}\right) = x_i, \quad i = 1, \cdots, k, & \text{for large } \hat{\Lambda}. \end{cases}$$

(when Λ has rank 1, then $\hat{\lambda}_1 = \dfrac{m-1}{2(1-x_1)} + O\left(\hat{\Lambda}^{-1}\right)$)

[V . 3] High dimensional asymptotics

(i) Background

Applications in compositional data and certain permutation distributions.

For Langevin distributions with $k=1$, it is known that practically, concentrations for $m=3$ are larger than those for $m=2$.

$$L(m,k;F) \doteqdot \text{uniform distribution } (F=0),$$

$$\text{as } m \to \infty \text{ (see later)} . \quad \therefore F = O\left(m^{\beta}\right)?$$

(ii) Asymptotic expansions for distributions

For $X_1, \ldots, X_n$ a random sample from $L(m, k; F)$, put

$Z = \sqrt{nm}\Psi' \overline{X}(q \times k)$, for fixed $\Psi(m \times q) \in V_{q,m}$, q fixed.

pdf $f_Z(Z) = \varphi^{(q,k)}(Z)\ \{1 + \sqrt{\frac{n}{m}} tr(F'\Psi Z)$

$$+ \frac{1}{4m}[\frac{1}{n}\sum_{\lambda \vdash 2} a_\lambda H_\lambda^{(q,k)}(Z) + 2n((tr\, F'\Psi Z)^2$$

$$- tr\, \Psi' F F' \Psi)] + O(m^{-3/2})\},$$

where $\begin{cases} \varphi^{q,k}(Z) = (2\pi)^{-qk/2} etr(-Z'Z/2), \\ H_\lambda^{(q,k)}(\cdot): \end{cases}$ Hermite polys. associated with the normal $N_{q,k}(0; I_q, I_k)$, defined by

$$H_\lambda^{(q,k)}(Z)\ \varphi^{(q,k)}(Z) = C_\lambda(\partial Z \partial Z')\ \varphi^{(q,k)}(Z)$$

with $\partial Z = \left(\dfrac{\partial}{\partial Z_{ij}}\right)$, for $Z = (Z_{ij})$.

$\therefore Z \sim_d N_{q,k}(0; I_q, I_k)$ not dependent on F,

$\therefore L(m,k;F) \sim$ uniform distribution as $m \to \infty$.

(iii) Generalized Stam's Limit Theorems

(iii, 1) Histrical background

de Finetti's theorem (1929): an infinite, exchangeable sequence of random variables is mixed i.i.d.

Poincare's theorem (1912), or Stam's (first) theorem (1982): asymptotic normality of the first k coordinates of a random point (uniform) on $V_{1,m}$, as $m \to \infty$.

If $X_1, \ldots, X_n$ is orthogonally invariant, $X_1, \ldots, X_n$ has a σ-mixture distribution of normal $N\left(0, \sigma^2\right)$, as $n \to \infty$. [Diaconis, Eaton, Freedman, Lauritzen (1980's)]

(iii, 2) Stam's first theorem

Theorem 0. [Stam (1982)] Asymptotic normality of a finite number of coordinates of the uniform variate on $V_{1,m}$, as $m \to \infty$.

Theorem 1. [Watson (1983)] If $X \underset{d}{\sim}$ uniform on $V_{k,m}$, a finite number of elements of $\sqrt{m}X$ are i.i.d. $N(0,1)$, as $m \to \infty$.

Extensions to more general distributions with pdf $f(P_\nu X)$

where P_ν : the orthogonal projection matrix onto a subspace ν of dimension q.

Ex: $\begin{cases} L(m,k;F=\Gamma\Lambda\Theta'(svd)), \text{ with pdf } \propto etr(F'P_\nu X) \\ B(m,k;B=\Gamma\Lambda\Gamma'(sd)), \text{ with pdf } \propto etr[(P_\nu X)'B(P_\nu X)] \end{cases}$

with $P_\nu = \Gamma\Gamma'$.

Theorem 2. [Chikuse (1991)] If $X \underset{d}{\sim} f\left(P_\nu X\right)$,

with $P_\nu = \Gamma\Gamma', \Gamma \in V_{q,m}\left(q \quad fixed \le m\right)$,

then $\sqrt{m}\Gamma' X\left(q \times k\right) \underset{d}{\dot{\sim}} N_{q,k}\left(0; I_q, I_k\right)$, as $m \to \infty$.

($q = m \leftrightarrow$ uniform case)

Theorem 3. [Chikuse] Suppose that $X \underset{d}{\sim} f\left(m^{\beta} P_{\nu} X\right)$.

(1) When $\beta < \frac{1}{2}$, $\sqrt{m}\Gamma' X \underset{d}{\dot\sim} N_{q,k}\left(0; I_q, I_k\right)$, as $m \to \infty$.

(ex: $\beta = 0\left(< \frac{1}{2}\right) \leftrightarrow$ Theorem 2.)

(2) When $\beta = \frac{1}{2}$, the limiting distribution depends on $f(\cdot)$.

Ex:

(i) When $X \underset{d}{\sim} L\left(m, k; \sqrt{m} F\right)$, $F = \Gamma\Lambda\Theta'$, then $\sqrt{m}\Gamma' X \underset{d}{\dot\sim} N_{q,k}\left(\Gamma' F; I_q, I_k\right)$, as $m \to \infty$.

(ii) When $X \underset{d}{\sim} B\left(m, k; mB\right)$, $B = \Gamma\Lambda\Gamma'$, then $\sqrt{m}\Gamma' X \underset{d}{\dot\sim} N_{q,k}\left(0; \left(I_q - 2\Lambda\right)^{-1}, I_k\right)$, as $m \to \infty$.

(iii, 3) Stam's second theorem (limit orthogonality of $X_1, \ldots, X_n$)

Theorem 0. [Stam (1982)] On $V_{1,m}$.

Theorem 1. [Watson (1983)] If $X_1, \ldots, X_n$ is a random sample from the uniform distribution on $V_{k,m}$, then $\sqrt{m} X_i' X_j (k \times k),\ 1 \le i < j \le n,$ are mutually independent normal $N_{k,k}(0; I_k, I_k)$, as $m \to \infty$.

Extensions to non-uniform distributions on $V_{k,m}$

Theorem 2. [Chikuse (1993)] If $X_1, \ldots, X_n$ is a random sample from $L(m,k;F)$ or $B(m,k;B)$, then $\sqrt{m}X_i^{'}X_j$, $1 \le i < j \le n$, are mutually independent normal $N_{k,k}(0; I_k, I_k)$, as $m \to \infty$.

Theorem 3. [Chikuse]

(1) If $X_1, \ldots, X_n$ is a r. sample from $L\left(m, k; m^{\beta} F\right)$, $\beta \le \frac{1}{2}$,
or $B\left(m, k; m^{2\beta} B\right)$, $\beta < \frac{1}{2}$, then $\sqrt{m} X_i' X_j$, $1 \le i < j \le n$,
are mutually independent normal $N_{k,k}\left(0; I_k, I_k\right)$, as $m \to \infty$.

(2) If $X_1, \ldots, X_n$ is a r. sample from $B\left(m, k; m^{2\beta} B\right)$, $\beta = \frac{1}{2}$,
then $\sqrt{m} X_i' \Sigma^{-1/2} X_j$, $1 \le i < j \le n$, are mutually
independent normal $N_{k,k}\left(0; I_k, I_k\right)$, as $m \to \infty$,
where $\Sigma = \left(I_m - 2B\right)^{-1} > 0$.

(ex: $\beta = 0 \left(< \frac{1}{2} \right) \leftrightarrow$ Theorem 2.)

Applications

Let $X_1, \ldots, X_n$ be a random sample from $L\left(m, k; m^{\beta} F\right), \beta \le \frac{1}{2},$ or $B\left(m, k; m^{2\beta} B\right), \ \beta < \frac{1}{2}.$

Then $\sqrt{m} X_i^{'} X_j \underset{d}{\sim} N_{k,k}\left(0; I_k, I_k\right)$ independent

as $m \to \infty$.

(i) $S_n = X_1 + \cdots + X_n,\ \| S_n \|^2 = trS_n' S_n = nk + 2\sum_{i<j}{}_1^n trX_i' X_j,$

where $u_{ij} = \sqrt{m} trX_i' X_j \underset{d}{\dot{\sim}} N_1(0, k)$.

$$\therefore \| S_n \| = \sqrt{nk} + \frac{1}{\sqrt{nkm}} \sum_{i<j} u_{ij} + O(m^{-1})$$

$$\therefore \| S_n \| \underset{d}{\dot{\sim}} N_1\left(\sqrt{nk}, \frac{n-1}{2m}\right),$$ as $m \to \infty$.

Ex: distance between 2 random points $X, Y \quad (n = 2)$

$$d^2 = tr(X - Y)'(X - Y)$$

$$d \underset{d}{\dot{\sim}} N_1\left(\sqrt{2k}, \frac{1}{2m}\right),$$ as $m \to \infty$.

(ii) $$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i X_i'$$

$$trM_n^2 = \frac{k}{n} + \frac{2}{n^2}\sum_{i<j} tr\left(X_i' X_j\right)\left(X_j' X_i\right),$$

where

$$m\sum_{i<j} tr\left(X_i' X_j\right)\left(X_j' X_i\right) \underset{d}{\dot{\sim}} \chi_q^2, \quad q = \frac{n(n-1)}{2}k^2.$$

$$\therefore m\left(trM_n^2 - \frac{k}{n}\right) \underset{d}{\dot{\sim}} \frac{2}{n^2}\chi_q^2,$$ as $m \to \infty$.

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