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Riemann-Hilbert Problems (RHP's): with applications

A picture: think of a RHP as a "picture" or "representation" of a system of interest, much as classical functions such as Bessel, Airy, Hermite functions... have integral representations. And just as Bessel, Airy... functions can be evaluated asymptotically as some parameter becomes large by the (classical) method of stationary phase / steepest descent, the same is true for systems repr. by a RHP, using the non-linear steep. desc. method.

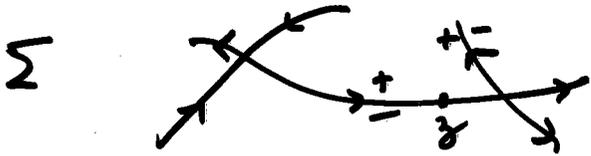
Many pictures: many systems can be repr'nted by a RHP

Outline of my talk:

- ① What is a RHP?
- ② Origins
- ③ Uses
 - (a) algebraic
 - (b) analytical
 - (c) asymptotics
 - (d) perturbation theory
 - (e) radiation, diffraction th^y
- ④ How do RHP's arise in applications
- ⑤ Steepest-descent method (3(c) above)

① What is a RHP?

Let Σ be an oriented contour in \mathbb{C}



Orientation means: move along contour in the direction of arrow, then (+)-side to left, (-)-side to right.

For any $k \in \mathbb{N}$, a jump matrix ν is a map $\nu: \Sigma \rightarrow GL(k, \mathbb{C})$ st $\nu, \nu^{-1} \in L^\infty(\Sigma)$.

A solution of the RHP (Σ, ν) is $l \times k$ matrix valued funct. $m = m(z)$ st

- $m(z)$ anal. in $\mathbb{C} \setminus \Sigma$
- $m_+(z) = m_-(z) \nu(z)$, $z \in \Sigma$

where $m_\pm(z) = \lim_{z' \rightarrow z, z' \in (\pm)\text{-side}} m(z')$

If in addition $l=k$, and

- $m(z) \rightarrow I = I_h$ as $z \rightarrow \infty$

say m solves the normalized RHP (Σ, σ) .

Many technicalities: solution exist? And in normalized case, unique? One needs to make precise in what sense the limits m_{\pm} and $m(z) \rightarrow I$ exist. Also, what happens

at pts of intersection? Will not say much about such issues (see Clancey-Yoh'berg for more info.): in all cases will assume

- $m(z)$ anal. in $\mathbb{C} \setminus \Sigma$ and contin. up to $\partial\Sigma$

so all limits are pt.-wise.

At the analytical level, what kind of a problem is a RHP?

Let

$$\underline{Ch(z)} = \int_{\Sigma} \frac{h(s)}{s-z} \frac{ds}{2\pi i}, \quad z \in \mathbb{C} \setminus \Sigma$$

denote the Cauchy oper. on Σ and let

$$\underline{C^{\pm}h(z)} = \lim_{\substack{z' \rightarrow z \\ z' \in (\pm)\text{-side}}} Ch(z'), \quad z \in \Sigma$$

Then for "reasonable" Σ , $C^{\pm} \in \mathcal{L}(L^p(\Sigma))$

for $1 < p < \infty$ and one has $\underline{C^+ - C^- = id.}$
(1)

Let

$$v(z) = (v_-(z))^{-1} v_+(z), \quad z \in \Sigma,$$

be any pt.-wise factorization of v , $v_{\pm}, (v_{\pm})^{-1} \in L^{\infty}(\Sigma)$

and set

$$\underline{w_+} = v_+ - I, \quad \underline{w_-} = I - v_- \quad \text{and} \quad \underline{w} = (w_+, w_-)$$

Define $\underline{C_w} \in \mathcal{L}(L^p(\bar{\Sigma}))$, $1 < p < \infty$, by

$$C_w h = C^+(h w_-) + C^-(h w_+), \quad h \text{ is } l \times k$$

and assume in addition

$$\underline{w_{\pm}} \in L^p(\bar{\Sigma}), \quad 1 < p < \infty.$$

Now let $\mu \in I + L^p$ solve

$$(2) \quad (1 - C_w)\mu = I$$

or more precisely, if $\underline{\mu = I + v}$

$$(2)' \quad (1 - C_w)v = C_w I = C^+ w_- + C^- w_+ \in L^p$$

with $v \in L^p$

Now set $\underline{m(z)} = I + C(\mu(w_+ + w_-))$

Then (1)(2) =>

$$m_+ = \mu \nu_+ \quad \text{and} \quad m_- = \mu \nu_-$$

so that $m_+ = (m_- \nu_-^{-1}) \circ \nu_+ = m_- \nu$ on Σ

Thus soln. of norm. RHP (Σ, ν) is governed

by the sing. integ. eqn. (2). In "good" cases

$1 - C_w$ is Fredholm, etc. So at the anal. level this is the kind of prob. we are dealing with.

2) Origins

RHP's go back to Riemann and Hilbert, who were interested in solving bary. value prob's for Laplacian in a planar region with baries (Mushkhvili). RHP's also arose in the context of the monodromy problem:

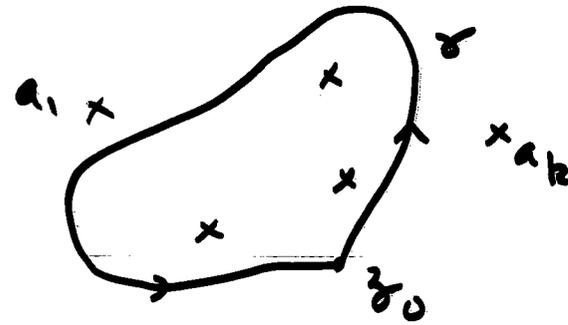
Given a linear system with simple poles

at $a_1, \dots, a_m \in \mathbb{C}$

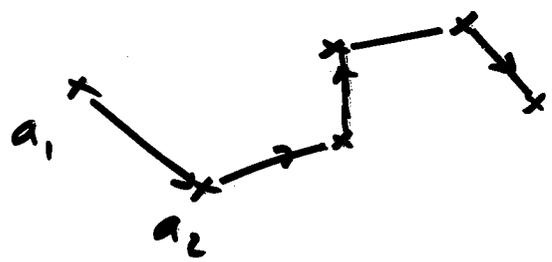
$$\frac{dY}{dz} = \left(\sum_{j=1}^m \frac{A_j}{z - a_j} \right) Y$$

a_1
 a_2
 \dots
 a_m
 z_0

Starting from a point $z_0 \notin \{a_1, \dots, a_m\}$ one can continue solutions around the poles



Starting from $Y(z_0) = I$, one returns to z_0 with value Y_γ , the monodromy matrix for the contour γ . One sees easily that the monodr. \Rightarrow RHP on the cont. $\Sigma: a_1 \rightarrow a_2 \rightarrow \dots \rightarrow a_m$



with const. jumps across the segments. Thus [9]
the mono. prob. is a special case of the more
general RHP. RHP's also arose in so-called Wigner ^(*)

In the th^y of integrable systems and all
the developments that followed, RHP's arose
in the suggestion of Shabat in 1975 that
the sol'n of the inv. scatt. prob. for the
Schrödinger oper. on \mathbb{R} could be rephrased
as a RHP. Because of the connection
Schrödinger \rightarrow KdV, this \Rightarrow

RHP's could be used to solve KdV.

In the case of the defocusing NLS eqn.,
Shabat's suggestion amounts to the following: let
 $q(x, t)$ be the sol'n of NLS on the line

Hopf th^y: radiation problems, diffraction problems. Provides
the structural basis out of which modern AH theory grew.

$$i q_t + q_{xx} - 2|q|^2 q = 0$$

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$$q(x, t=0) = q_0(x) \rightarrow 0 \text{ suff. rapidly as } x \rightarrow \pm \infty.$$

Just as KdV is assoc. with the Schröd. eq., NLS is assoc. with a 1st order 2x2 scatt. system

$$(3) \quad \frac{d\psi}{dx} = \frac{iz}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi + \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix} \psi, \quad x \in \mathbb{R}$$

Let $r = r(z)$ be the reflec. coef. for (3) with $q = q_0$.

Now let $m = m(z; x, t)$ be the solution of the normal. RHP $(\Sigma = \mathbb{R}, \nu_{x,t}) \xrightarrow{\quad} \mathbb{R}$

where

$$\nu_{x,t}(z) = \begin{pmatrix} 1 - |r(z)|^2 & r e^{i\Theta} \\ -\bar{r} e^{-i\Theta} & 1 \end{pmatrix}, \quad z \in \mathbb{R}$$

and $\Theta = xz - tz^2$

Let $m_1(x, t)$ be the residue of m at $z = \infty$ ||

$$m(z; x, t) = I + \frac{m_1(x, t)}{z} + O\left(\frac{1}{z^2}\right)$$

Then

$$(4) \quad q(x, t) = -i (m_1(x, t))_2$$

Certainly a remarkable formula!

the response to Shabat's observation, strangely muted. People in inv. scatt. community were used to working with elegant Gel'fand - ~~Levitan~~ - Marchenko eqn. with smooth kernels etc and did not see the pt. in suddenly having to work with singular eqns. On the other side, people working in RHP's were just pleased to see one more example in the already long list of applicat'n's

of the gen'l, and venerable, th' of RHP's! 112

But there was more to it! Things were happen'g.
Zakharov had shown that the Boussinesq
eqn. could be linearized via a 3rd order diff'l
operator and efforts to find an assoc. GLM
eqn. had failed. However, it was suddenly
clear that the scatt/inverse scatt. problem for
the 3rd order oper. could be phrased as a RHP
on a contour of 6 rays



On the other hand it was clear that one
could, in the case of kdV , or NLS, obtain the
GLM eqn. by taking a Fourier transform of
the sing. integ. eqn. (2). Thus one could only
expect a GLM theory in cases where the underlying
contour was a group, thereby expln'g failure to find

GLM eqn. for Boussinesq. Also people were interested in evaluating the long-time behavior of KdV, NLS etc. From above RHP for NLS one anticipates that one should be dealing with some kind of stationary phase pt analysis

$$\frac{d\theta}{dz} = x - zt\gamma = 0$$

$$\text{stat. pt.} : z_0 = x/zt$$

Recalling that GLM is a four. trans. of the RHP, we see that such micro-local behavior is washed out. To do long-time analysis one needs to stay with the micro-local variable is. stay with the RHP!

From the classical RHP pt. of view, these questions raised in the inv. scatt. community were new : in particular

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- very little was known about RHP's on contours with pt's of self-interaction 14
- very little was known about the behavior of solns of RHP's with highly oscillatory jump matrices eg. $\begin{pmatrix} \cdot & r e^{i\theta} \\ \cdot & 1 \end{pmatrix}, \theta = x_3 - t_3^2$
 $t \rightarrow \infty$

These were the 2 principal prob's that had to be overcome to implement the RHP methodology successfully.

First: overcome by Beals and Coifman with later simplifications by Beals-D-Tomei

Second: overcome by D-Zhou with the intro. of the non-linear steepest descent method for RHP's

③ Uses

In current technology, how are RHP's used? Four

■ uses:

- (a) algebraic \Rightarrow identities, eqns
- (b) analytic
- (c) asymptotics
- (d) perturbation th^y : LP theory (e) radiation, diffrac. th^y

(a) There is a mantra in the business which can be traced all way back to the original work of Gel'fand - Levitan (early '50's) v.i.z.

"If the jump matrix for the RHP is indep. of a parameter, then diff. wrt that param. leads to an eqn."

For NLS, for exple, it works like this. Let us solve the norm. RHP $(\mathbb{R}, u_{x,t})$ above and

2nd set

$$\psi = m e^{i \frac{\theta}{2} \sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then on $\Sigma = \mathbb{R}$,

$$\psi_+ = \psi_- \begin{pmatrix} 1 - |r|^2 & r \\ -\bar{r} & 1 \end{pmatrix}$$

indep. of x (and z)

Diff. w.r.t $x \Rightarrow$

$$\psi_{x+} = \psi_{x-} \begin{pmatrix} 1 - |r|^2 & r \\ -\bar{r} & 1 \end{pmatrix}$$

from which it follows that $T = \psi_x \psi^{-1}$ has no jump across \mathbb{R} , and hence is entire. But as $z \rightarrow \infty$

$$T = m_x m^{-1} + m \frac{i z}{2} \sigma_3 m^{-1} = i z \frac{\sigma_3}{2} + A + O\left(\frac{1}{z}\right)$$

for some const. matrix A . But then by Liouville's th^m, must have $T = i z \frac{\sigma_3}{2} + A$ or

$$(4) \quad \psi_x = i z \frac{\sigma_3}{2} \psi + A \psi$$

which is the original scatt. prob. for NLS

Diff. wrt t gives a similar eqn in t

(5) $\psi_t = (\dots)\psi$

Cross-diffng (4) (5) gives rise to NLS.

This is a mantra in the subject: all diff. eqns, difference eqns, recurrence relations, string eqns, arise in same way through "differentiating" a RHP

For exple: let $\pi_k = z^k + \dots, k \geq 0$ denote the monic orthog. poly's wrt a measure $w(x)dx$ with finite moments on \mathbb{R} , $\int_{\mathbb{R}} \pi_k(x) \pi_l(x) w(x) dx = \delta_{kl}$

$k, l \geq 0$. Then (following Fokas, Its, Kitaev) let $\gamma^{(k)}$ solve the RHP on $\mathbb{R} \rightarrow$

- $\gamma^{(k)}(z)$ anal. on $\mathbb{C} \setminus \mathbb{R}$

- $\gamma_+^{(k)} = \gamma_-^{(k)} \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}, z \in \mathbb{R}$

normalized st

- $\gamma^{(k)}(z) \begin{pmatrix} z^{-k} & 0 \\ 0 & z^k \end{pmatrix} \rightarrow I$ as $z \rightarrow \infty$

Then

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$$\pi_k(z) = Y_{11}^{(k)}(z)$$

By the mantra, we observe that $Y^{(k+1)}$ solves same jump as $Y^{(k)}$. Hence $Y^{(k+1)}(Y^{(k)})^{-1}$

is entire and evaluating asymptotics as $z \rightarrow \infty$, obtain eqn

$$Y^{(k+1)} = (Az + B)Y^{(k)}$$

for some const. matrices A, B . This yields the celebrated 3-term recurrence relation for OP's.

b) Analysis

Consider, for example, the 6 Painlevé eqns. They are determined as follows: They solve 2nd order ode of the form

$$\frac{d^2 y}{dz^2} = F(z, y, \frac{dy}{dz})$$

$$(y(z_0), y'(z_0)) = (a, b)$$

Here F is mer. in z and rational in y & y' .

Solutions $y = y(z; a, b)$ have following distinguishing prop.: as we change the initial data (a, b) , only the poles of $y = y(z; a, b)$ as a function of z , can move. Essential sing's, branch pts, if they \exists , must remain fixed eg if $F = y$ then $y'' = y$ and the ess. sing is always at $z = \infty$, indep of (a, b) . This situation is somewhat reminiscent of Weyl's th^m for ess. spectrum: $\text{ess spec}(A + B) = \text{ess spec}(A)$ if B is (relat.) compact.

There are 50 such eqns, but only 6 of them (the 6 Painlevé transcendents) are "new". For exple P_{II} has the form

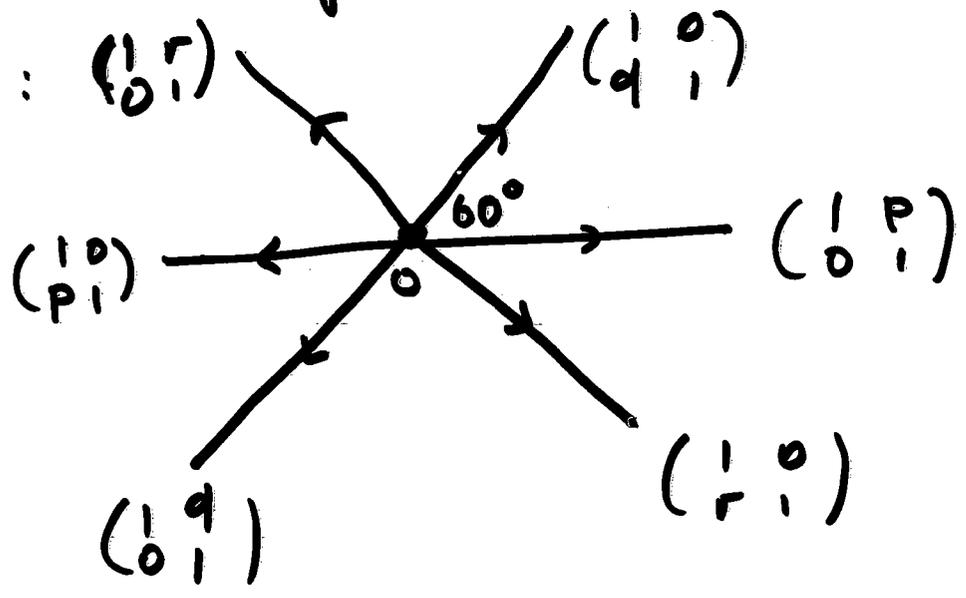
$$y'' = 2y^3 + zy$$

Now, how does one show that P_{II} has Painlevé property?

It turns out that PII (or PI, III, ...) can be solved via a RHP (Miwa, Jimbo, Ueno) (Flashka, Newell). Consider p, q, r satisfying

$$p + q + r + pqr = 0$$

Fix $x \in \mathbb{C}$, and let $\psi(z; x)$ solve the RHP on $\Sigma = \{6 \text{ rays}\}$ with jump matrix V as indicated:



subject to

$$\psi e^{i(\frac{4}{3}z^3 + xz)\sigma_3} \rightarrow I \text{ as } z \rightarrow \infty$$

Let $m_1(x)$ denote the residue of $\psi e^{i(\frac{4}{3}z^3 + xz)\sigma_3}$ at $z = \infty$,

$$\psi e^{i(\frac{4}{3}z^3 + xz)\sigma_3} = I + \frac{m_1(x)}{z} + O(\frac{1}{z^2})$$

Then

$$u(x) = -2i (m_1(x))_{21}$$

solves PII (remember ν is indep of x so must get an eqn!)

How do we check the Painlevé prop.

for $u(x)$? Recall that the solut'n of a RHP is governed by a sing. integ. eqn. $\mu = (1 - \omega)^{-1} I$

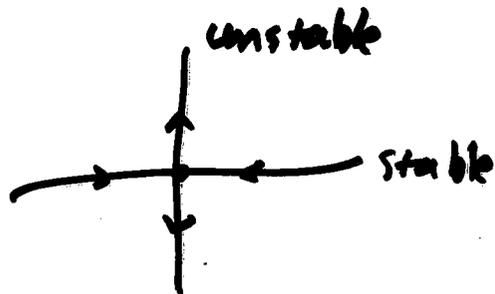
Here ω depends on x in an anal. fashion, and it then follows by analytic Fredholm th^y (the same result needed for Weyl's ess. spec. th^m!) that the only sing's of $(1 - \omega)^{-1}$ are poles of finite order that can accumulate only at ∞ . Thus solutions of PII cannot have movable ess. sing's!

Another example: Boussinesq eqn

$$u_{tt} = u_{xxxx} + \text{nonlinear}$$

Linear'd eqns: forward + backwards heat eqn

Thus, at the linear level, have a stable-unstable manifold decomp in phase space



By RHP, obtain the same result in the fully non-linear case (Beals-D-Trubowitz).

Need RH techniques.

(c) Asymptotics

From the RHP for NLS we see that soln $q(x,t)$ of NLS can be written in the form

$$q(x,t) = \mathcal{F}(r(\cdot) e^{i\theta(\cdot)})$$

$$\theta(\cdot) = x(\cdot) - t(\cdot)^3$$

where \mathcal{F} is some (nonlinear) functional of $r e^{i\theta}$.

If r is "small", as $t \rightarrow \infty$

$$\mathcal{F}(r e^{i\theta}) \sim \text{const.} \int_{\mathbb{R}} r(z) e^{i\theta} dz = \text{const.} \int_{\mathbb{R}} r(z) e^{i(x_3 - t_3^3)} dz$$

and we are dealing with a classical stpst
disc. problem. What we need is a steep. desc.
meth. for the fully nonlinear functional. More
later!

(d) Perturbation th^y

In analyzing pert's of NLS

$$iq_t + q_{xx} - 2|q|^2q - \varepsilon W(|q|)q = 0$$

$$q(x, t=0) = q_0(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty$$

($W(|q|) \sim |q|^l$, $l > 2$ as $|q| \rightarrow 0$) the scatt.

map $\mathcal{S} : q \rightarrow r$ plays the same role as the
linear Four. trans. in analyzing more standard

situations $iq_t + q_{xx} - \varepsilon W(|q|)q = 0$. To be

successful the estimates one gets for \mathcal{S} must
be as fine as one gets for the Four. trans.,

and this is precisely what one can obtain via RHP

In particular, get L^p estimates, $p \geq 2$, as $t \rightarrow \infty$.

(4) How do RHP's arise?

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One systematic way is from integrable eqns. Here there is a lin. oper. L , say, assoc. with the eqn as in the case of KdV, NLS, Boussinesq as above. One singles out special eigensolutions ψ (so-called Beals-Coifman solutions) of the operator e.g. for $z \in \mathbb{C} \setminus \mathbb{R}$ in case NLS

$$\frac{d\psi}{dx} = i \frac{z}{2} \sigma_3 \psi + \begin{pmatrix} 0 & q(x) \\ \overline{q(x)} & 0 \end{pmatrix} \psi$$

$$\begin{cases} \psi(x, z) e^{-i \frac{z}{2} \sigma_3 x} \rightarrow I \text{ as } x \rightarrow +\infty \\ \psi(x, z) e^{-i \frac{z}{2} \sigma_3 x} \rightarrow \text{is bndd as } x \rightarrow -\infty. \end{cases}$$

Then one immediately shows that these ψ solve a RHP on a contour Σ determined by L ($\Sigma = \mathbb{R}$ for NLS, $\Sigma = \times$ for Boussinesq etc.). This is one way RHP's arise.

Sometimes RHP's appear just out of the blue. This was the case for the above RHP's for OP's and the Painlevé eqns. In this serendipitous world we find

(i) random matrix th': it just so happens that the interesting stats. for the stand. ensembles can be expressed in terms of OP's : hence RH applies

(ii) Toeplitz, Hankel determinants : classical relaths Toeplitz, Hankel \leftrightarrow OP's. But combinatorial problems often have soluth in terms of Toeplitz or Hankel dets : hence OP's, hence RH applies True, in particular, for Ulam's longest incr. subseq. probl (Gessel: Baik-D-Johansson)

There is, however, another useful systematic in the business: theory of ~~matrix~~ integrable op's.

Special cases : Tracy, McCoy et al (60s)

elements of gen'l th': Saknovic (late 60s)

But full general theory of ~~matrix~~

such op's. due to Its, Izergin, Korepin and Slavnov in early 90's. [26]

Let Σ be oriented contour in \mathbb{C} . We say that an oper. K is integrable if it has a kernel of the form

$$(6) \quad K(z, z') = \frac{\sum_{j=1}^N f_j(z) g_j(z')}{z - z'}, \quad z, z' \in \Sigma$$

for some N and $f_j, g_j \in L^\infty(\Sigma)$

$$(Kf)(z) = \int_{\Sigma} K(z, z') f(z') dz'$$

Op's K form an algebra, but also have following remarkable prop: if $K = \frac{\sum_{j=1}^N f_j g_j}{z - z'}$

is integ, then so is $(1 - K)^{-1} - I$ and

if we write

$$(1 - K)^{-1} = I + \left(\sum_{j=1}^N F_j(z) G_j(z') \right) / (z - z')$$

then F_j, G_j can be obtained by solving a

naturally assoc. RHP on Σ viz. (for simplicity assume $\sum_{j=1}^N f_j(z) g_j(z) = 0$). Let $m = m(z)$ solve the norm. RHP (Σ, ν) where

$$\nu = I - 2\pi i f g^T$$

$$f = (f_1, \dots, f_N)^T, \quad g = (g_1, \dots, g_N)^T$$

Then $F = m_{\pm} f$ and $G = (m_{\pm}^T)^{-1} g$

Often one is faced with having to compute a determinantal quantity

$$\alpha = \det(1 - k)$$

And it turns out, very often, that k is integrable. Then

$$\begin{aligned} \log \alpha &= \log \det(1 - k) = \int_0^1 \frac{d}{dt} \log \det(1 - tk) dt \\ &= \int_0^1 \frac{d}{dt} \text{tr} \log(1 - tk) dt = - \int_0^1 \text{tr} \left(\frac{1}{1 - tk} tk \right) \frac{dt}{t} \end{aligned}$$

But $(1-tk)^{-1} tk = (1-tk)^{-1} - I$

is then expressible in terms of a RHP, to which the steepest descent method can be applied.

In this way one can prove, for exple., the celebrated Strong Szegő limit \mathcal{T}^n for Toeplitz det's, and also eval. asymp's of Hankel det's.

Here is another exple: Consider the spin- $\frac{1}{2} \times Y$ model in a (critical) magnetic field with Hamiltonian

$$H = -\frac{1}{2} \sum_{l \in \mathbb{Z}} (\sigma_l^x \sigma_{l+1}^x + \sigma_l^z)$$

Then it turns out that the auto-correlation function of the first spin component at inv. temp. β

$$\chi(t) = \langle \sigma_0^x(t) \sigma_0^x \rangle_{\beta}$$

can be written (McCoy, Perk, Shrock) in the form

$$\chi(t) = e^{-t^2/2} \det(1 - K_t)$$

where

$$K_t(z, z') = \varphi(z) \frac{\sin[it(z-z')]}{\pi(z-z')}$$

for $-1 \leq z, z' \leq 1$, and $\varphi(z) = \tanh(\beta \sqrt{1-z^2})$

Clearly integrable op, and following above route, can show that as $t \rightarrow \infty$

$$\chi(t) = \exp\left\{\frac{t}{\pi} \int_{-1}^1 \log |\tanh \beta s| ds + O(\log t)\right\}$$

(D-zhou)

(e) radiation, diffraction problems: Wiener-Hopf th.7

5) Finally, some words about the steepest descent method

All the phenomena of the class. steep desc. meth are present in the non-lin. method for RHP's:

• for NLS, have $e^{i\theta} = e^{i(xz - tz^2)}$ 30
only 1 stat. phase pt $z = x/2t$

• for MKdV and kdv have $e^{i\theta} = e^{i(xz + tz^3)}$
now 2 stat. phase pts $z = \pm \sqrt{\frac{-x}{2t}}$

(if $x/2t < 0$)

The 2 stat. phase pts do not interact if we are in the space-time region

$$-1 < \frac{x}{t} < -\frac{1}{M}$$

But in the region where $x/t \rightarrow 0$ we have the non-linear analog of caustics: here the solution of MKdV, or kdv, looks like a self-similar version of P II.

So far theory proceeds in analogy with linear st. desc. method.

• but new phenomena, beyond the scope of

lin. th^y, start to appear. The new pheno-
mena have the prop^y that in place of stat.
phase pts., or coalescing stat. phase pts.
as in caustic situations, one now has

"lines of stationary phase"

each pt. of which contributes equally
to the leading asymp's of the prob."

This => instead of linear (or modulated linear)
type oscillations, e.g. in the long-time behavior
of solutions of MKdV

$$q(x,t) \sim \frac{\lambda(x/t)}{t^{1/2}} \sin\left(t\left(\frac{x}{t}\right)^2 + \nu\left(\frac{x}{t}\right)\log t\right),$$

one now has genuinely non-linear oscillations
expressed e.g. in terms of the Jacobi sn or cn
functions in place of sin, etc. Situation arises eg

- i) universality for R+T
- ii) semi-class. limit
- iii) continuum limit of Toda lattice etc.