Universality of distribution functions
in random matrix theory

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Overview

- Universality
  - Local eigenvalue statistics
  - Fluctuations of the largest eigenvalue
- Connections outside RMT
  - Zeros of Riemann zeta function
  - Non-intersecting Brownian paths
  - Tiling problem
- Unitary ensembles
  - Determinantal point process
  - Precise formulation of universality
  - Universality in regular cases
- Universality classes in singular cases
  - Singular cases I and II and Painlevé equations
  - Spectral singularity
Simplest ensembles are **Gaussian ensembles**.

Matrix entries have normal distribution with mean zero. The entries are independent up to the constraints that are imposed by the symmetry class.

- **Gaussian Unitary Ensemble** \( \text{GUE} \): complex Hermitian matrices
- **Gaussian Orthogonal Ensemble** \( \text{GOE} \): real symmetric matrices
- **Gaussian Symplectic Ensemble** \( \text{GSE} \): self-dual quaternionic matrices

Where are the eigenvalues?
Wigner’s semi-circle law

△ Histogram of eigenvalues of large Gaussian matrix, size $10^4 \times 10^4$

△ After scaling of eigenvalues with a factor $\sqrt{n}$, there is a limiting mean eigenvalue distribution, known as Wigner’s semi-circle law

△ This is special for Gaussian ensembles (non-universal). Other limiting distributions for Wishart ensembles, Jacobi ensembles,...
Universality 1: Local eigenvalue statistics

▲ Global statistics of eigenvalues depend on the particular random matrix ensemble in contrast to local statistics. Distances between consecutive eigenvalues show regular behavior.

▲ Rescale eigenvalues around a certain value so that mean distance is one.

plot shows only a few rescaled eigenvalues of a very large GUE matrix

▲ This is the same behavior as seen in energy spectra in quantum physics.

▲ The repulsion between neighboring eigenvalues is very different from Poisson spacings.
Universality 1: Local eigenvalue statistics

- This local behavior of eigenvalues is not special for GUE.

- It holds for large class of unitary ensembles
  \[
  \frac{1}{Z_n} e^{-n \text{Tr} V(M)} dM
  \]
  these are ensembles that have the same symmetry property as GUE.

  Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

- Local eigenvalue statistics is different for GOE and GSE which have different symmetry properties. Proof of universality for orthogonal and symplectic ensembles is more recent result

  Deift, Gioev (arxiv 2004)

- Universality fails at special points, such as end points of the spectrum, or points where eigenvalue density vanishes.

  This gives rise to new universality classes.
Fluctuations of the largest eigenvalues of random matrices also show a universal behavior (depending on the symmetry class).

For $n \times n$ GUE matrix, the largest eigenvalue grows like $\sqrt{2n}$ and has a standard deviation of the order $n^{-1/6}$.

Centered and rescaled largest eigenvalue
\[ \sqrt{2n^{1/6}} \left( \lambda_{\text{max}} - \sqrt{2n} \right) \]

converges in distribution as $n \to \infty$ to a random variable with the Tracy-Widom distribution, described by Tracy, Widom in 1994.

Same limit holds generically for unitary random matrix ensembles.

Different TW-distributions for orthogonal and symplectic ensembles.
Tracy-Widom distribution

There is no simple formula for the Tracy-Widom distribution $F(s)$.

First formula is as a Fredholm determinant:

$$F(s) = \det(I - A_s)$$

where $A_s$ is the integral operator acting on $L^2(s, \infty)$ with kernel

$$\frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y}$$

and $\text{Ai}$ is the Airy function.

Second formula

$$F(s) = \exp\left(-\int_s^\infty (x - s) q^2(x) dx\right)$$

where $q(s)$ is a special solution of the Painlevé II equation

$$q''(s) = sq(s) + 2q^3(s)$$
Distribution functions of random matrix theory appear in various other domains of mathematics and physics.

- **Number theory**
  - Riemann zeta-function, $L$-functions, ...  

- **Representation theory**
  - Young tableaux, large classical groups, ...  

- **Random combinatorial structures**
  - random permutations, random tilings, ...  

- **Growth models** in statistical physics
  - last passage percolation, polynuclear growth, ...  

- ... , as well as in applications in statistics, finance, information theory, ...
The zeta function \( \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \) has an analytic continuation to the complex plane.

The non-trivial zeros of the zeta function are believed to be on the line \( \Re s = 1/2 \).
(Riemann hypothesis)

Computational evidence: no non-real zeros have been found off the critical line.

1,500,000,000 zeros have been found on the critical line.
Riemann zeta function

The zeta function \( \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \) has an analytic continuation to the complex plane.

The non-trivial zeros of the zeta function are believed to be on the line \( \text{Re } s = 1/2 \).
(Riemann hypothesis)

More computational evidence: Spacings between consecutive zeros on the critical line \( \text{Re } s = 1/2 \) (after appropriate scaling) show the same behavior as the spacings between eigenvalues of a large GUE matrix.

Zeros of \( \zeta(s) \) on the critical line have the same local behavior as the eigenvalues of a large random matrix.
Non-intersecting Brownian motion paths

△ Take \( n \) independent 1-dimensional Brownian motions with time in \([0, 1]\) conditioned so that:

△ All paths start and end at the same point.

△ The paths do not intersect at any intermediate time.

Five non-intersecting Brownian bridges
Non-intersecting Brownian motion

**Remarkable fact:** At any intermediate time the positions of the paths have exactly the same distribution as the eigenvalues of an $n \times n$ GUE matrix (up to a scaling factor).

Positions of five non-intersecting Brownian paths behave the same as the eigenvalues of a $5 \times 5$ GUE matrix

This interpretation is basic for the connection of random matrix theory with growth models of statistical physics.
A random tiling problem

- Hexagonal tiling with rhombi.

- May also be viewed as packing of boxes in a corner.

- Take a tiling at random.
  - What does a typical tiling look like, if the number of rhombi increases?
Observation:

- frozen regions near the corners,
- disorder in the center.
Non-intersecting random walk

Consider only blue and red rhombi.
Non-intersecting random walk

- We can connect the left and the right with non-intersecting paths.

- A random tiling is the same as a number of non-intersecting random walks.

- As size increases: Tracy-Widom distribution governs the transition between frozen region and disordered region.

  Baik, Kriecherbauer, McLaughlin, Miller, (arxiv 2003)
Other examples

- Longest increasing subsequence of random permutations
  [Baik, Deift, Johansson (1999)]

- Polynuclear growth model (PNG)
  Totally asymmetric exclusion process (TASEP)
  [Praehofer, Spohn, Ferrari
   Imamura, Sasamoto]

- Buses in Cuernavaca, Mexico
  [Krbalek, Seba
   Baik, Borodin, Deift, Suidan]

- Airplane boarding problem
  [Bachmat]
Universality classes in unitary ensembles

- Probability measure on $n \times n$ Hermitian matrices
  
  $$
  \frac{1}{Z_n} e^{-n \text{Tr} V(M)} dM
  $$

  where $dM = \prod_j dM_{jj} \prod_{j<k} d\text{Re}M_{jk} d\text{Im}M_{jk}$

- This is a Gaussian ensemble for $V(x) = \frac{1}{2} x^2$

- Joint eigenvalue density
  
  $$
  \frac{1}{Z_n} \prod_{i<j} |x_i - x_j|^2 \prod_{j=1}^n e^{-nV(x_j)}
  $$

  where
It is special about unitary ensembles that the eigenvalues follow a determinantal point process. This means that there is a kernel $K_n(x, y)$ so that all eigenvalue correlation functions are expressed as determinants

$$R_m(x_1, x_2, \ldots, x_k) = \det[K_n(x_i, x_j)]_{i,j=1,\ldots,m}$$

$\int_a^b K_n(x, x) dx$ is expected number of eigenvalues in $[a, b]$

$\int_a^b \int_c^d \begin{vmatrix} K_n(x, x) & K_n(x, y) \\ K_n(y, x) & K_n(y, y) \end{vmatrix} dxdy$ is expected number of pairs of eigenvalues in $[a, b] \times [c, d]$, etc.
Orthogonal polynomial kernel

Let $P_{k,n}(x)$ be the $k$th degree monic orthogonal polynomial with respect to $e^{-nV(x)}$

$$\int_{-\infty}^{\infty} P_{k,n}(x)P_{j,n}(x)e^{-nV(x)}\,dx = h_{k,n}\delta_{j,k}.$$ 

Correlation kernel is equal to

$$K_n(x,y) = e^{-\frac{1}{2}n(V(x)+V(y))} \sum_{k=0}^{n-1} \frac{1}{h_{k,n}} P_{k,n}(x)P_{k,n}(y)$$

$$= e^{-\frac{1}{2}n(V(x)+V(y))} \frac{h_{n,n}}{h_{n-1,n}} \frac{P_{n,n}(x)P_{n-1,n}(y) - P_{n-1,n}(x)P_{n,n}(y)}{x-y}$$

Christoffel-Darboux formula
Asymptotical questions

▲ All information is contained in the correlation kernel $K_n$.

▲ Asymptotic questions deal with the global regime

$$\rho_V(x) = \lim_{n \to \infty} \frac{1}{n} K_n(x, x)$$

▲ and with the local regime

▲ Choose $x^*$ and center and scale eigenvalues around $x^*$

$$\lambda \mapsto (cn)^\gamma (\lambda - x^*)$$

▲ This is a determinantal point process with rescaled kernel

$$\frac{1}{(cn)^\gamma} K_n \left( x^* + \frac{x}{(cn)^\gamma}, x^* + \frac{y}{(cn)^\gamma} \right)$$

▲ Determine $\gamma$ and calculate limit of rescaled kernel. Limits turn out to be universal, depending only on the nature of $x^*$ in the global regime.
Global regime

\[ \text{In limit } n \to \infty, \text{ the mean eigenvalue density has a limit } \rho = \rho_V \text{ which minimizes} \]

\[ \iint \log \frac{1}{|x-y|} \rho(x) \rho(y) \, dx \, dy + \int V(x) \rho(x) \, dx \]

among density functions \( \rho \geq 0, \int \rho(x) \, dx = 1. \)

\[ \text{Equilibrium conditions} \]

\[ 2 \int \log \frac{1}{|x-y|} \rho(y) \, dy + V(x) = \text{const on support of } \rho \]

\[ 2 \int \log \frac{1}{|x-y|} \rho(y) \, dy + V(x) \geq \text{const outside support} \]
Regular and singular cases

- If $V$ is real analytic, then
  - $\text{supp}(\rho)$ is a finite union of disjoint intervals,
  - $\rho(x)$ is analytic on the interior of each interval
  - $\rho(x) \sim |x - a|^{2k + 1/2}$ at an endpoint $a$ for some $k = 0, 1, 2, \ldots$

- Regular case: positive in interior, square root vanishing at endpoints, and strict inequality in

$$2 \int \log \frac{1}{|x - y|} \rho(y) dy + V(x) > \text{const} \text{ outside the support of } \rho$$
Singular cases correspond to possible change in number of intervals if parameters in the external field $V$ change.

- **Singular case I:** $\rho$ vanishes at an interior point
- **Singular case II:** $\rho$ vanishes to higher order than square root at an endpoint.
- **Singular case III:** equality in equilibrium inequality somewhere outside the support
Local regime

▲ Limit of rescaled kernel

\[
\frac{1}{(cn)^\gamma} K_n \left( x^* + \frac{x}{(cn)^\gamma}, x^* + \frac{y}{(cn)^\gamma} \right)
\]

▲ For \( x^* \) in the bulk, we take \( \gamma = 1, c = \rho(x^*) \), and limit is the sine kernel

\[
\frac{\sin \pi(x - y)}{\pi(x - y)}
\]


Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

▲ Scaling limit of kernel follows from detailed asymptotics of the orthogonal polynomials \( P_{n,n} \) and \( P_{n-1,n} \) as \( n \to \infty \), which is available in the GUE case since then the orthogonal polynomials are Hermite polynomials.

▲ For more general cases a powerful new technique was developed: steepest descent analysis of Riemann-Hilbert problems
For a regular endpoint of the support, we take \( \gamma = \frac{2}{3} \), and the scaling limit is the \textit{Airy kernel}

\[
\frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y}
\]

This always gives rise to the Tracy Widom distribution for the fluctuations of the extreme eigenvalues.
Singular cases

Unitary ensemble

\[ \frac{1}{Z_n} e^{-n \text{Tr} V(M)} dM \]

Reference point \( x^* \)

- Regular interior point:
- Regular endpoint:
- Singular case I:
- Singular case II:
- Singular case III:
Singular cases

Unitary ensemble

\[
\frac{1}{Z_n} e^{-n \text{Tr} V(M)} dM
\]

Reference point \( x^* \)

- Regular interior point: sine kernel
- Regular endpoint: Airy kernel
- Singular case I:
- Singular case II:
- Singular case III:
Singular cases

Unitary ensemble

\[ \frac{1}{Z_n} e^{-n \text{Tr} V(M)} dM \]

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- Regular interior point: sine kernel
- Regular endpoint: Airy kernel
- Singular case I: kernels built out of \( \psi \) functions associated with the Hastings-Mcleod solution of Painlevé II

Claeys, AK (2006)

- Singular case II:

- Singular case III:
Singular cases

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    Claeys, AK (2006)

- Singular case II: second member of Painlevé I hierarchy
  
    Claeys, Vanlessen (in progress)

- Singular case III:
Singular cases

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- Singular case III: ??
Spectral singularity

Extra factor in random matrix model

\[
\frac{1}{Z_n} \left| \det M \right|^{2\alpha} e^{-n \operatorname{Tr} V(M)} dM
\]

The extra factor does not change the global behavior but it does change the local behavior around the reference point \( x^* = 0 \)

- Regular interior point:
- Regular endpoint:

- Singular case I:
- Singular case II:
- Singular case III:
Extra factor in random matrix model

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\frac{1}{Z_n} |\det M|^{2\alpha} e^{-n \text{Tr} V(M)} dM
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The extra factor does not change the global behavior but it does change the local behavior around the reference point \( x^* = 0 \)

- **Regular interior point:** Bessel kernel \( \text{AK, Vanlessen (2003)} \)
- **Regular endpoint:**

- **Singular case I:**
- **Singular case II:**
- **Singular case III:**
Spectral singularity

Extra factor in random matrix model

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- **Regular interior point:** **Bessel kernel**
  - AK, Vanlessen (2003)

- **Regular endpoint:** general **Painlevé II with parameter** \( 2\alpha + \frac{1}{2} \)
  - Its, AK, Östensson (in progress)

- **Singular case I:**

- **Singular case II:**

- **Singular case III:**
Spectral singularity

Extra factor in random matrix model

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The extra factor does not change the global behavior but it does change the local behavior around the reference point \( x^* = 0 \)

- **Regular interior point:** Bessel kernel  
  \[ q'' = s q + 2 q^3 - 2 \alpha - \frac{1}{2} \]  
  AK, Vanlessen (2003)

- **Regular endpoint:** general Painlevé II with parameter \( 2\alpha + \frac{1}{2} \)  
  Its, AK, Östensson (in progress)

- **Singular case I:** Painlevé II with parameter \( \alpha \)  
  Claeys, AK, Vanlessen (arxiv 2005)

- **Singular case II:**

- **Singular case III:**
Spectral singularity

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- **Singular case I:** Painlevé II with parameter \( \alpha \)
  - Claeys, AK, Vanlessen (arxiv 2005)

- **Singular case II:** ??

- **Singular case III:** ??
Singular case I

- Quartic external field $V(x) = \frac{1}{4}x^4 - x^2$ is simplest singular case I.

- Transition from two-interval to one-interval. If

\[ V_t(x) = \frac{1}{t} \left( \frac{1}{4}x^4 - x^2 \right) \]

then for $t < 1$: two intervals, and for $t > 1$: one interval

- Consider singular case in double scaling limit where we rescale eigenvalues $\lambda \mapsto (c_1 n)^{1/3}(\lambda - x^*)$ and we let $t \to 1$ as $n \to \infty$

\[ n^{2/3}(t - 1) = c_2 s \]
Double scaling limit in singular case I

- One-parameter family of limiting kernels, depending on $s$, but independent of $V$
  \[ \frac{\Phi_1(x; s)\Phi_2(y; s) - \Phi_2(x; s)\Phi_1(y; s)}{2\pi i(x - y)} \]

- $\Phi_1$ and $\Phi_2$ satisfy a differential equation
  \[
  \frac{d}{dx} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} -4ix^2 - i(s + 2q^2) & 4xq + 2ir \\ 4xq - 2ir & 4ix^2 + i(s + 2q^2) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}
  \]
  with parameters $s$, $q$ and $r$ that are such that $q = q(s)$ satisfies Painlevé II:
  \[ q'' = sq + 2q^3 \text{ and } r = r(s) = q'(s). \]
  for real analytic $V$: Claeys, AK (arxiv 2005)
  for less smooth, even $V$: Shcherbina (arxiv 2006)

- Our proof uses the fact that $\Phi_1$ and $\Phi_2$ solve a RH problem that can be used as a local parametrix in the steepest descent analysis.