Applications of Random Matrices to Complex Systems

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Outline

- **setting the seen**
  - ***crude* definitions of random matrix theory and complex systems**
  - **complex systems as networks**
  - **a few examples of applications of random matrices to complex systems**

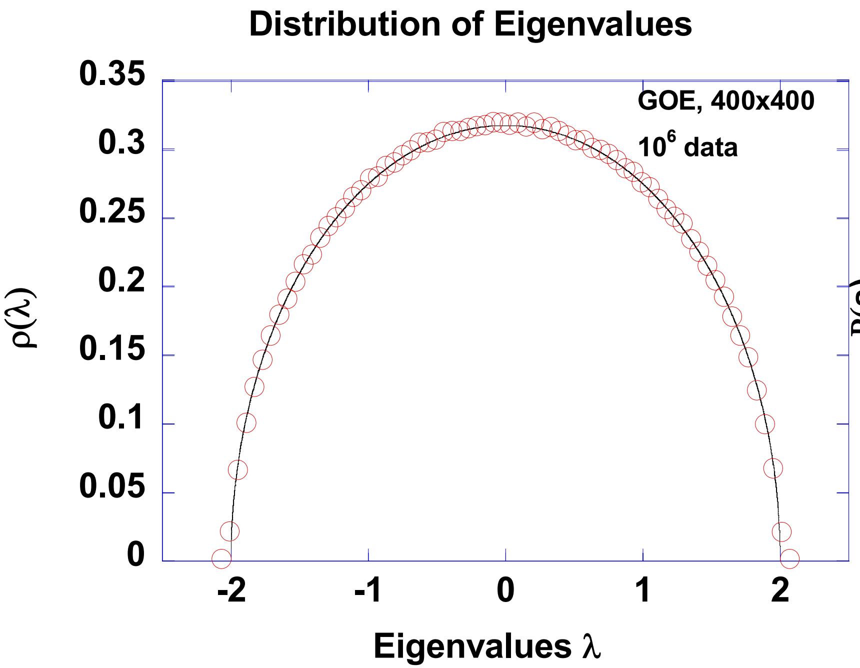
- **a generic model to study coupled systems or a system with modules**
  - **No inter-coupling**
  - **Contrasted intra-coupling**
  - **Strong inter-coupling**

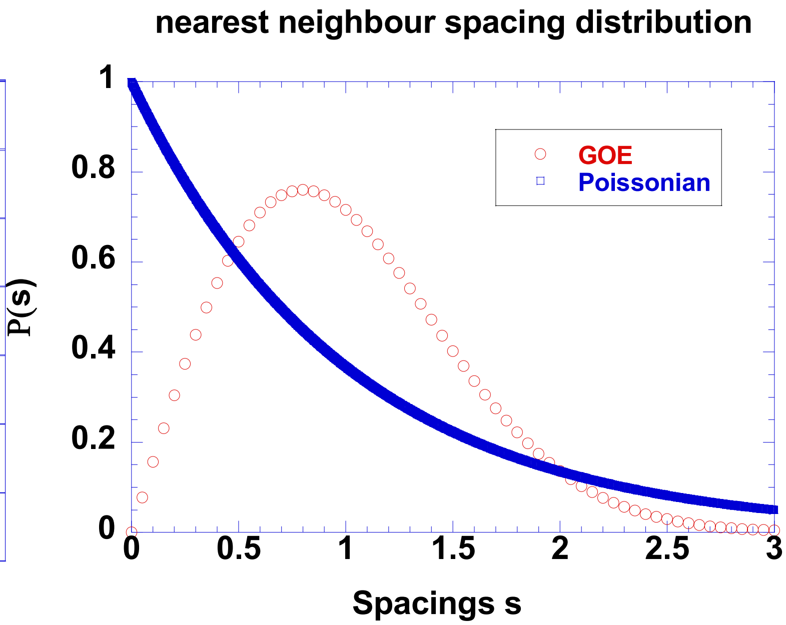
- **analytical steps**
- **remarks**

crude definition of random matrix theory (RMT)

RMT is a method of studying some statistical properties of *complex systems*, by defining an ensemble of large random matrices, in which all possible laws of interactions are equally likely.

two most studied statistics





one important point about random matrices: the matrix elements are i.i.d but the eigenvalues are correlated

crude definition of a complex system

A system that has a large number of variables, processes, or components with nontrivial interactions leading to an emergent behaviour is said to be complex.

a few examples of complex systems (with no particular order):

interactions of ecological species

social interactions

Interactions of gas particles in a box

motion of a single particle in a box

human brain

electrical, mechanical and thermal properties of composite materials

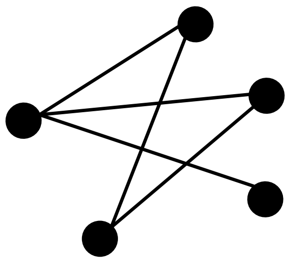
standing waves on infinite depth

and the list goes on

one way of representing complex systems is the use of graphs

A graph is a collection of vertices (points)

and edges (lines)

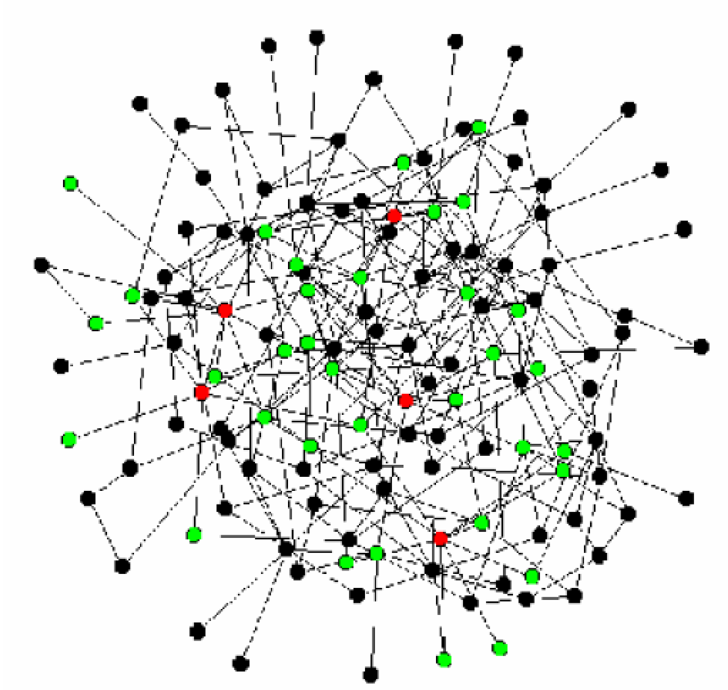


***Vertices* denote the components of the systems, such as: the webpages in the www, the individuals in social networks, gas particles etc.**

***Edges* give the connections, relationships, or interactions of these components**

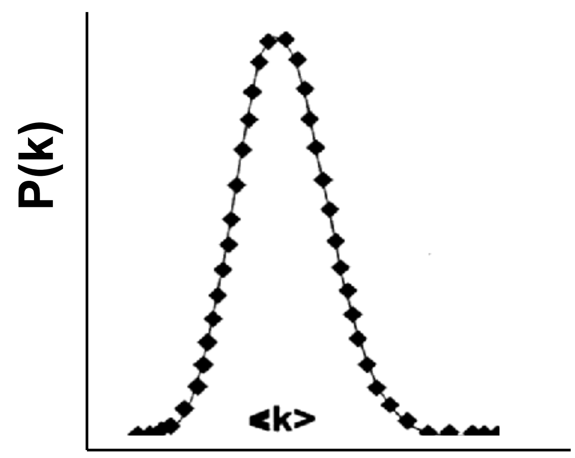
Some systems are built by random interactions

- **take n number of vertices**
- **connect them at random with some probability**



Homogeneous system

exponential distribution

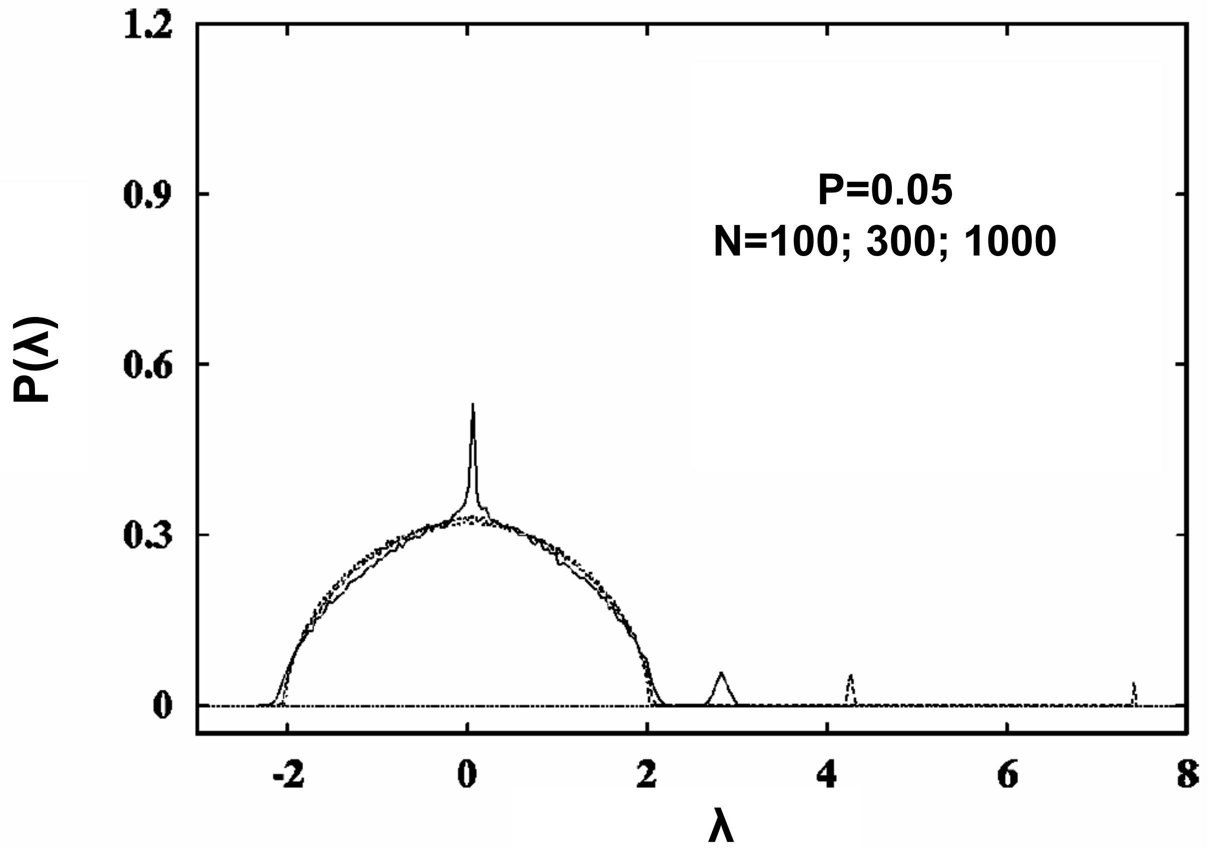


Barabasi A.-L. and Albert R., Emergence of scaling in random networks, Science 286, 509 (1999).

spectrum of random networks with undirected links

Adjacency matrix A of an undirected (bidirectional) random graph defined as:

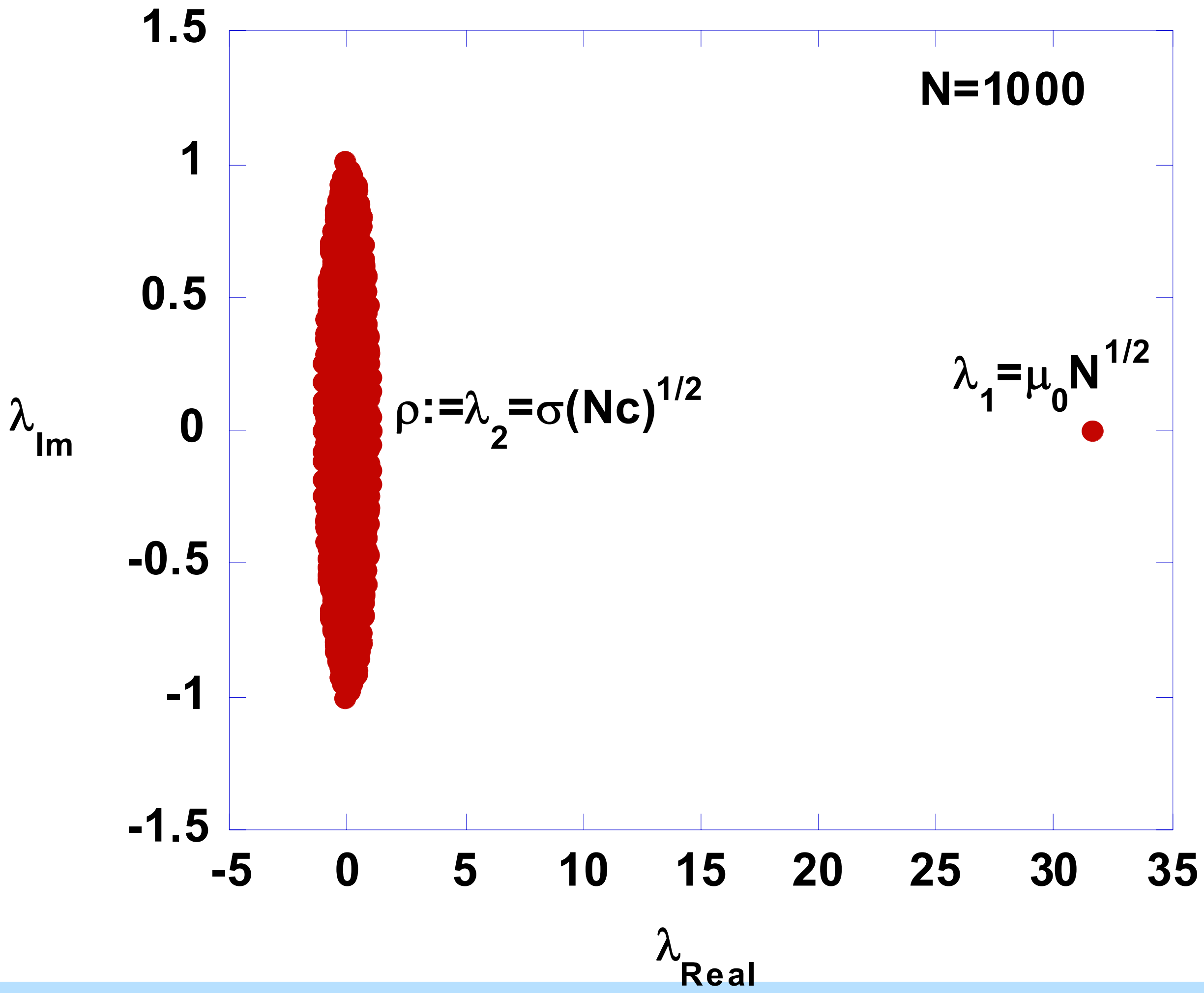
$a_{ij}=1$ if i and j are connected, 0 otherwise, and $a_{ii}=0$ (no self loops)



Farkas et al, Spectra of "real-world" graphs: Beyond the semicircle law, Phys. Rev. E, 64, 026704 (*figure modified*)

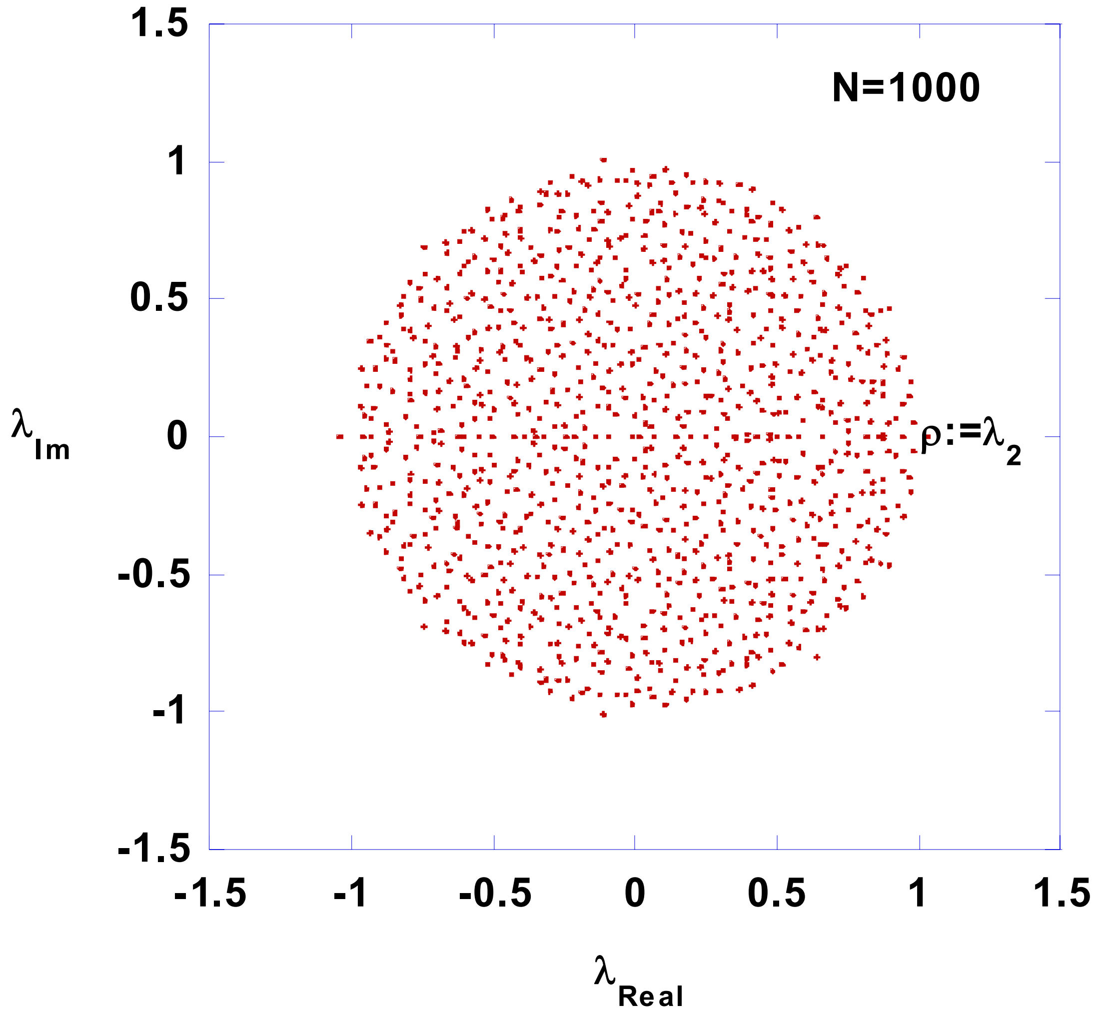
Eigenvalues of random networks with directed links

Random Gaussian asymmetric mean shifted



Eigenvalues of random networks with directed links

Random Gaussian asymmetric

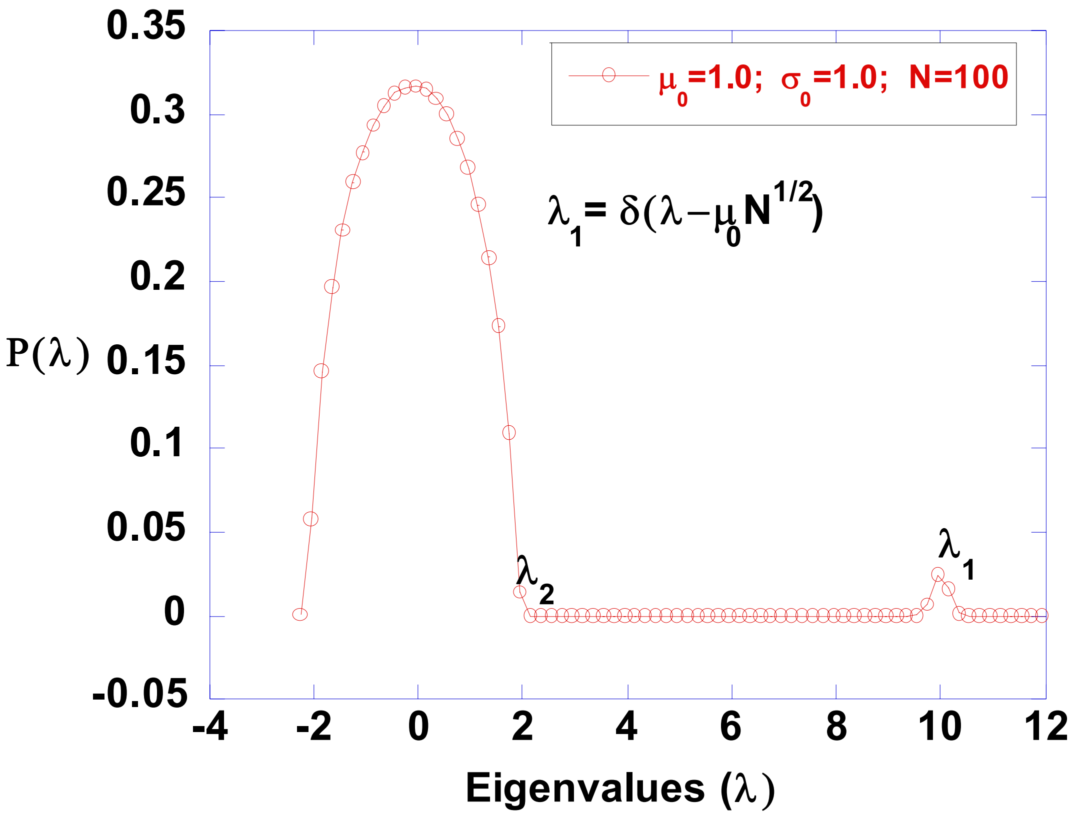


Related theorems:

Peron-Frobenius Theorem: If A is a positive matrix, then there exists a unique eigenvalue of A, λ_1 which has the greatest absolute value, and its associated eigenvector may be taken as positive.

Girko's Circular law: Eigenvalues of a set of NxN random real matrices(i.i.d elements taken from a standard normal distribution) are uniformly distributed on the unit disk in the complex plane as $\mathbf{n} \to \infty$

Gaussian symmetric with mean shifted



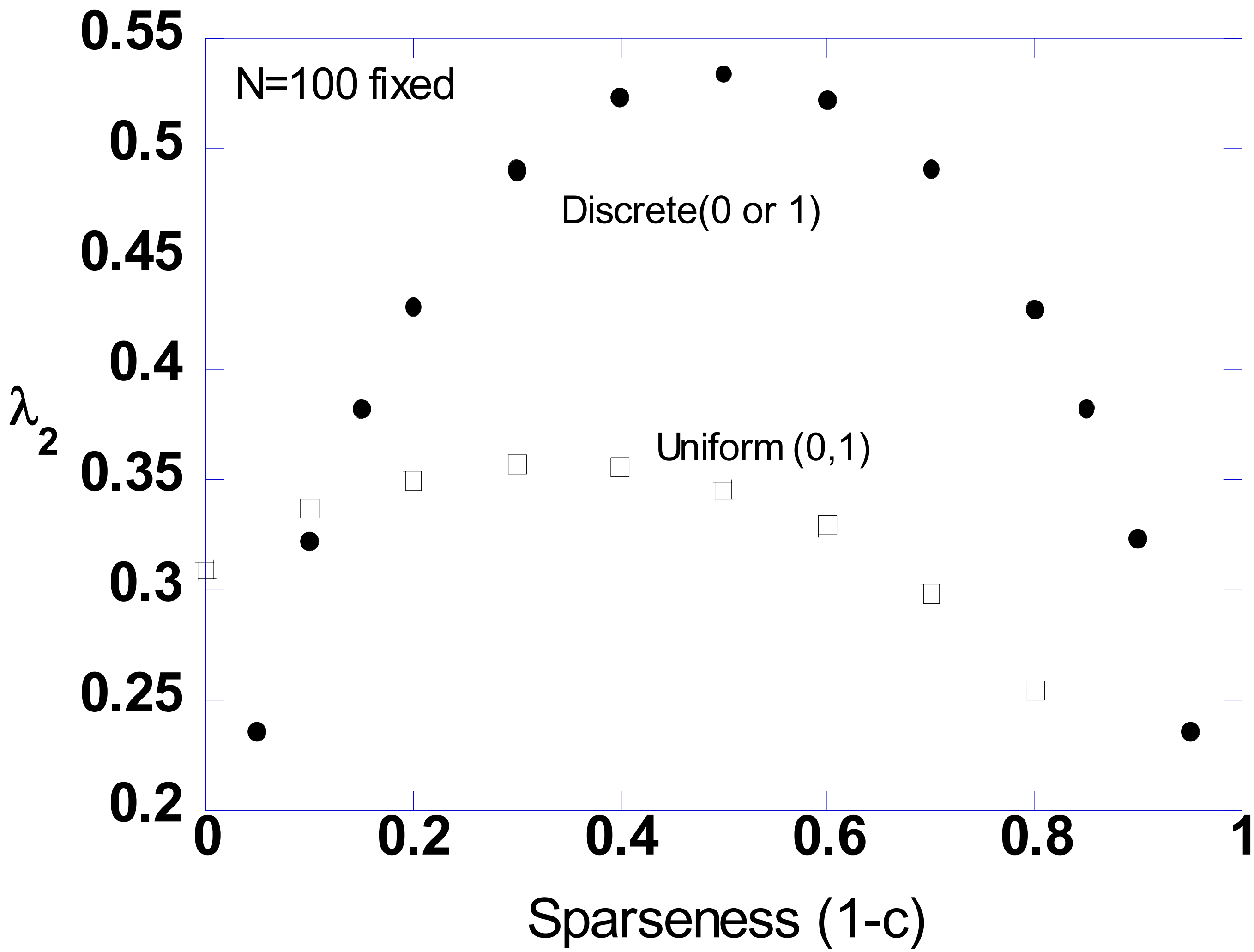
In uncorrelated random graphs

λ_1 gives the average of what ever the edges are describing

λ_2 related to complexity, stability or conductivity of the system of interest

Ergün G., "When does the large eigenvalue separate from the bulk of the spectrum"

How does the diameter (or the second largest eigenvalue) change with sparseness (1-connectivity)?



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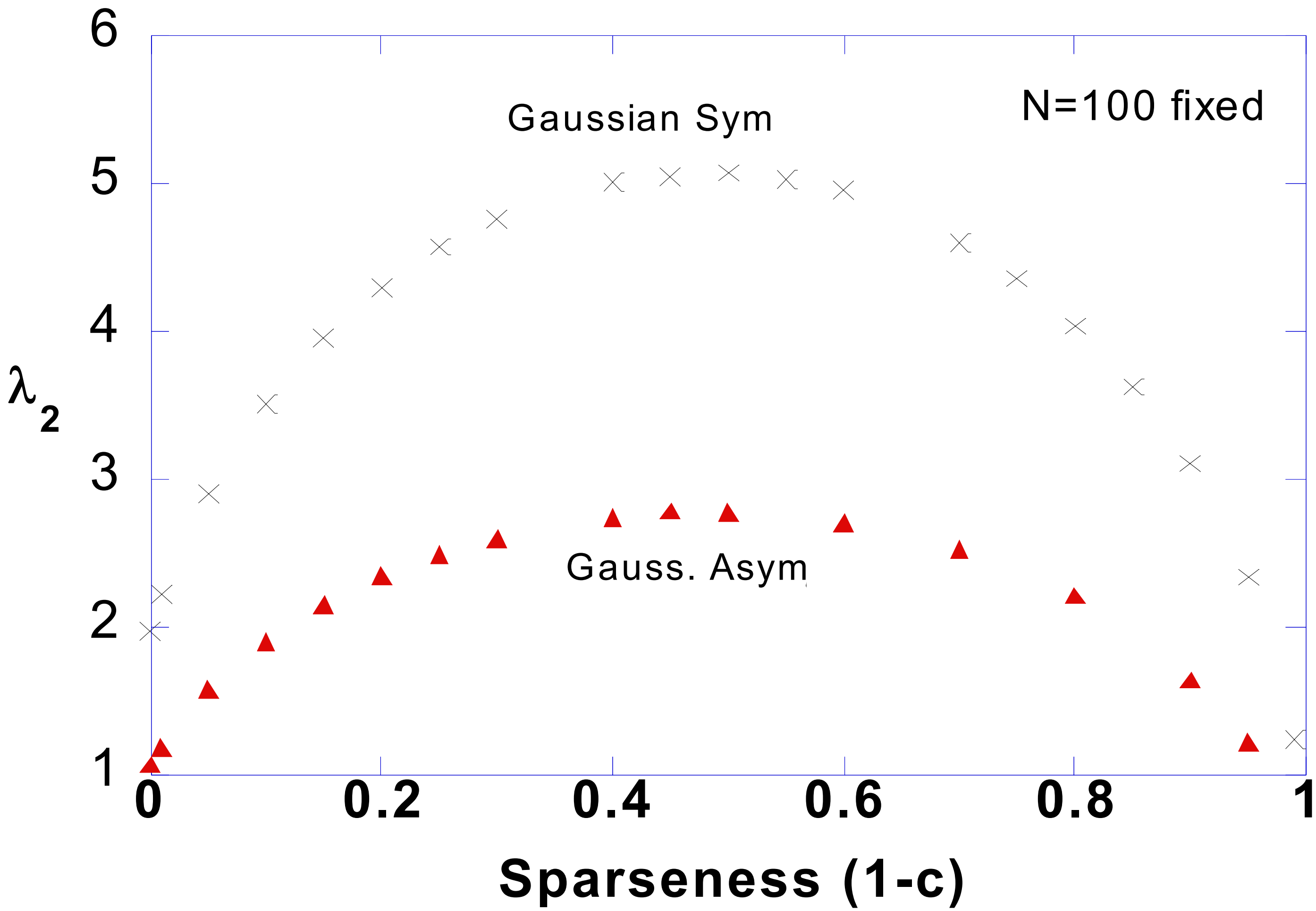
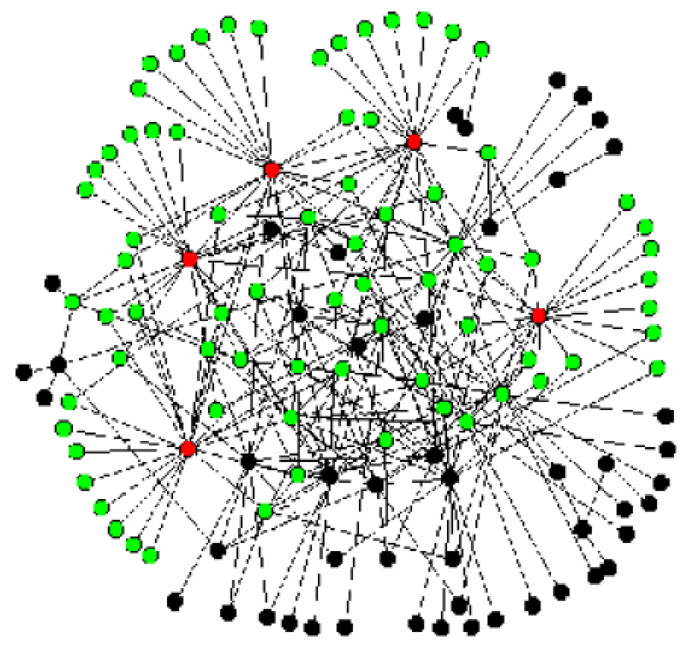
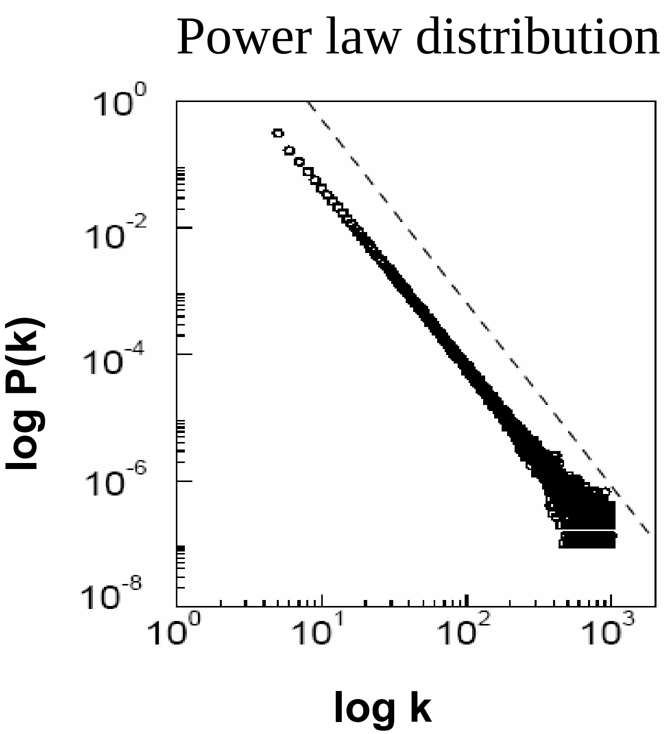


some systems have hierarchical structures

- **Number of vertices are not fixed**
- **Vertices with high number of edges more likely to acquire new edges**

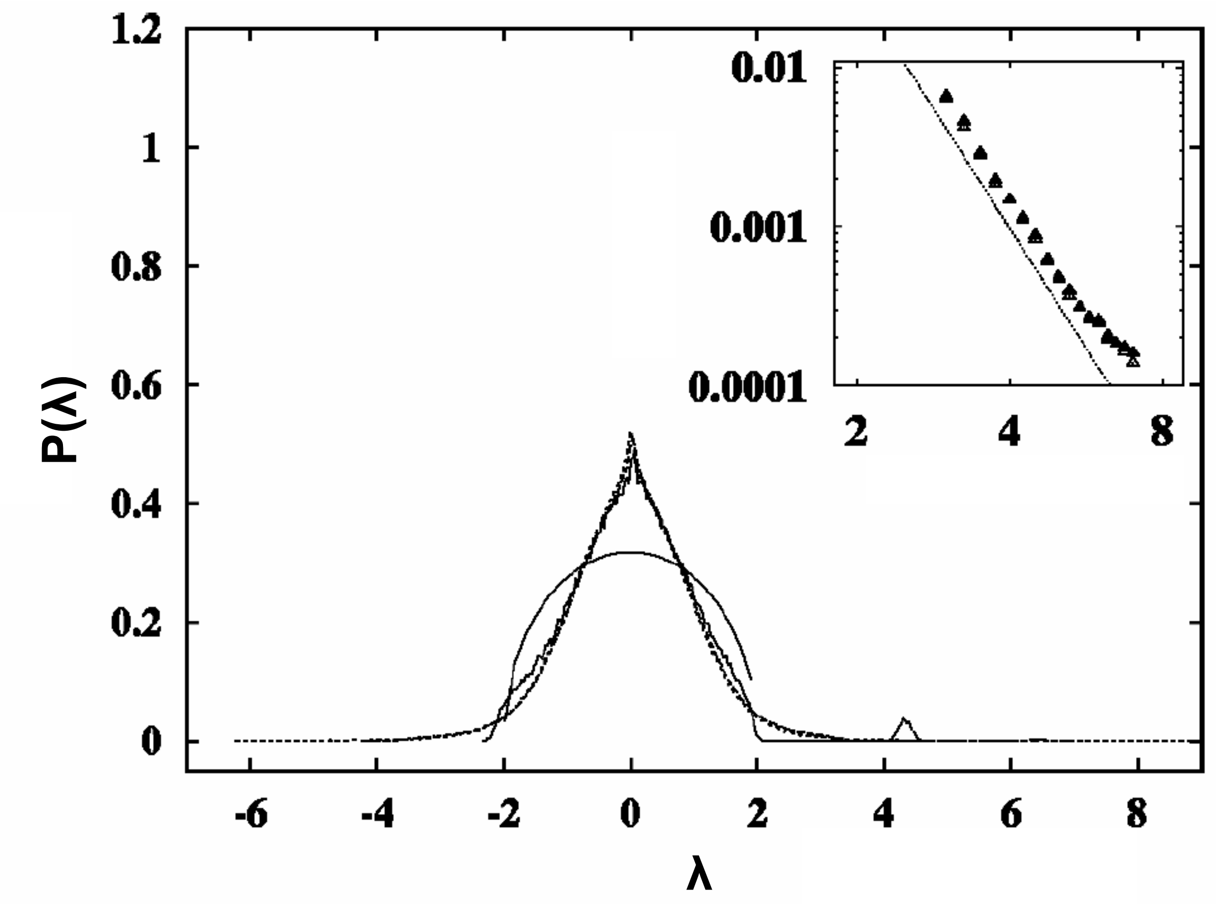




Barabasi A.-L. and Albert R., Emergence of scaling in random networks, Science 286, 509 (1999).

spectrum of scale free networks

Numerical studies show triangular distribution



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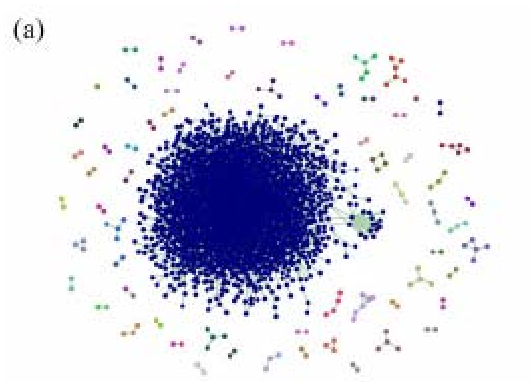
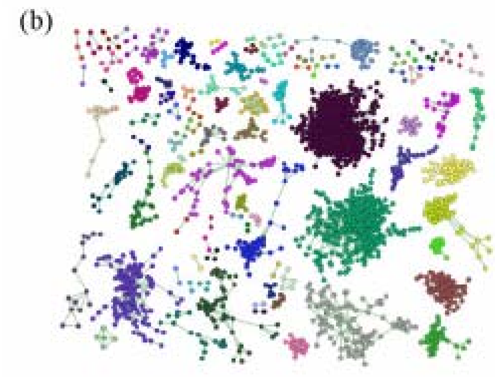
protein-protein interactions in yeast

Each point represents a different protein and each line indicates that the two proteins are capable of binding to one another. Only the largest cluster, which contains ~78% of all proteins, is shown. The colour of a node signifies the phenotypic effect of removing the corresponding protein (red, lethal; green, non-lethal; orange, slow growth; yellow, unknown).

H. Jeong, S. P. Mason, A.-L. Barabási & Z. N. Oltvai, "Lethality and centrality in protein networks", Nature, Vol 411, p41, 3 May 2001

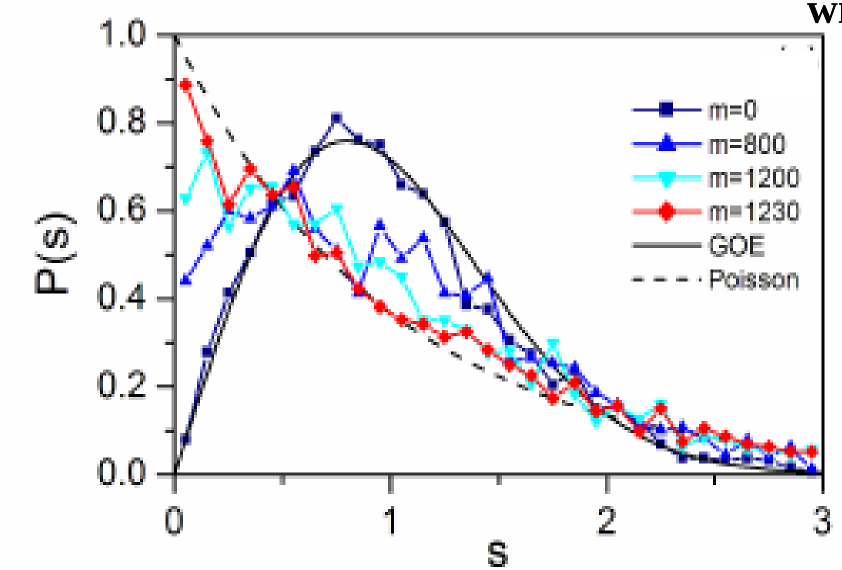
motivational example: application of RMT to biological systems





(a) yeast core protein interaction network.

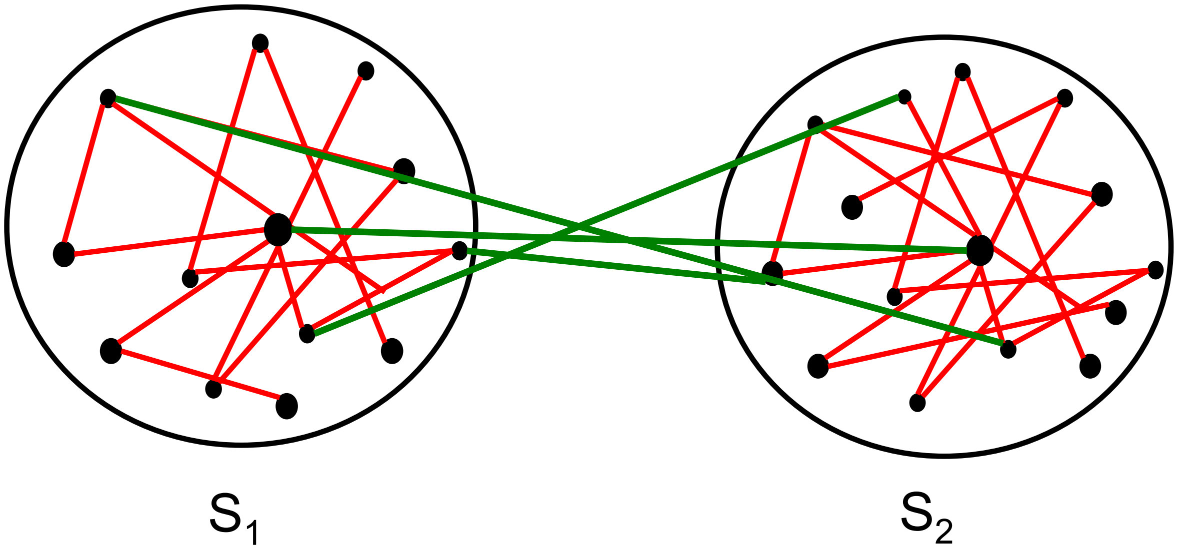
(b) yeast core protein interaction network with 1230 links removed



q-bio.MN/0503035

A generic model to study coupled systems or a system with modules

Two systems S_1 and S_2 with intra interactions σ_1 and σ_2 respectively



Introduce inter interactions with probability c and strength σ_v

Matrix representation

A block diagonal matrix perturbed by an off diagonal sparse matrix

$$\mathcal{M} = \hat{A} + \hat{B}$$

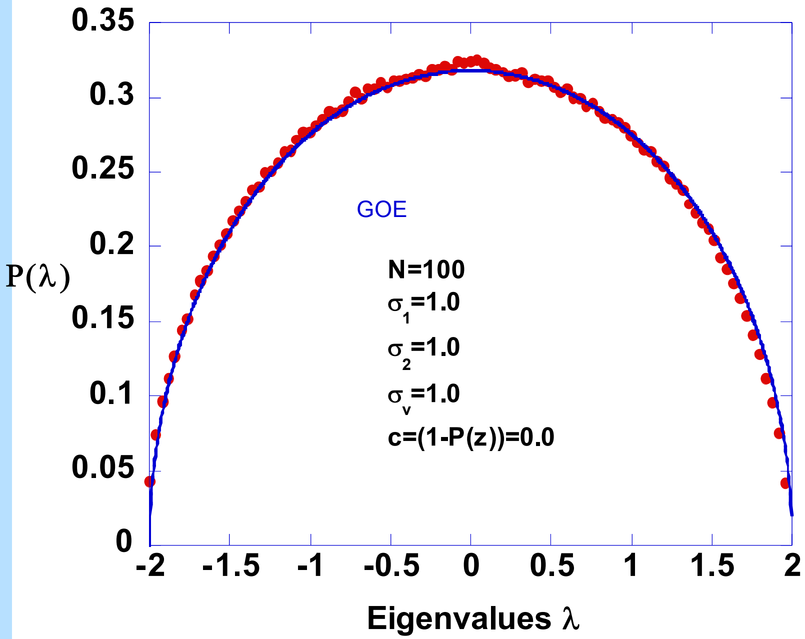
$$\hat{A} = \begin{pmatrix} \hat{S}_1 & \oslash \\ \oslash & \hat{S}_2 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \oslash & \hat{V} \\ \hat{V}^{\mathrm{T}} & \oslash \end{pmatrix}$$

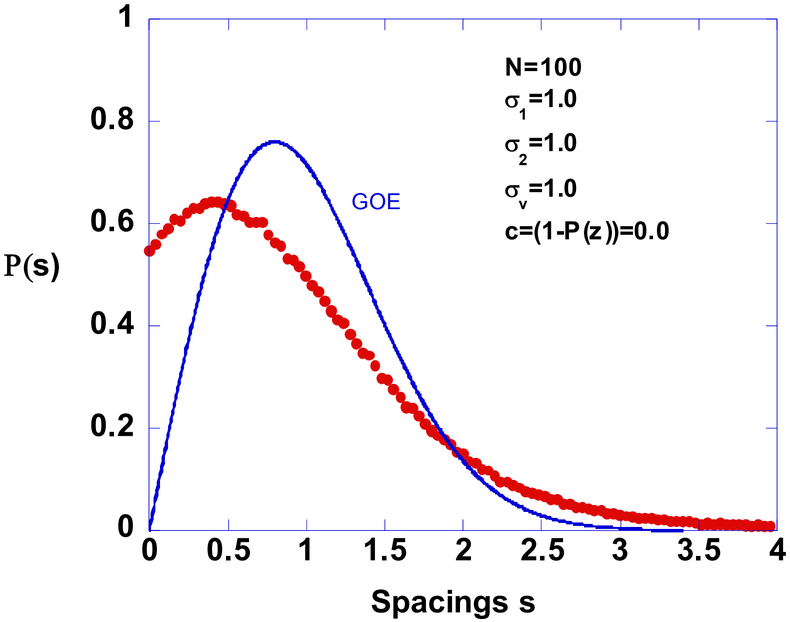
- **all elements are i.i.d. from a Gaussian distribution,**
- **all V_{ij} are non-zero with probability c**

Case 1: no coupling; c=0 and σ1 = σ2,

the sequence of eigenvalues are not mixed

$$\mathcal{M} = \begin{pmatrix} \lambda_1^1 & & & & & \\ & \ddots & & & & \\ & & \lambda_N^1 & & & \\ & & & \lambda_1^2 & & \\ & & & & \ddots & \\ & & & & & \lambda_N^2 \end{pmatrix}$$

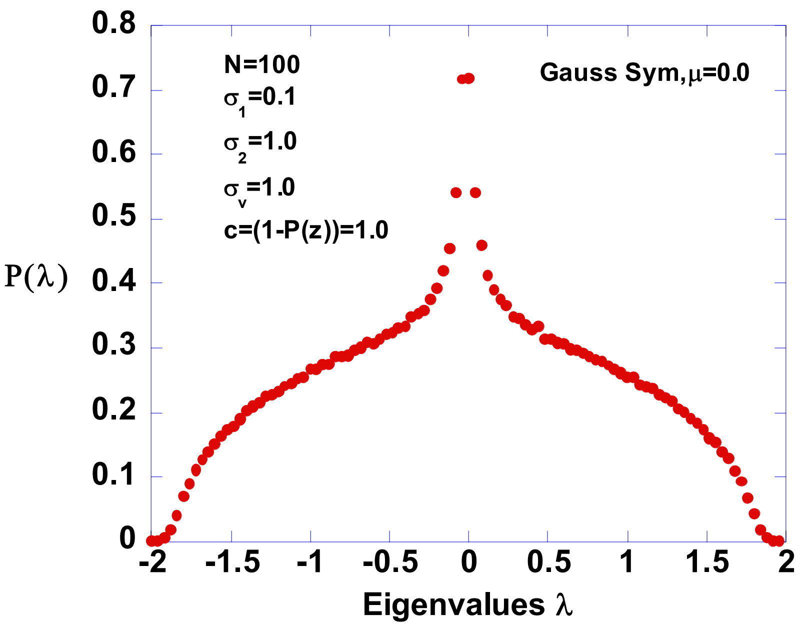


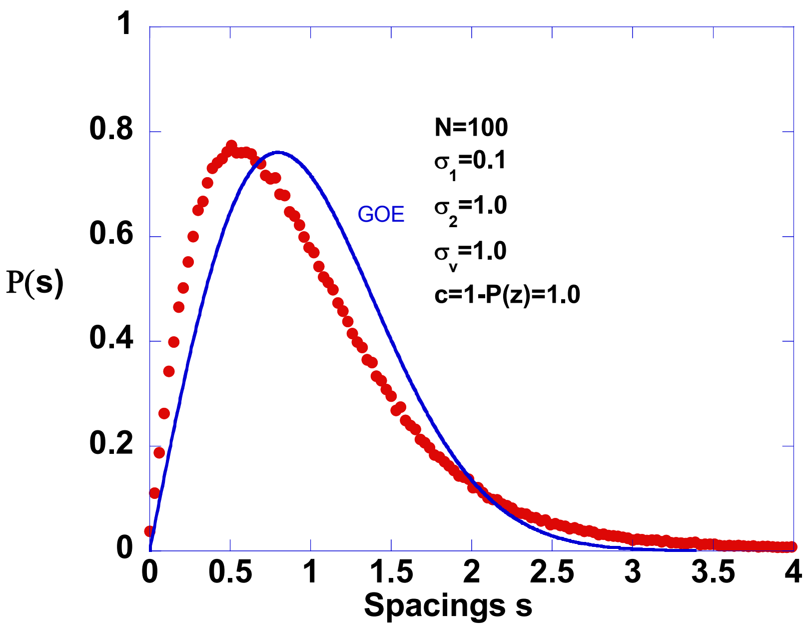


Case 2: full coupling; c=1 but $\sigma_2 >> \sigma_1$,

eigenvalues are correlated

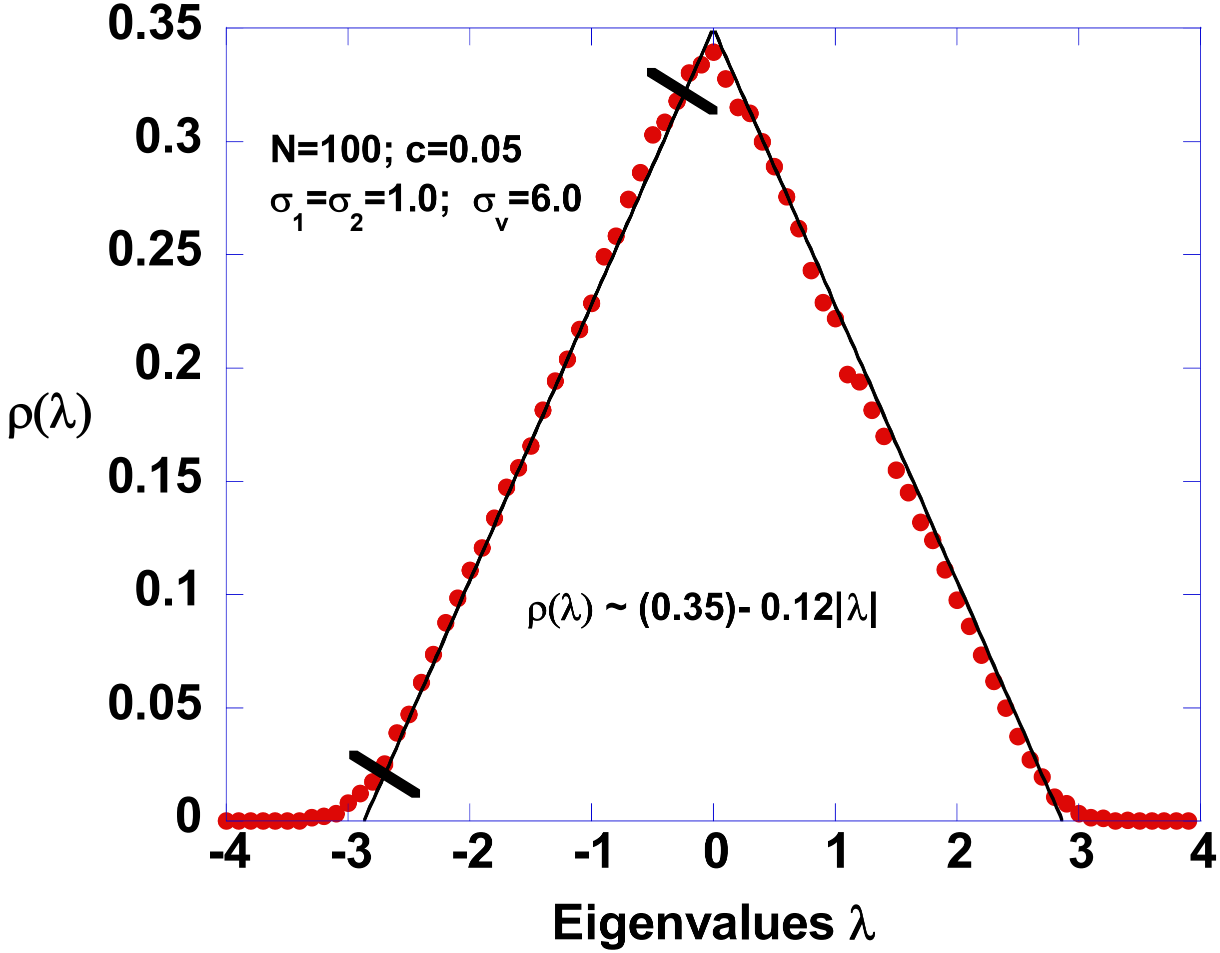
$$\mathcal{M} = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_N & & & \\ & & & \lambda_{N+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_{2N} \end{pmatrix}$$





Case 3: partial coupling; c=0.05, and $\sigma_v >> \sigma_1 = \sigma_2$

very few strong inter coupling between S_1 and S_2



Ergün G., “Random Matrix Spectra: Semicircle to Triangular Distribution”

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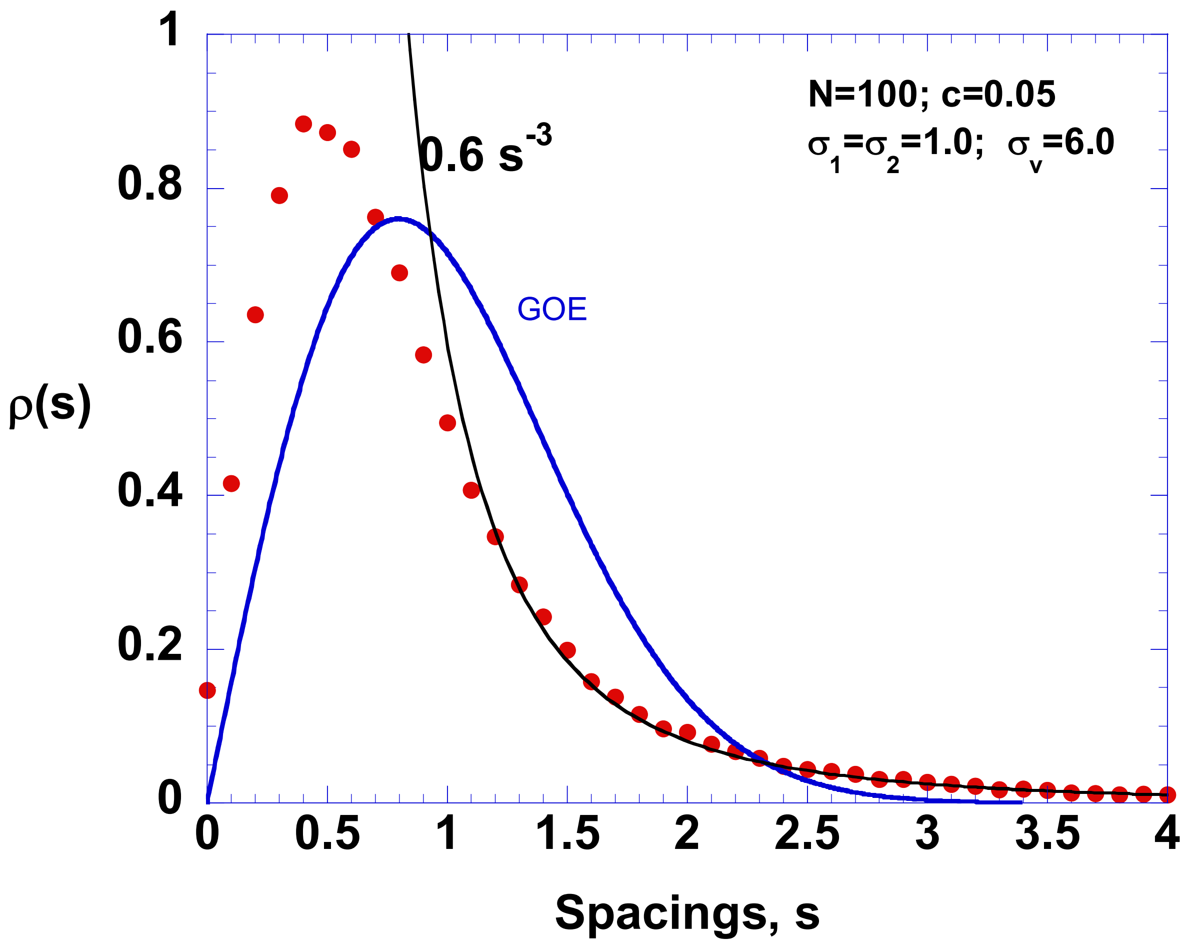

Analytical steps

The matrix M considered here is 2N x2N real symmetric.
Normalised spectrum of M will be

$$\rho(\lambda) = \frac{1}{2N} \sum_{k=1}^{2N} \delta(\lambda - \lambda_k)$$

representing the delta function as a Lorentzian curve with vanishing widths

$$\delta(\lambda - \lambda_k) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \frac{1}{(\lambda - \lambda_k)^2 + \varepsilon^2}$$

substituting this latter relation to the former and some manipulations give

$$\rho(\lambda) = \frac{1}{2N\pi} \mathrm{Im}\mathrm{Tr} \left(\frac{1}{\lambda \hat{I} - \mathcal{M}} \right)$$

Analytical steps

To proceed with the supersymmetric calculations we introduce a generating function

$$\mathcal{Z}(\Lambda,\Lambda_b) \equiv \frac{\det^{1/2}(\Lambda\hat{I}-\mathcal{M})}{\det^{1/2}(\Lambda_b\hat{I}-\mathcal{M})}$$

So that

$$\frac{\partial}{\partial x}\mathcal{Z}(\Lambda,\Lambda_b)|_{x\to 0} = \frac{1}{2N}\mathrm{ImTr}\left(\frac{1}{\Lambda\hat{I}-\mathcal{M}}\right)$$

$$\Lambda_b \equiv \Lambda \pm ix/2N.$$

$$\frac{\det^{1/2}(\Lambda\hat{I}-\mathcal{M})}{\det^{1/2}(\Lambda_b\hat{I}-\mathcal{M})} \equiv \frac{\det(\Lambda\hat{I}-\mathcal{M})}{\det^{1/2}(\Lambda_b\hat{I}-\mathcal{M})\det^{1/2}(\Lambda\hat{I}-\mathcal{M})}$$

$$\det(\Lambda\hat{I}-\mathcal{M}) \sim \int d\chi^\dagger\, d\chi\, \exp\left\{\frac{i}{2}\chi^\dagger(\Lambda\hat{I}-\mathcal{M})\chi\right\}$$

$$\det^{-1/2}(\Lambda\hat{I}-\mathcal{M}) \sim \int d\mathrm{x}_1\, \exp\left\{-\frac{i}{2}\mathrm{x}_1^{\mathrm{T}}(\Lambda\hat{I}-\mathcal{M})\mathrm{x}_1\right\},$$

$$\det^{-1/2}(\Lambda_b\hat{I}-\mathcal{M}) \sim \int d\mathrm{x}_2\, \exp\left\{-\frac{i}{2}\mathrm{x}_2^{\mathrm{T}}(\Lambda_b\hat{I}-\mathcal{M})\mathrm{x}_2\right\}$$

Ergün G., “Random Matrix Spectra: Semicircle to triangular distribution”

Analytical steps

$$\frac{\det^{1/2}(\Lambda \hat{I} - \mathcal{M})}{\det^{1/2}(\Lambda_b \hat{I} - \mathcal{M})} \sim \int d\boldsymbol{\chi}^\dagger \, d\boldsymbol{\chi} \, d\mathbf{x}_1 \, d\mathbf{x}_2$$

$$\exp\left\{\frac{i}{2}(\mathbf{x}_1^{\mathrm{T}}\Lambda\mathbf{x}_1 + \mathbf{x}_2^{\mathrm{T}}\Lambda_b\mathbf{x}_2 - \boldsymbol{\chi}^\dagger\Lambda\boldsymbol{\chi}) + \frac{i}{2}(\mathbf{x}_1^{\mathrm{T}}\mathcal{M}\mathbf{x}_1 + \mathbf{x}_2^{\mathrm{T}}\mathcal{M}\mathbf{x}_2 - \boldsymbol{\chi}^\dagger\mathcal{M}\boldsymbol{\chi})\right\}$$

aim is to find the average of this quantity over all realisations of S_μ and V

In the model we have $\mathcal{M} = \hat{A} + \hat{B}$ **with**

$$\hat{A} = \begin{pmatrix} \hat{S}_1 & \oslash \\ \oslash & \hat{S}_2 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \oslash & \hat{V} \\ \hat{V}^{\mathrm{T}} & \oslash \end{pmatrix}$$

$$\mathbf{x}_1 = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}; \quad \mathbf{x}_2 = \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}; \quad \boldsymbol{\chi} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Analytical steps

$$\Big\langle \cdots \Big\rangle_{\hat{S}_1,\hat{S}_2,\hat{V}} \propto \int \mathrm{d}\mathbf{y}_1\, \mathrm{d}\mathbf{y}_2 \mathrm{d}\mathbf{z}_1\, \mathrm{d}\mathbf{z}_2\, \mathrm{d}\boldsymbol{\eta}_1^\dagger\, \mathrm{d}\boldsymbol{\eta}_1 \boldsymbol{\eta}_2^\dagger\, \mathrm{d}\boldsymbol{\eta}_2 \times \exp\Bigg\{ \frac{i}{2} \Bigg[\Bigg(\mathbf{y}_1^\mathrm{T} \lambda^{(1)} \mathbf{y}_1 + \mathbf{y}_2^\mathrm{T} \lambda^{(2)} \mathbf{y}_2 \Bigg)$$
$$+ \Bigg(\mathbf{z}_1^\mathrm{T} \lambda_b^{(1)} \mathbf{z}_1 + \mathbf{z}_2^\mathrm{T} \lambda_b^{(2)} \mathbf{z}_2 \Bigg) + \Bigg(\boldsymbol{\eta}_1^\dagger \lambda^{(1)} \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2^\dagger \lambda^{(2)} \boldsymbol{\eta}_2 \Bigg) \Bigg] \Bigg\}$$
$$\times \Bigg\langle \exp\Bigg\{ \frac{i}{2} \Bigg[\Bigg(\mathbf{y}_1^\mathrm{T} \hat{S}_1 \mathbf{y}_1 + \mathbf{y}_2^\mathrm{T} \hat{V}^\mathrm{T} \mathbf{y}_1 + \mathbf{y}_1^\mathrm{T} \hat{V} \mathbf{y}_2 + \mathbf{y}_2^\mathrm{T} \hat{S}_2 \mathbf{y}_2 \Bigg)$$
$$+ \Bigg(\mathbf{Z}_1^\mathrm{T} \hat{S}_1 \mathbf{Z}_1 + \mathbf{z}_2^\mathrm{T} \hat{V}^\mathrm{T} \mathbf{z}_1 + \mathbf{z}_1^\mathrm{T} \hat{V} \mathbf{z}_2 + \mathbf{z}_2^\mathrm{T} \hat{S}_2 \mathbf{Z}_2 \Bigg)$$
$$- \Bigg(\boldsymbol{\eta}_1^\dagger \hat{S}_1 \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2^\dagger \hat{V} \mathrm{T} \boldsymbol{\eta}_1 + \boldsymbol{\eta}_1^\dagger \hat{V} \boldsymbol{\eta}_2 + \boldsymbol{\eta}_2^\dagger \hat{S}_2 \boldsymbol{\eta}_2 \Bigg) \Bigg] \Bigg\} \Bigg\rangle_{\hat{S}_1,\hat{S}_2,\hat{V}}$$

Analytical steps

the joint probability distribution of S*μ* where *μ*=1,2 and V

$$\mathrm{P}(\hat{S}_\mu) = \left(\frac{N}{2\pi j_\mu^2}\right)^{N(N+1)/2} \exp\left\{-\frac{N}{2j_\mu^2}\mathrm{Tr}\hat{S}^2\right\}$$

$$\widetilde{\mathrm{P}}(\hat{V}) = (1-c)\delta(\hat{V}) + c\mathrm{P}(\hat{V})$$ where

$$\mathrm{P}(\hat{V}) = \left(\frac{N}{2\pi j_v^2}\right)^{N^2/2} \exp\left\{-\frac{N}{2j_v^2}\mathrm{Tr}\hat{V}^{\mathrm{T}}\hat{V}\right\}$$

Summary and remarks

Spectra of random matrices can be used to study structures of various modular complex networks.

If there are no interaction between systems/modules then the spacing distribution of the consecutive eigenvalues will indicate this

If there are contrasted intra- interactions within systems then the density of states should be also looked at

Full couplings (inter interactions) between two systems of contrasted intra-interactions changes the Semi Circular shape of the density of states (DOS) to an Onion Dome shape. Whereas a few but strong couplings yield a triangular shape