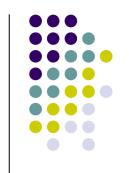
Asymptotic Statistical Analysis on Special Manifolds (Stiefel and Grassmann manifolds)

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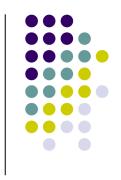
[1] Stiefel manifold $V_{k,m}$ $(k \le m)$



 $\stackrel{\text{def}}{=}$ { k - frames in R^m ; k - frame = a set of k orthonormal vectors in R^m }

$$:= \{X(m \times k) ; X'X = I_k\}.$$

Dimension of
$$V_{k,m} = km - \frac{1}{2}k(k+1)$$
.



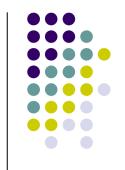
Ex: (i)
$$O(m) = V_{m,m}$$
: orthogonal group.
(ii) $V_{1,m}$: unit hypersphere in R^m .
($m = 2$ circle, $m = 3$ sphere)

Applications in Earth Sciences, Medicine, Astronomy, Meteorology, Biology.

 $\begin{cases} k=1 \text{ : A large literature exists for analysis of directional statistics.} \\ 2 \le k \le m \le 3 \text{ : } \begin{cases} \text{vector cardiogram orbits of comets.} \end{cases}$

$$2 \le k \le m \le 3$$
: vector cardiogram orbits of comets.

[II] Grassmann manifold $G_{k,m-k} (k \leq m)$

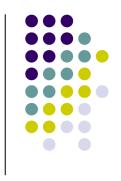


 $\stackrel{\text{def}}{=}$ { k - planes in R^m , i.e., k - dimensional hyperplanes containing the origin in R^m }

To each "k - plane" $\in G_{k,m-k}$, corresponds a unique " $m \times m$ orthogonal projection matrix P idempotent of rank k" $\in P_{k,m-k}$.

$$\therefore G_{k,m-k} \equiv P_{k,m-k}$$

We carry out statistical analysis on $P_{k,m-k}$.

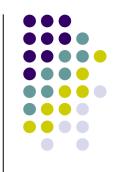


Ex: $G_{1,m-1}$ ={axes or undirected lines through $\vec{0}$ }

Applications: in the signal processing of radar with m elements observing k targets. A rather new sample space.

We confine our discussion mainly on $V_{k,m}$ in the following.

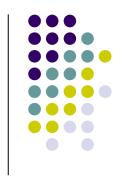
[III] Population distributions on $V_{k,m}$



1. Uniform distribution (normalized invariant measure) [dX]

invariant under
$$X \rightarrow H_1XH'_2$$
,

$$H_1 \in O(m), \quad H_2 \in O(k).$$



$$[dX] = (dX) / \int_{V_x} (dX)$$
, where the differential form

$$(dX) = \bigwedge_{j=1}^{m-k} \bigwedge_{i=1}^{k} x'_{k+j} d \stackrel{\rightarrow}{x_i} \bigwedge_{i < j}^{k} x'_{j} d \stackrel{\rightarrow}{x_i}, \text{ for } X = (\stackrel{\rightarrow}{x_1} \cdots \stackrel{\rightarrow}{x_k})$$

choosing $\overset{i < j}{\underset{x_{k+1}}{\rightarrow}} \cdots \overset{i < j}{\underset{m}{\rightarrow}} \text{ s.t. } (\overset{j}{\underset{x_1}{\cdots}} \overset{j}{\underset{x_k}{\rightarrow}} \overset{j}{\underset{x_{k+1}}{\cdots}} \overset{j}{\underset{x_m}{\rightarrow}} \circ O(m),$

$$\int_{V_{k,m}} (dx) = 2^k \pi^{km/2} / \Gamma_k (m/2), \text{ with}$$

$$\Gamma_k(a) = \int_{S>0} etr(-S) |S|^{a-(k+1)/2} (dS) = \pi^{k(k-1)/4} \prod_{i=1}^k \Gamma(a-(i-1)/2).$$

(multivariate Gamma function)



2. Matrix Langevin L(m,k;F) distribution

pdf $f_x(X) = etr(F'X)/_0F_1(m/2; F'F/4)$

for $F m \times k$ matrix, w.r.t. [dX].

 $(:: X \in N_{m,k}(M; I_m, \Sigma), \text{ with } F = M\Sigma^{-1} \mid X'X = I_k).$

 $_{0}F_{1}$: hypergeometric function with matrix argument.

Exponential type distribution.

Parametrization of $F = \Gamma \Lambda \Theta'$ (svd), where

$$\Gamma \in \widetilde{V}_{k,m}$$
, $\Theta \in O(k)$, and $\Lambda = diag(\lambda_1 \cdots \lambda_k)$, (with first row positive) $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$.



$$\begin{array}{ll} \text{Mode:} X_0 = \Gamma\Theta'(\in V_{k,m}) & (\because)etr(F'X) = etr(\Lambda\Gamma'X\Theta) \\ \text{Concentrations:} \ \Lambda & (\because)etr(F'X_0) = etr\Lambda. \\ \end{array}$$

Assume rank F = k, unless otherwise stated.

F = 0 (or $\Lambda = 0$): uniform distribution.

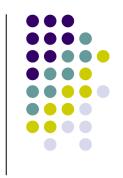
Historical background:

k = 1: directional disrtibution

$$\begin{cases} m=2 \text{ von Mises distribution (1918)} \\ m=3 \text{ Fisher distribution (1953)} \\ m\geq 2 \text{ Langevin (1905)} \end{cases}$$

generalization by Watson & Williams (1956).

Matrix-variate normal $N_{m,k}(M; \Sigma_1, \Sigma_2)$ distribution

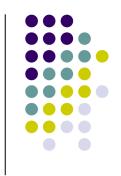


$$Z(m\times k) \approx N_{m,k}(0;I_m,I_k) \text{ if } Z \text{ has the pdf}$$

$$\varphi^{(m,k)}(Z) = (2\pi)^{-km/2} \exp[tr(-Z'Z/2)]$$
 i.e., all elements of Z are i.i.d. $N(0,1)$.

Let
$$Y = \Sigma_1^{1/2} Z \ \Sigma_2^{1/2} + M$$
 [i.e., $Z = \Sigma_1^{-1/2} (Y - M) \Sigma_2^{-1/2}$] for $M(m \times k)$, $\Sigma_1(m \times m) > 0$, $\Sigma_2(k \times k) > 0$, then $Y(m \times k) \sim N_{m,k}(M; \Sigma_1, \Sigma_2)$.

Hypergeometric functions with symmetric matrix argument.



Starting with $_{0}F_{0}(S) = etr(S)$, for $S = m \times m$ symmetric,

(Laplace transforms)
$$\frac{1}{\Gamma_{m}(a)} \int_{S>0} etr(-S) |S|^{a-(m+1)/2} {}_{p} F_{q}(a_{1} \cdots a_{p}; b_{1} \cdots b_{q}; YS)(dS)$$

$$= {}_{p+1} F_{q}(a_{1} \cdots a_{p}a; b_{1} \cdots b_{q}; Y).$$

$$\begin{aligned}
& = \sum_{l=0}^{\infty} \sum_{\lambda \vdash l} \frac{(a_1)_{\lambda} \cdots (a_p)_{\lambda}}{(b_1)_{\lambda} \cdots (b_q)_{\lambda}} \frac{C_{\lambda}(S)}{l!}
\end{aligned}$$



where $\lambda = (l_1,...,l_m)$ ordered partition of l,

$$(l_1 \ge \cdots \ge l_m \ge 0, \sum_{i=1}^m l_i = l)$$

$$(a)_{\lambda} = \prod_{i=1}^{m} \left(a - \frac{i-1}{2} \right)_{l_i}$$
, with

$$(a)_l = a(a+1) \cdot \cdot \cdot (a+l-1), \ \ (a)_0 = 1,$$

 $C_{\lambda}(S)$: zonal polynomial with $m \times m$ symmetric matrix S.

3. Matrix Bingham B(m,k;B) distribution

pdf
$$f_x(X) = etr(X'BX) / {}_1F_1(k/2; m/2; B)$$

for $B m \times m$ symmetric matrix.

$$(:: X \in N_{m,k}(0; \Sigma, I_k), \text{ with } B = -\frac{1}{2}\Sigma^{-1} \mid X'X = I_k).$$
 Exponential type distribution.

Parametrization of $B = \Gamma \Lambda \Gamma'$ (sd), where

$$\Gamma \in \widetilde{O}(m)$$
, and $\Lambda = diag(\lambda_1 \cdots \lambda_m)$, $\lambda_1 \ge \cdots \ge \lambda_m$.

Multi-mode
$$\Gamma_1 H_2$$
, for $\Gamma=\left(\Gamma_1 \Gamma_2\right)$ with $\Gamma_1 \left(m \times k\right)$ and any $H_2 \in O(k)$

centrations:

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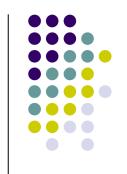
4. A general form of distributions

pdf
$$f_x(X) = \frac{{}_p F_q(a_1 \cdots a_p; b_1 \cdots b_q; X'BX)}{{}_{p+1} F_{q+1}(a_1 \cdots a_p, k/2; b_1 \cdots b_q, m/2; B)}$$

for $B: m \times m$ symmetric matrix.

Ex:
$$_{0}F_{0}(X'BX) = etr(X'BX),$$
 $_{1}F_{0}(b;X'BX) = |I_{k} - X'BX|^{-b}.$

5. Distributions with pdf of the form $f(P_{\nu}X)$,



where v: a subspace of R^m of dimension q $P_{v}:$ orthogonal projection matrix onto v.

Ex:
$$\begin{cases} L(m,k;F=\Gamma\Lambda\Theta'), & \text{with pdf} & \propto etr(F'X) \\ = etr(\Theta\Lambda\Gamma'\Gamma\Gamma'X) = etr(F'P_{\nu}X), \\ B(m,k;B=\Gamma\Lambda\Gamma'), & \text{with pdf} & \propto etr(X'BX) \\ = etr(X'\Gamma\Gamma'\Gamma\Lambda\Gamma'\Gamma'X) = etr[(P_{\nu}X)'B(P_{\nu}X)], \\ \text{with } \begin{cases} P_{\nu} = \Gamma\Gamma' \\ \nu \text{: subspace spanned by the the columns of } & \Gamma. \end{cases}$$

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6. A method to generate a new family of distributions



For an $m \times k$ random matrix Z,

$$Z = Z(Z'Z)^{-1/2} \cdot (Z'Z)^{1/2} = H_Z \cdot T_Z^{-1/2}$$

$$T_Z^{1/2} \quad \text{(polar decomposition of } Z\text{)}.$$

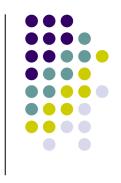
 $H_Z \in V_{k,m}$: orientation of Z.

What is the distribution of H_Z ?

using
$$(dZ) = \frac{\pi^{km/2}}{\Gamma_k(m/2)} [dH_Z] (dT_Z),$$

where (dZ), (dT_Z) : Lebesgue measures,

 $[dH_Z]$: normalized invariant measure on $V_{k,m}$.



Ex: (i) When
$$Z \sim N_{m,k}(0; \Sigma(m \times m), I_k)$$
, then

$$f_{H_z}(H_z) = |\Sigma|^{-k/2} |H_z|^{-1} H_z^{-1}$$
:

Matrix angular central Gaussian [MACG(Σ)] distribution. (MACG(I_m) = uniform distribution)

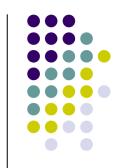
(ii) More generally, when $f_Z(Z)=|\Sigma|^{-k/2}g(Z'\Sigma^{-1}Z)=|\Sigma|^{-k/2}g(H'Z'\Sigma^{-1}ZH)$,

$$f_Z(Z) = |\Sigma| \quad g(Z'\Sigma'Z) = |\Sigma| \quad g(H'Z'\Sigma'ZH)$$

for $H \in O(k)$,

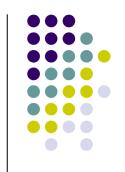
then $H_Z \sim MACG(\Sigma)$.

[IV] Inference for $F = \Gamma \Lambda \Theta'$ of Langevin distribution



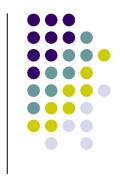
Let X_1, \dots, X_n a random sample from L(m, k; F). $\overline{X} = \sum X_i / n$, then the log-likelihood is $l(F; X_1, \dots, X_n) = n \left[tr(F'\overline{X}) - \log_0 F_1(m/2; \Lambda^2/4) \right]$ $= n \left[tr \left(\overline{H}_{2} \Theta \Lambda \Gamma' \overline{H}_{1} \overline{X}_{d} \right) - \log_{0} F_{1} \left(m / 2; \Lambda^{2} / 4 \right) \right],$ where $\overline{X} = \overline{H}_1 \overline{X}_d \overline{H}'_2$ (svd), with $\overline{H}_1 \in \widetilde{V}_{k,m}$, $\overline{H}_{2} \in O(k), \ \overline{X}_{d} = diag(x_{1} \cdots x_{k}), \ x_{1} > \cdots > x_{k} > 0.$ M.I.e.'s $\hat{\Gamma} = \overline{H}_1$, $\hat{\Theta} = \overline{H}_2$, and $\hat{\Lambda} = diag(\hat{\lambda}_1 \cdots \hat{\lambda}_k)$ where $\frac{\partial \log_0 F_1(m/2; \hat{\Lambda}^2/4)}{\partial \hat{\lambda}} = x_i, \quad i = 1, \dots, k.$ (*)

[v] Asymptotic theorems in Inference and disrtibution theory



 $\begin{cases} \text{large sample } n \\ \text{large or small concentrations } \Lambda \\ \text{(near uniformity)} \end{cases}$ high dimension m





Testing

asymptotically Rayleigh-style, likelihood ratio, Rao score, locally best invariant tests.



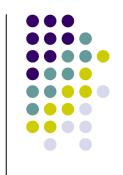
[v.2] Small or large Λ asymptotics

Solve (*) by using the asymptotic expansion for the ${}_{0}F_{1}(m/2; \hat{\Lambda}^{2}/4)$ with $\hat{\Lambda}$ small or large:

$$\begin{cases} \hat{\lambda}_i = mx_i + O(\hat{\Lambda}^3), & i = 1, \cdots, k, \text{ for small } \hat{\Lambda}. \\ 1 - \frac{m - k}{2\hat{\lambda}_i} - \frac{1}{2} \sum_{j=1, j \neq i}^k \frac{1}{\hat{\lambda}_i + \hat{\lambda}_j} + O(\hat{\Lambda}^{-2}) = x_i, & i = 1, \cdots, k, \\ & \text{for large } \hat{\Lambda}. \end{cases}$$

(when Λ has rank 1, then $\hat{\lambda}_1 = \frac{m-1}{2(1-x_1)} + O(\hat{\Lambda}^{-1})$)





(i) Background

Applications in compositional data and certain permutation distributions.

For Langevin distributions with k = 1, it is known that practically, concentrations for m = 3 are larger than those for m = 2.

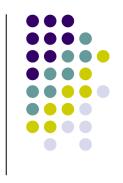
$$L(m,k;F)$$
 in uniform distribution $(F=0)$, as $m \to \infty$ (see later). $\therefore F = O(m^{\beta})$?



(ii) Asymptotic expansions for distributions

For $X_1,...,X_n$ a random sample from L(m,k;F), put $Z = \sqrt{nm} \Psi' \overline{X}(q \times k)$, for fixed $\Psi(m \times q) \in V_{q,m}$, q fixed.

$$\text{pdf } f_{Z}(Z) = \varphi^{(q,k)}(Z) \left\{ 1 + \sqrt{\frac{n}{m}} tr(F'\Psi Z) + \frac{1}{4m} \left[\frac{1}{n} \sum_{\lambda \vdash 2} a_{\lambda} H_{\lambda}^{(q,k)}(Z) + 2n((tr F'\Psi Z)^{2} - tr \Psi' F F' \Psi) \right] + O(m^{-3/2}) \right\},$$



where
$$\begin{cases} \varphi^{q,k}(Z) = (2\pi)^{-qk/2} etr(-Z'Z/2), \\ H_{\lambda}^{(q,k)}(\cdot) \end{cases}$$
: Hermite polys. associated with the normal $N_{q,k}(0;I_q,I_k)$, defined by $H_{\lambda}^{(q,k)}(Z) \; \varphi^{(q,k)}(Z) = C_{\lambda} \left(\partial Z \partial Z'\right) \; \varphi^{(q,k)}(Z)$ with $\partial Z = \left(\frac{\partial}{\partial Z_{ii}}\right)$, for $Z = (Z_{ij})$.

- $\therefore Z \underset{d}{\sim} N_{q,k}(0; I_q, I_k)$ not dependent on F,
- $\therefore L(m,k;F) \stackrel{*}{\sim}$ uniform distribution as $m \rightarrow \infty$.



(iii, 1) Histrical background

de Finetti's theorem (1929): an infinite, exchangeable sequence of random variables is mixed i.i.d.

Poincare's theorem (1912), or Stam's (first) theorem (1982): asymptotic normality of the first k coordinates of a random point (uniform) on $V_{1,m}$, as $m \to \infty$.

If $X_1,...,X_n$ is orthogonally invariant, $X_1,...,X_n$ has a σ -mixture distribution of normal $N(0,\sigma^2)$, as $n\to\infty$. [Diaconis, Eaton, Freedman, Lauritzen (1980's)]

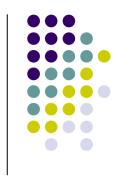


(iii, 2) Stam's first theorem

Theorem 0. [Stam (1982)] Asymptotic normality of a finite number of coordinates of the uniform variate on $V_{1,m}$, as $m \to \infty$.

Theorem 1. [Watson (1983)] If X and uniform on $V_{k,m}$, a finite number of elements of $\sqrt{m}X$ are i.i.d. N (0,1), as $m \to \infty$.

Extensions to more general distributions with pdf $f(P_{\nu}X)$



where P_{ν} : the orthogonal projection matrix onto a subspace ν of dimension q_{ν}

Ex:
$$\int L(m,k;F = \Gamma\Lambda\Theta'(svd))$$
, with pdf $\propto etr(F'P_{\nu}X)$
 $B(m,k;B = \Gamma\Lambda\Gamma'(sd))$, with pdf $\propto etr[(P_{\nu}X)'B(P_{\nu}X)]$
with $P_{\nu} = \Gamma\Gamma'$.

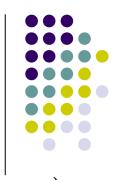


Theorem 2. [Chikuse (1991)] If $X \in f(P_{\nu}X)$,

with
$$P_{\nu} = \Gamma \Gamma', \Gamma \in V_{q,m} (q \text{ fixed } \leq m),$$

then
$$\sqrt{m}\Gamma'X(q\times k) \approx N_{q,k}(0;I_q,I_k)$$
, as $m\to\infty$.

$$(q = m \leftrightarrow uniform case)$$

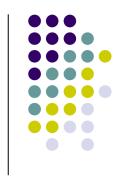


Theorem 3. [Chikuse] Suppose that $X \in f(m^{\beta}P_{\nu}X)$.

- (1) When $\beta < \frac{1}{2}$, $\sqrt{m}\Gamma'X \approx N_{q,k}(0; I_q, I_k)$, as $m \to \infty$. (ex: $\beta = 0 \left(< \frac{1}{2} \right) \leftrightarrow \text{Theorem 2.}$)
- (2) When $\beta = \frac{1}{2}$, the limiting distribution depends on $f(\cdot)$.

$$\begin{split} & \text{Ex:} \bigg(\text{(i) When } X \underset{\text{d}}{\sim} L\Big(m,k;\sqrt{m}F\Big) \,, \quad F = \Gamma\Lambda\Theta \text{ ', then} \\ & \sqrt{m}\Gamma'X \underset{\text{d}}{\sim} N_{q,k}\Big(\Gamma'F;I_q,I_k\Big) \,, \text{ as } m \longrightarrow \infty. \\ & \text{(ii) When } X \underset{\text{d}}{\sim} B\Big(m,k;mB\Big), \quad B = \Gamma\Lambda\Gamma \text{ ', then} \\ & \sqrt{m}\Gamma'X \underset{\text{d}}{\sim} N_{q,k}\Big(0;\big(I_q-2\Lambda\big)^{\!-1},I_k\Big) \,, \text{ as } m \longrightarrow \infty. \\ & \quad \ _{29} \end{split}$$

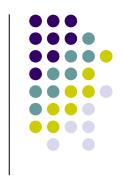
(iii, 3) Stam's second theorem (limit orthogonality of $X_1,...,X_n$)



Theorem 0. [Stam (1982)] On $V_{1,m}$.

Theorem 1. [Watson (1983)] If X_1, \ldots, X_n is a random sample from the uniform distribution on $V_{k,m}$, then $\sqrt{m}X_i'X_j(k\times k)$, $1\leq i < j \leq n$, are mutually independent normal $N_{k,k}(0;I_k,I_k)$, as $m \to \infty$.

Extensions to non-uniform distributions on $V_{{\scriptscriptstyle k,m}}$



Theorem 2. [Chikuse (1993)] If $X_1, ..., X_n$ is a random sample from L(m,k;F) or B(m,k;B), then $\sqrt{m}X_iX_j$, $1 \le i < j \le n$, are mutually independent normal $N_{k,k}(0;I_k,I_k)$, as $m \to \infty$.



Theorem 3. [Chikuse]

- (1) If $X_1, ..., X_n$ is a r. sample from $L(m, k; m^{\beta} F)$, $\beta \leq \frac{1}{2}$, or $B(m, k; m^{2\beta} B)$, $\beta < \frac{1}{2}$, then $\sqrt{m} X_i X_j$, $1 \leq i < j \leq n$, are mutually independent normal $N_{k,k}(0; I_k, I_k)$, as $m \rightarrow \infty$.
- (2) If $X_1,...,X_n$ is a r. sample from $B\left(m,k;m^{2\beta}B\right)$, $\beta=\frac{1}{2}$, then $\sqrt{m}X'_i \Sigma^{-1/2}X_j$, $1 \le i < j \le n$, are mutually independent normal $N_{k,k}(0;I_k,I_k)$, as $m \to \infty$, where $\Sigma = (I_m 2B)^{-1} > 0$.

$$(ex: \beta = 0 \left(< \frac{1}{2} \right) \leftrightarrow Theorem 2.)$$



Applications

Let $X_1, ..., X_n$ be a random sample from $L(m, k; m^{\beta}F)$, $\beta \leq \frac{1}{2}$, or $B(m, k; m^{2\beta}B)$, $\beta < \frac{1}{2}$. Then $\sqrt{m}X_i'X_j \stackrel{.}{\rightleftharpoons} N_{k,k}(0; I_k, I_k)$ independent as $m \rightarrow \infty$.



(i)
$$S_n = X_1 + \dots + X_n$$
, $||S_n||^2 = trS_n S_n = nk + 2\sum_{i < j} {}^n_i trX_i X_j$, where $u_{ij} = \sqrt{mtr}X_i X_j \rightleftharpoons N_1(0,k)$.

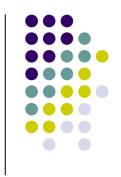
$$\therefore \parallel S_n \parallel = \sqrt{nk} + \frac{1}{\sqrt{nkm}} \sum_{i < j} u_{ij} + O(m^{-1})$$

$$\therefore \|S_n\| \stackrel{.}{\approx} N_1 \left(\sqrt{nk}, \frac{n-1}{2m} \right), \text{ as } m \longrightarrow \infty.$$

Ex: distance between 2 random points $X, Y \quad (n = 2)$

$$d^{2} = tr(X - Y)'(X - Y)$$

$$d \stackrel{.}{\approx} N_{1}\left(\sqrt{2k}, \frac{1}{2m}\right), \text{ as } m \rightarrow \infty.$$



(ii)
$$M_n = \frac{1}{n} \sum_{i=1}^n X_i X_i^{'}$$

$$tr M_n^2 = \frac{k}{n} + \frac{2}{n^2} \sum_{i < j} tr (X_i^{'} X_j) (X_j^{'} X_i),$$

where

$$m\sum_{i < j} tr(X_i X_j)(X_j X_i) \stackrel{.}{\rightleftharpoons} \chi_q^2, \ q = \frac{n(n-1)}{2}k^2.$$

$$\therefore m\left(trM_n^2 - \frac{k}{n}\right) \stackrel{.}{\rightleftharpoons} \frac{2}{n^2}\chi_q^2, \text{ as } m \longrightarrow \infty.$$





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