Global fluctuations for the β-Hermite and β-Laguerre ensembles

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joint work with **Ofer Zeitouni**

University of Washington (starting 09/06)

July 13, 2006

Outline.

1. β-ensembles: Hermite as generalized *GOE, GUE, GSE*, Laguerre as generalized *Wishart real, complex, quaternion.*

2. Semicircular laws and global fluctuations.

3. Sketch of proof for the Hermite case.

Gaussian orthogonal, unitary, and symplectic ensembles.

- $\mathbb{R}$, $\boldsymbol{\beta = 1}$: $A = \text{randn(n)}$, $S = (A + A')/2$;
- $\mathbb{C}$, $\boldsymbol{\beta = 2}$: $A = \text{randn(n)} + i * \text{randn(n)}$, $H = (A + A')/2$;
- $\mathbb{H}$, $\boldsymbol{\beta = 4}$: $A = \text{randn(n)} + i * \text{randn(n)} + j * \text{randn(n)} + k * \text{randn(n)}$,
  $D = (A + A')/2$.

Eigenvalue ($\Lambda := (\lambda_1, \ldots, \lambda_n)$) distributions:

$$\rho_{\boldsymbol{\beta}}(\Lambda) \propto \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^{\boldsymbol{\beta}} \; e^{-\sum_{i=1}^{n} \lambda_i^2/2}$$

$$\equiv |\Delta(\Lambda)|^{\boldsymbol{\beta}} \; e^{-\sum_{i=1}^{n} \lambda_i^2/2} .$$

General β-Hermite (β-Gaussian) ensembles.

Defined through eigenvalue p.d.f. for $\boldsymbol{\beta} > 0$:

$$\propto \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^{\boldsymbol{\beta}} \quad e^{-\sum_{i=1}^{n} \lambda_i^2/2}$$

$$\equiv \quad |\Delta(\Lambda)|^{\boldsymbol{\beta}} \quad e^{-\sum_{i=1}^{n} \lambda_i^2/2} .$$

General β-Hermite (β-Gaussian) ensembles.

...or, equivalently, through their matrix model (with eigenvalue p.d.f. shown before):

$$H_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_{\beta} \\ & & & \chi_{\beta} & N(0,2) \end{pmatrix}$$

(D. and Edelman, 2001)

Wishart real, complex, and quaternion ensembles.

- $\mathbb{R}$, $\boldsymbol{\beta} = \mathbf{1}$: $\quad A = \mathrm{randn(m,n)};\ W_1 = AA';$
- $\mathbb{C}$, $\boldsymbol{\beta} = \mathbf{2}$: $\quad A = \mathrm{randn(m,n)} + i{*}\mathrm{randn(m,n)};\ W_2 = AA';$
- $\mathbb{H}$, $\boldsymbol{\beta} = \mathbf{4}$: $\quad A = \mathrm{randn(m,n)} + i{*}\mathrm{randn(m,n)} + \ldots$
  $j{*}\mathrm{randn(m,n)} + k{*}\mathrm{randn(m,n)};\ W_4 = AA'.$

Eigenvalue ($\Lambda := (\lambda_1, \ldots, \lambda_m)$) distributions:

$$\rho_{\boldsymbol{\beta},n}(\Lambda) \propto \prod_{1\le i<j\le m} |\lambda_i - \lambda_j|^{\boldsymbol{\beta}} \prod_{i=1}^{m} \lambda_i^{\boldsymbol{\beta}(n-m+1)/2-1} \quad e^{-\sum_{i=1}^{m} \lambda_i/2}$$

$$\equiv |\Delta(\Lambda)|^{\boldsymbol{\beta}} \quad \det(\Lambda)^{\boldsymbol{\beta}(n-m+1)/2-1} \quad e^{-\sum_{i=1}^{m} \lambda_i/2} \; .$$

General β-Laguerre (β-Wishart) ensembles.

Defined through eigenvalue p.d.f. for $\beta > 0$, $a > \frac{\beta}{2}(n-1)$:

$$
\begin{aligned}
\propto \quad & \prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^{\boldsymbol{\beta}} \prod_{i=1}^{n} \lambda_i^{a-\frac{\boldsymbol{\beta}}{2}(n-1)-1} \, e^{-\lambda_i/2} \\
\equiv \quad & |\Delta(\Lambda)|^{\boldsymbol{\beta}} \prod_{i=1}^{n} \lambda_i^{a-\frac{\boldsymbol{\beta}}{2}(n-1)-1} \, e^{-\lambda_i/2} \, .
\end{aligned}
$$

General β-Laguerre (β-Wishart) ensembles.

...or, equivalently, through their matrix model (with eigenvalue p.d.f. shown before):

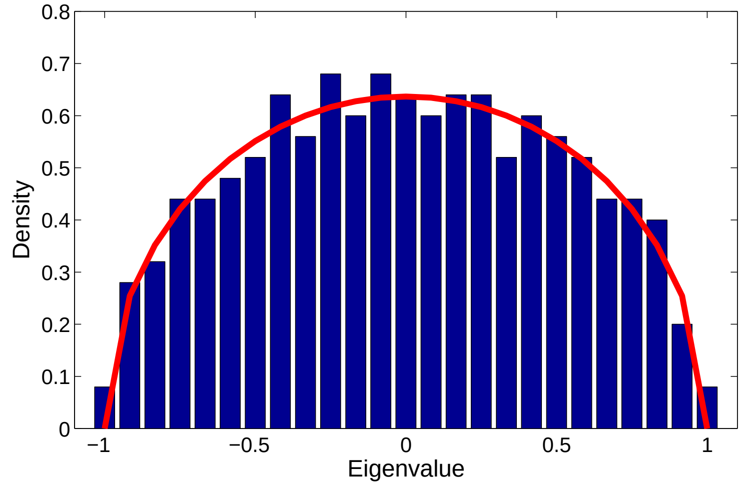
$$L_\beta = B_\beta B_\beta^T \ ,$$

where

$$B_\beta \sim \begin{pmatrix} \chi_{2a} & & & \\ \chi_{\beta(m-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a-\beta(m-1)} \end{pmatrix}$$

(D. and Edelman, 2001)

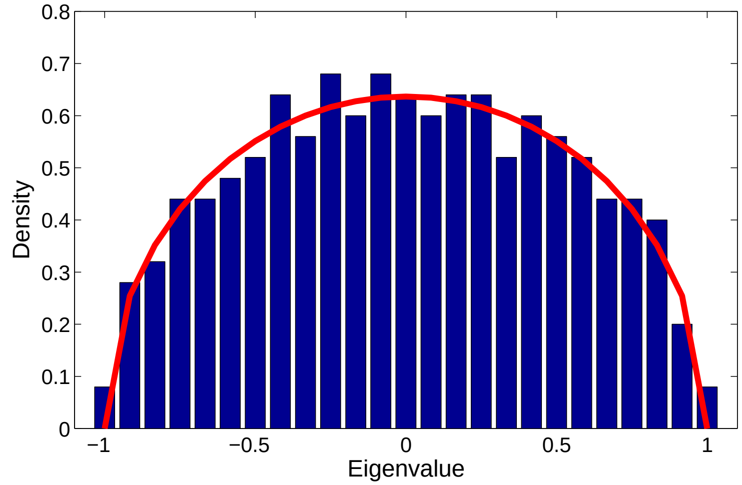
Semicircle Law.



Histogram of eig(S), where $n = 300$, $A =$randn(n), $S = (A + A')/(2\sqrt{2n})$

Works for all β-Hermite ensembles!

Semicircle Law.

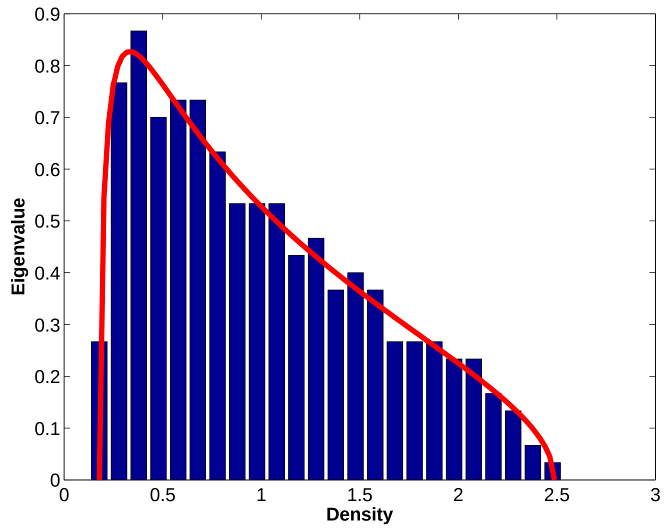


Histogram of eig(S), where $n = 300$, $A =$randn(n), $S = (A + A')/(2\sqrt{2n})$

Rigorously: Given any suitable function $h : [-1, 1] \longrightarrow \mathbb{R}$,

$$\frac{1}{n}\sum_{i=1}^{n} h(\lambda_i) \;\rightarrow\; \frac{2}{\pi}\int_{-1}^{1} h(x)\sqrt{1-x^2}\,dx$$

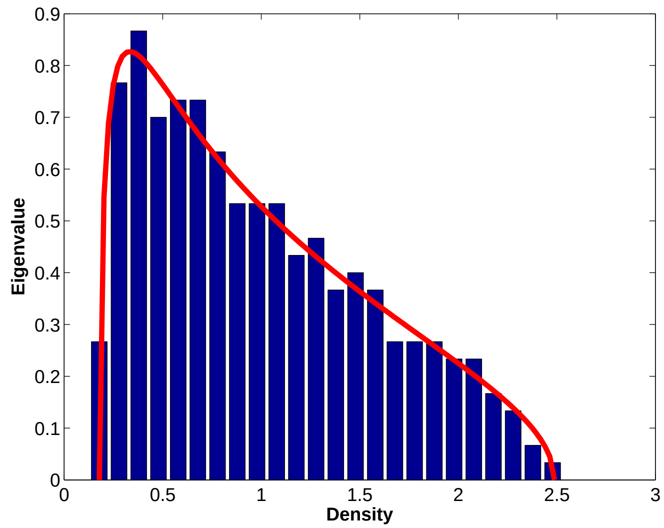
Marčenko-Pastur Law.



Histogram of eig(W), where $m = 300,\ n = 900,\ \gamma = 1/3,\ A =$randn$(m, n), W = AA'\ \sqrt{\gamma/m}$

Works for all β-Laguerre ensembles!

Marčenko-Pastur Law.



Histogram of eig(W), where $m = 300$, $n = 900$, $\gamma = 1/3$, A =randn(m, n), $W = AA' \sqrt{\gamma/m}$

Rigorously: Given any suitable function $h : [-1, 1] \longrightarrow \mathbb{R}$, let

$a = (1 - \sqrt{\gamma})^2$, $b = (1 + \sqrt{\gamma})^2$, then

$$\frac{1}{n}\sum_{i=1}^{n} h(\lambda_i) \;\rightarrow\; \frac{1}{2\pi\gamma}\int_a^b h(x)\frac{\sqrt{(b-x)(x-a)}}{x}\,dx$$

What about fluctuations from semicircle and Marčenko-Pastur laws?...

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Fluctuations define Gaussian process.

Global fluctuations from semicircle and Marčenko-Pastur Laws.

CLT for Hermite. (Zeitouni and D., -)
Let h be a "nice" function, and λ_i, $i = \overline{1,n}$ the *scaled* eigenvalues of the β-Hermite ensemble. Then

$$\sum_{i=1}^{n} h(\lambda_i) - \frac{2n}{\pi}\int_{-1}^{1} h(x)\sqrt{1-x^2}\mathrm{d}x \quad \Longrightarrow$$

$$\Longrightarrow \quad N\left(\left(1-\frac{2}{\beta}\right)\mu(h), \frac{2}{\beta}\,\sigma^2(h)\right) \; .$$

"Nice" means **Lipschitz** on $[-1-\epsilon, 1+\epsilon]$ for some $\epsilon > 0$.

Works for **all** β.

Global fluctuations from semicircle and Marčenko-Pastur Laws.

CLT for Laguerre. (Zeitouni and D., -)
Let h be a "nice" function, and λ_i, $i = \overline{1,n}$ the *scaled* eigenvalues of the β-Laguerre ensemble (γ, a, b as before). Then

$$\sum_{i=1}^{n} h(\lambda_i) - \frac{n}{2\pi\gamma}\int_a^b h(x)\frac{\sqrt{(b-x)(x-a)}}{x}\mathrm{d}x \quad \Longrightarrow$$

$$\Longrightarrow \quad N\left(\left(1-\frac{2}{\beta}\right)\mu(h), \frac{2}{\beta}\,\sigma^2(h)\right) \; .$$

"Nice" means **Lipschitz** on $[-1-\epsilon, 1+\epsilon]$ for some $\epsilon > 0$.

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What was known before.

- **Johansson, 1998**

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- **Kusalik, Mingo, & Speicher, 2005**

  CLT for Wishart complex ($\beta = 2$), explicit calculation of covariance matrix, h polynomial. *Calculation of covariances for any finite number of independent matrices.*

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- **D. and Edelman, 2006**

  CLT for all β, Hermite and Laguerre, h polynomial. *Use of matrix models for all β.*

Outline of rest of the talk.

Use matrix methods, together with the "h polynomial" result of Edelman and D., to extend the class of h for which the Gaussian process is defined (in the talk, we only show this for Hermite).

1. Reduction to compactly supported functions h (Gershgorin).

2. Proof for a slightly different matrix model (Guionnet-Zeitouni).

3. Tying up the loose ends (Lidskii, Gershgorin, Chebyshev).

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Sketch of idea.

Would like to approximate a "nice" function h by a sequence of polynomials p_n (on compact sets); remainder should be small with high probability.

Let $p_n \to h$ on $C \in \mathbb{R}$, $n||p_n - h|| \to 0$ (plus some other regularity). Then

$$\int_{-1}^{1} p_n(x)\sqrt{1-x^2}\mathrm{d}x \to \int_{-1}^{1} h(x)\sqrt{1-x^2}\mathrm{d}x \ ;$$

Need to show (e.g.) that for any C,

$$\Pr\left[\left| \sum_{i=1}^{n} h(\lambda_i) - \sum_{i=1}^{n} p_n(\lambda_i) \right| > C\right] \leq \frac{c}{C^2}\, ||p_n - h||\ .$$

Reduction to compactly supported functions.

We are essentially concerned with $\sum_{i=1}^{n} h(\lambda_i)$; we know (semicircle law) that

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \longrightarrow \mu_s \ ,$$

where $d\mu_s(x) = \frac{2}{\pi}\sqrt{1-x^2}$.

Enough to show that the probability of at least one of the eigenvalues being large (outside $[-1-\epsilon, 1+\epsilon]$, for fixed $\epsilon > 0$) decays *exponentially* in n.

To show this we use Gershgorin's theorem.

Reduction to compactly supported functions.

Recall the *scaled* matrix

$$H_{\beta} \sim \frac{1}{2\sqrt{n\beta}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_{\beta} \\ & & & \chi_{\beta} & N(0,2) \end{pmatrix}$$

All Gershgorin disks have centers distributed like $\tilde{c}\, N(0,1)/\sqrt{n}$, typical Gershgorin radius is distributed like $c\, \chi_{\alpha n}/\sqrt{n}$, $0 < \alpha < 1$.

But, for $\epsilon > 0$,

$$\Pr\left[(\tilde{c} |\, N(0,1)\,| \; + \; c\, \chi_{\alpha n}) \; > \; (1+\epsilon)\sqrt{n} \right] \leq c_1 e^{-c_2 n} \, ,$$

q.e.d.

A very useful result.

Theorem. (Guionnet-Zeitouni, 2000)
Let $A = (a_{ij})$ be a random matrix such that a_{ij} have distributions ν which satisfy the log-Sobolev inequality with uniform constant c, i.e., for all differentiable f,

$$\int f^2 \log \frac{f^2}{\int f^2 \mathrm{d}\nu} \mathrm{d}\nu \leq 2c \int |f'|^2 \mathrm{d}\nu \ ,$$

then for any Lipschitz function h, for any C,

$$\Pr \left[\left| \sum_{i=1}^{n} h(\lambda_i) - E\left[\sum_{i=1}^{n} h(\lambda_i) \right] \right| > C \right] \leq 2e^{-\frac{c_1}{h_{\mathcal{L}}} C^2} \ ,$$

where $h_{\mathcal{L}}$ is the Lipschitz constant for h.

Sounds perfect!

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except...

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the χ function does not satisfy log-Sobolev constraints.

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the χ function does not satisfy log-Sobolev constraints.

But the Gaussian does.

Fixup: approximate χ by Gaussian:

$$\chi_r \sim N(c_r, \frac{1}{\sqrt{2}}) + \quad small \ .$$

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A slightly perturbed model.

The Guionnet-Zeitouni theorem works for

$$\tilde{H}_\beta \sim \frac{1}{2\sqrt{n\beta}} \begin{pmatrix} N(0,2) & N(c_{n-1},\frac{1}{2}) & & & \\ N(c_{(n-1)},\frac{1}{2}) & N(0,2) & N(c_{(n-2)},\frac{1}{2}) & & \\ & \ddots & \ddots & \ddots & \\ & & N(c_2,\frac{1}{2}) & N(0,2) & N(c_1,\frac{1}{2}) \\ & & & N(c_1,\frac{1}{2}) & N(0,2) \end{pmatrix}$$

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Tying up loose ends.

Note that $H_{\text{diff}} = H_\beta - \tilde{H}_\beta$ looks, entry by entry, like

$$H_{\text{diff}} = O\left(\begin{pmatrix} 0 & \frac{1}{\sqrt{n(n-1)}} & & & \\ \frac{1}{\sqrt{n(n-1)}} & 0 & \frac{1}{\sqrt{n(n-2)}} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{\sqrt{2n}} & 0 & \frac{1}{\sqrt{n}} \\ & & & \frac{1}{\sqrt{n}} & 0 \end{pmatrix}\right)$$

and the variables have uniformly bounded variance.

Tying up loose ends.

If λ_i are eigenvalues of H_β and $\tilde{\lambda}_i$ are eigenvalues of $\tilde{H}_\beta$, then for a Lipschitz h (with $h_{\mathcal{L}}$)

$$\left| \sum_{i=1}^{n} h(\lambda_i) - \sum_{i=1}^{n} h(\tilde{\lambda}_i) \right| \le h_{\mathcal{L}} \sum_{i=1}^{n} |\lambda_i - \tilde{\lambda}_i| \, .$$

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$$\left|\sum_{i=1}^{n} h(\lambda_i) - \sum_{i=1}^{n} h(\tilde{\lambda}_i)\right| \;\le\; h_{\mathcal{L}} \sum_{i=1}^{n} |\lambda_i - \tilde{\lambda}_i| \;.$$

By **Lidskii's theorem**,

$$\sum_{i=1}^{n} |\lambda_i - \tilde{\lambda}_i| \le \sum_{i=1}^{n} |\lambda_{\mathrm{diff}_i}| \;,$$

where $\lambda_{\mathrm{diff}_i}$ are the eigenvalues of H_{diff}.

Tying up loose ends.

By **Gershgorin**,

$$\sum_{i=1}^{n} |\lambda_i - \tilde{\lambda}_i| \leq \sum_{i=1}^{n} O\left(\frac{1}{\sqrt{n}\sqrt{i}}\right) \ ,$$

and thus by **Chebyshev**,

$$\Pr\left[\left|\sum_{i=1}^{n} h(\lambda_i) - \sum_{i=1}^{n} h(\tilde{\lambda}_i)\right| \geq C\right] \ \leq \ h_{\mathcal{L}} \frac{c}{C^2} \ .$$

Wrap-up.

We obtained a (slightly) stronger CLT for β-Hermite and β-Laguerre ensembles, using a matrix-based analysis.

The largest class of functions we could define the Gaussian process on is **Lipschitz** on $[-1-\epsilon, 1+\epsilon]$, for some $\epsilon > 0$.

Natural technical condition: $\sigma^2(h) < \infty$.

Can regularity be relaxed to match this?...