Universality of distribution functions in random matrix theory

Arno Kuijlaars

Katholieke Universiteit Leuven, Belgium

Overview

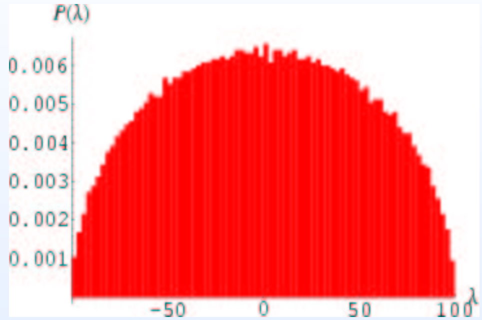
- Universality
  - Local eigenvalue statistics
  - Fluctuations of the largest eigenvalue
- Connections outside RMT
  - Zeros of Riemann zeta function
  - Non-intersecting Brownian paths
  - Tiling problem
- Unitary ensembles
  - Determinantal point process
  - Precise formulation of universality
  - Universality in regular cases
- Universality classes in singular cases
  - Singular cases I and II and Painlevé equations
  - Spectral singularity

Gaussian ensembles

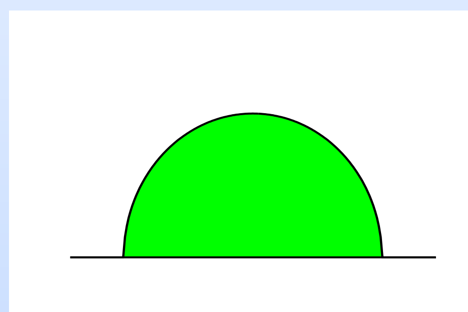
- **Simplest ensembles are Gaussian ensembles.**
- **Matrix entries have normal distribution with mean zero. The entries are independent up to the constraints that are imposed by the symmetry class.**
  - **Gaussian Unitary Ensemble GUE: complex Hermitian matrices**
  - **Gaussian Orthogonal Ensemble GOE: real symmetric matrices**
  - **Gaussian Symplectic Ensemble GSE: self-dual quaternionic matrices**
- **Where are the eigenvalues?**

Wigner's semi-circle law

- Histogram of eigenvalues of large Gaussian matrix, size $10^4 \times 10^4$



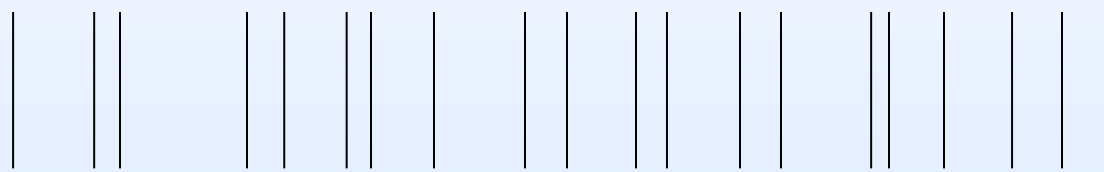
- After scaling of eigenvalues with a factor $\sqrt{n}$, there is a limiting mean eigenvalue distribution, known as **Wigner's semi-circle law**



- This is special for Gaussian ensembles (non-universal). Other limiting distributions for Wishart ensembles, Jacobi ensembles,...

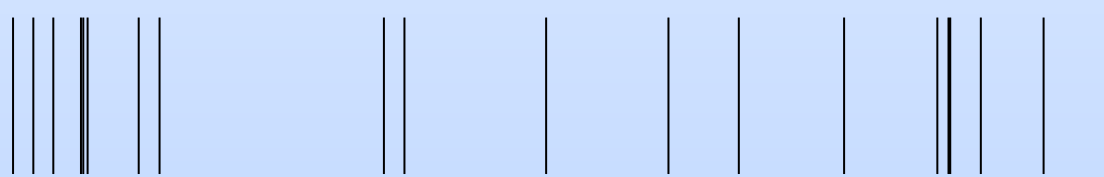
Universality 1: Local eigenvalue statistics

- **Global statistics of eigenvalues depend on the particular random matrix ensemble in contrast to local statistics. Distances distances between consecutive eigenvalues show regular behavior.**
- **Rescale eigenvalues around a certain value so that mean distance is one.**



plot shows only a few rescaled eigenvalues of a very large GUE matrix

- **This is the same behavior as seen in energy spectra in quantum physics.**
- **The repulsion between neighboring eigenvalues is very different from Poisson spacings.**



Universality 1: Local eigenvalue statistics

- This local behavior of eigenvalues is not special for GUE.
  - It holds for large class of unitary ensembles $\boxed{\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM}$

    these are ensembles that have the same symmetry property as GUE.

    Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)
- Local eigenvalue statistics is different for GOE and GSE which have different symmetry properties. Proof of universality for orthogonal and symplectic ensembles is more recent result Deift, Gioev (arxiv 2004)
- Universality fails at special points, such as end points of the spectrum, or points where eigenvalue density vanishes.
  - This gives rise to new universality classes.

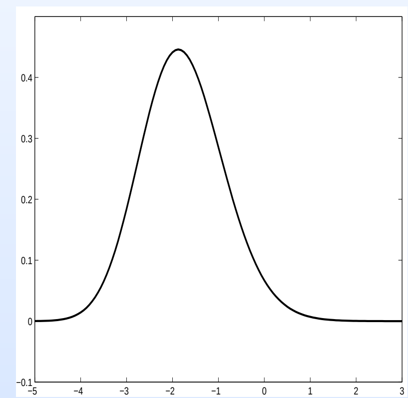
Universality 2: Largest eigenvalue

- Fluctuations of the largest eigenvalues of random matrices also show a universal behavior (depending on the symmetry class).
- For $n \times n$ GUE matrix, the largest eigenvalue grows like $\sqrt{2n}$ and has a standard deviation of the order $n^{-1/6}$.
- Centered and rescaled largest eigenvalue

$$\sqrt{2}n^{1/6}\left(\lambda_{\max} - \sqrt{2n}\right)$$

converges in distribution as $n \to \infty$ to a random variable with the Tracy-Widom distribution, described by Tracy, Widom in 1994.

- Same limit holds generically for unitary random matrix ensembles.
- Different TW-distributions for orthogonal and symplectic ensembles.



Density of Tracy-Widom distribution. The density is non-symmetric with top at -1.8 and different decay rates for $x \to +\infty$ and $x \to -\infty$

Tracy-Widom distribution

- There is **no simple formula** for the Tracy-Widom distribution $F(s)$.
- First formula is as a **Fredholm determinant**:

$$F(s) = \det(I - A_s)$$

where A_s is the integral operator acting on $L^2(s, \infty)$ with kernel

$$\frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y}$$

Airy kernel

and Ai is the Airy function.

- Second formula

$$F(s) = \exp\left(-\int_s^\infty (x - s)q^2(x)dx\right)$$

where $q(s)$ is a special solution of the **Painlevé II equation**

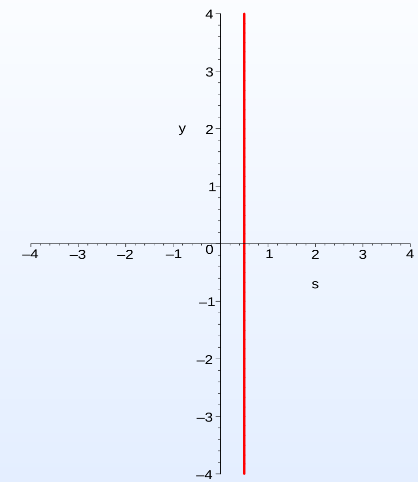
$$q''(s) = sq(s) + 2q^3(s)$$

Limit laws outside RMT

- Distribution functions of random matrix theory appear in various other domains of mathematics and physics.
  - **Number theory**
    - Riemann zeta-function, L-functions, ...
  - **Representation theory**
    - Young tableaux, large classical groups, ...
  - **Random combinatorial structures**
    - random permutations, random tilings, ...
  - **Growth models** in statistical physics
    - last passage percolation, polynuclear growth, ...
- ..., as well as in applications in statistics, finance, information theory, ...

Riemann zeta function

- The zeta function $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$ has an analytic continuation to the complex plane.
- The non-trivial zeros of the zeta function are believed to be on the line $\operatorname{Re} s = 1/2$. (Riemann hypothesis)



- **Computational evidence**: no non-real zeros have been found off the critical line.
- $1,500,000,000$ zeros have been found on the critical line.

Riemann zeta function

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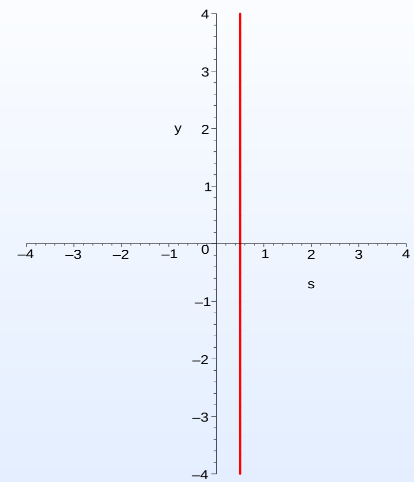
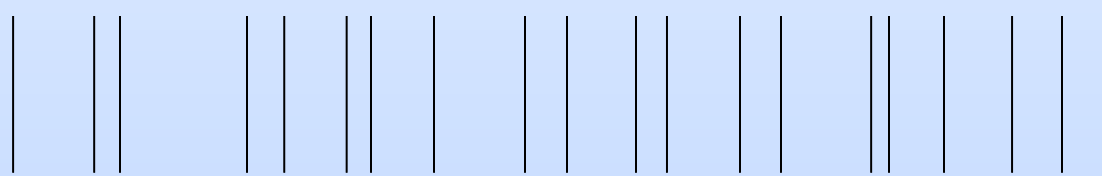


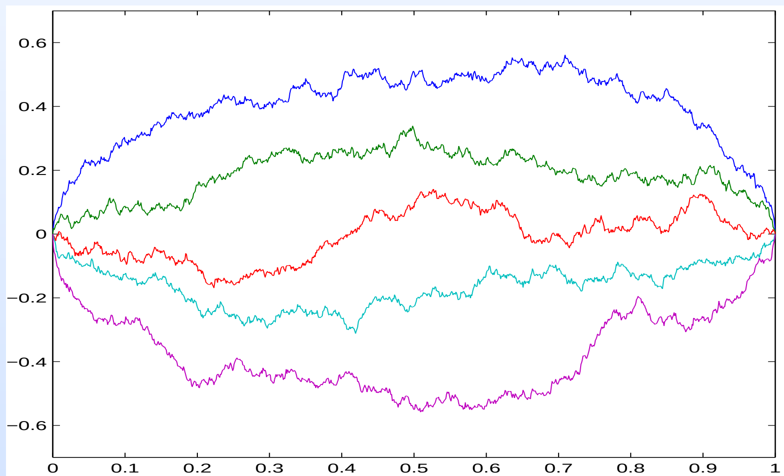
- **More computational evidence**: Spacings between consecutive zeros on the critical line $\mathrm{Re}\, s = 1/2$ (after appropriate scaling) show the same behavior as the spacings between eigenvalues of a large GUE matrix



Zeros of $\zeta(s)$ on the critical line have the same local behavior as the eigenvalues of a large random matrix

Non-intersecting Brownian motion paths

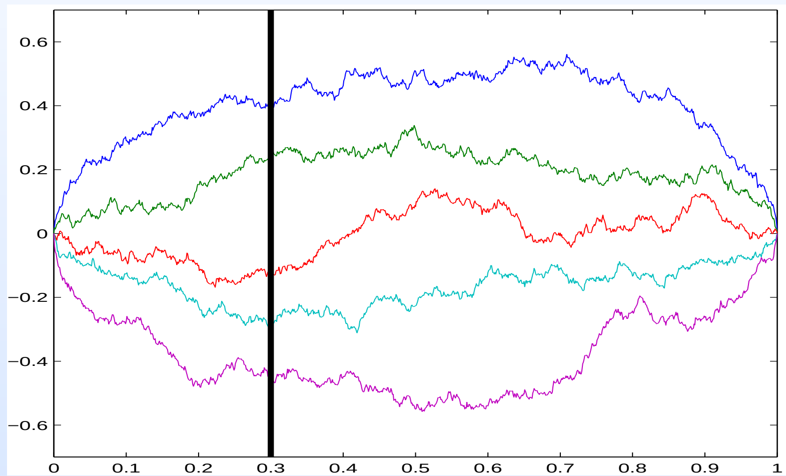
- Take n independent 1-dimensional Brownian motions with time in $[0,1]$ conditioned so that:
  - All paths start and end at the same point.
  - The paths **do not intersect** at any intermediate time.



Five non-intersecting Brownian bridges

Non-intersecting Brownian motion

- **Remarkable fact: At any intermediate time the positions of the paths have exactly the same distribution as the eigenvalues of an $n \times n$ GUE matrix (up to a scaling factor).**

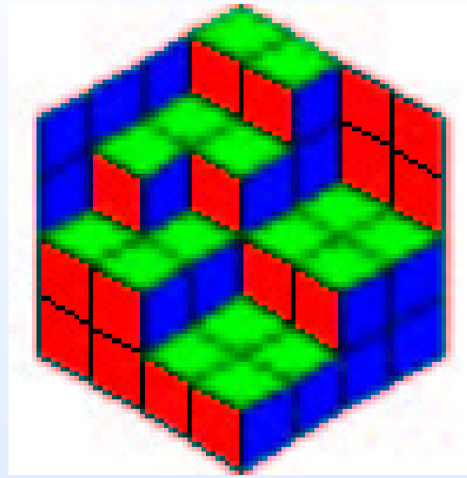


Positions of five non-intersecting Brownian paths behave the same as the eigenvalues of a 5×5 GUE matrix

- **This interpretation is basic for the connection of random matrix theory with growth models of statistical physics.**

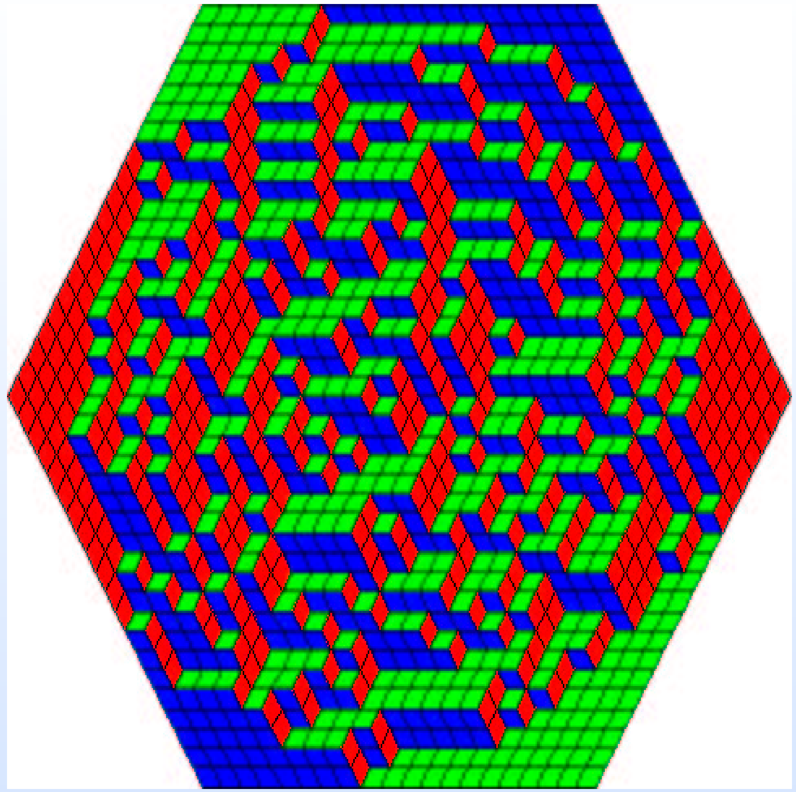
A random tiling problem

- Hexagonal tiling with rhombi.



- May also be viewed as packing of boxes in a corner.
- Take a tiling at random.
  - What does a typical tiling look like, if the number of rhombi increases?

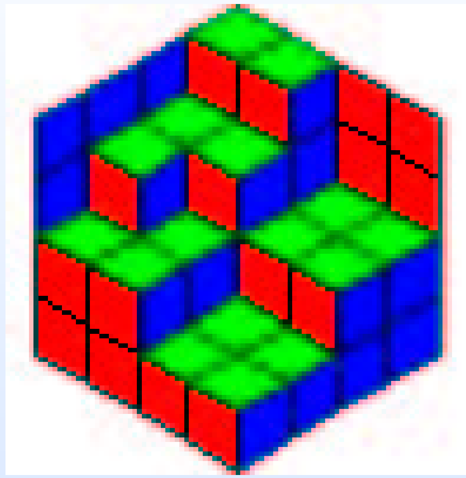
Typical random tiling

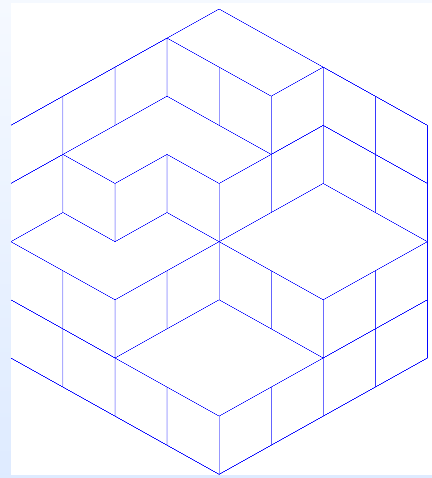


- **Observation:**
  - **frozen regions near the corners,**
  - **disorder in the center.**

Non-intersecting random walk

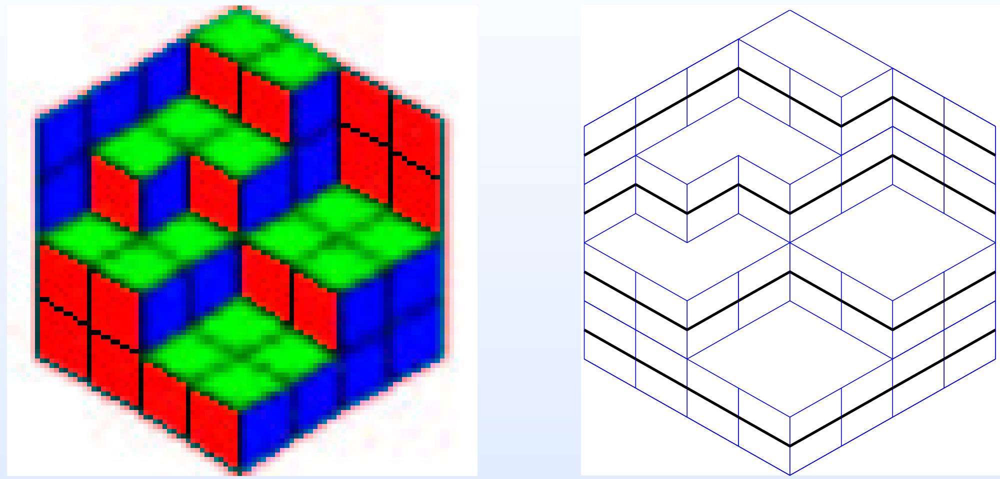
- Consider only blue and red rhombi.





Non-intersecting random walk

- We can connect the left and the right with non-intersecting paths.



- A random tiling is the same as a number of non-intersecting random walks.
- As size increases: Tracy-Widom distribution governs the transition between frozen region and disordered region.

Baik, Kriecherbauer, McLaughlin, Miller, (arxiv 2003)

Other examples

- **Longest increasing subsequence of random permutations**

  Baik, Deift, Johansson (1999)

- **Polynuclear growth model (PNG)**

  Totally asymmetric exclusion process (TASEP) **Praehofer, Spohn, Ferrari**

  Imamura, Sasamoto

- **Buses in Cuernavaca, Mexico** **Krbalek, Seba**

  Baik, Borodin, Deift, Suidan

- **Airplane boarding problem** **Bachmat**

Universality classes in unitary ensembles

- Probability measure on $n \times n$ Hermitian matrices

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM$$

where $dM = \prod_j dM_{jj} \prod_{j<k} d \operatorname{Re} M_{jk} d \operatorname{Im} M_{jk}$

  - This is a Gaussian ensemble for $V(x) = \frac{1}{2}x^2$

- Joint eigenvalue density

$$\frac{1}{\tilde{Z}_n} \prod_{i<j} |x_i - x_j|^2 \prod_{j=1}^{n} e^{-nV(x_j)}$$

wher

Determinantal point process

- It is special about unitary ensembles that the eigenvalues follow a **determinantal point process**. This means means that there is a kernel $K_n(x,y)$ so that all eigenvalue correlation functions are expressed as determinants

$$\mathcal{R}_m(x_1, x_2, \ldots, x_k) = \det\left[K_n(x_i, x_j)\right]_{i,j=1,\ldots,m}$$

  - $\int_a^b K_n(x,x)dx$ is expected number of eigenvalues in $[a,b]$
  - $\int_a^b \int_c^d \begin{vmatrix} K_n(x,x) & K_n(x,y) \\ K_n(y,x) & K_n(y,y) \end{vmatrix} dxdy$ is expected number of pairs of eigenvalues in $[a,b] \times [c,d]$, etc.

Orthogonal polynomial kernel

- **Let $P_{k,n}(x)$ be the kth degree monic orthogonal polynomial with respect to $e^{-nV(x)}$**

$$\int_{-\infty}^{\infty} P_{k,n}(x)P_{j,n}(x)e^{-nV(x)}dx = h_{k,n}\delta_{j,k}.$$

- **Correlation kernel is equal to**

$$K_n(x,y) = e^{-\frac{1}{2}n(V(x)+V(y))}\sum_{k=0}^{n-1}\frac{1}{h_{k,n}}P_{k,n}(x)P_{k,n}(y)$$

$$= e^{-\frac{1}{2}n(V(x)+V(y))}\frac{h_{n,n}}{h_{n-1,n}}\frac{P_{n,n}(x)P_{n-1,n}(y) - P_{n-1,n}(x)P_{n,n}(y)}{x-y}$$

Christoffel-Darboux formula

Asymptotical questions

- **All information is contained in the correlation kernel K_n.**
- **Asymptotic questions deal with the global regime**

$$\rho_V(x) = \lim_{n\to\infty} \frac{1}{n} K_n(x,x)$$

- **and with the local regime**
  - **Choose x^* and center and scale eigenvalues around x^***

$$\lambda \mapsto (cn)^\gamma (\lambda - x^*)$$

  - **This is a determinantal point process with rescaled kernel**

$$\frac{1}{(cn)^\gamma} K_n\left(x^* + \frac{x}{(cn)^\gamma}, x^* + \frac{y}{(cn)^\gamma}\right)$$

  - **Determine γ and calculate limit of rescaled kernel. Limits turn out to be universal, depending only on the nature of x^* in the global regime.**

Global regime

- In limit $n \to \infty$, the mean eigenvalue density has a limit $\rho = \rho_V$ which minimizes

$$\iint \log \frac{1}{|x-y|} \rho(x)\rho(y)dxdy + \int V(x)\rho(x)dx$$

among density functions $\rho \geq 0$, $\int \rho(x)dx = 1$.

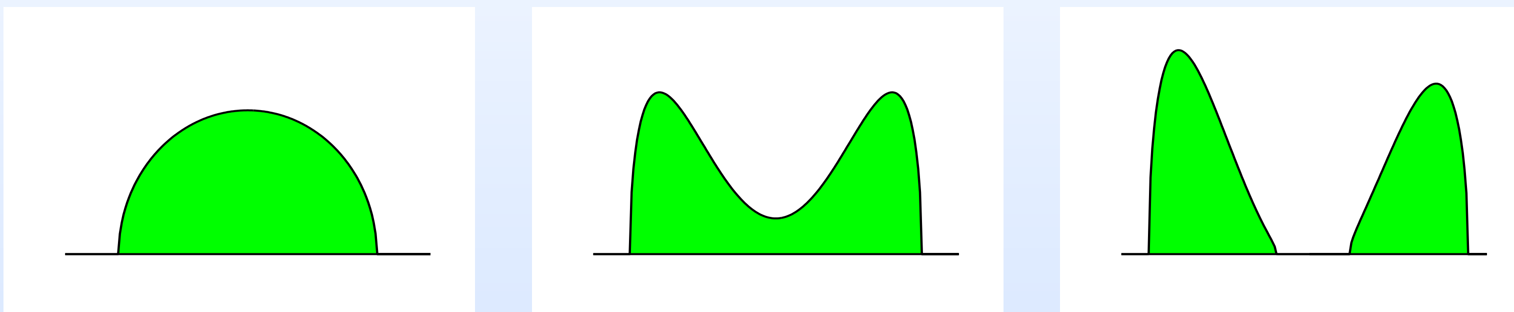
- Equilibrium conditions

$$2\int \log \frac{1}{|x-y|} \rho(y)dy + V(x) = const \text{ on support of } \rho$$

$$2\int \log \frac{1}{|x-y|} \rho(y)dy + V(x) \geq const \text{ outside support}$$

Regular and singular cases

- If V is real analytic, then **Deift, Kriecherbauer, McLaughlin (1998)**
  - $\mathrm{supp}(\rho)$ is a finite union of disjoint intervals,
  - $\rho(x)$ is analytic on the interior of each interval
  - $\rho(x) \sim |x-a|^{2k+1/2}$ at an endpoint a for some $k = 0, 1, 2, \ldots$

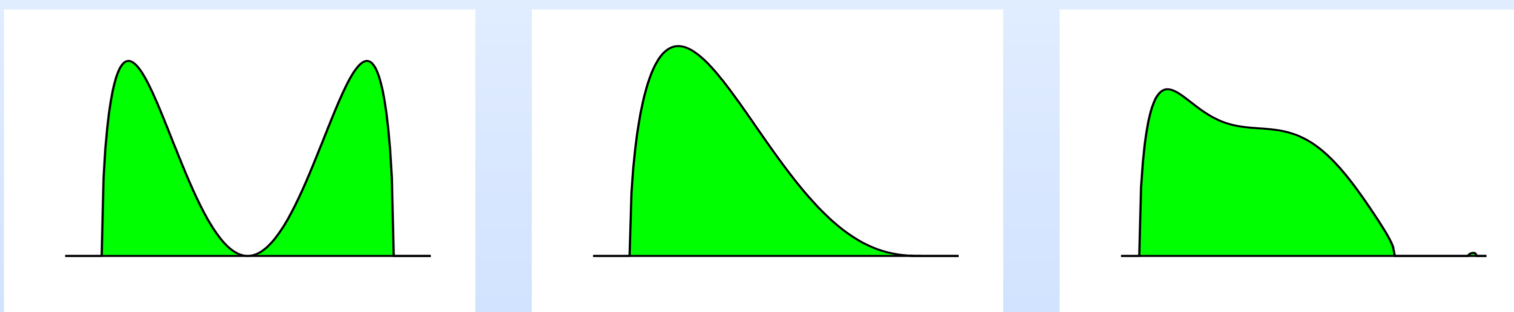


- Regular case: positive in interior, square root vanishing at endpoints, and strict inequality in

$$2\int \log \frac{1}{|x-y|}\rho(y)dy + V(x) > const \text{ outside the support of } \rho$$

Singular cases

- Singular cases correspond to possible change in number of intervals if parameters in the external field V change.
  - Singular case I: ρ vanishes at an interior point
  - Singular case II: ρ vanishes to higher order than square root at an endpoint.
  - Singular case III: equality in equilibrium inequality somewhere outside the support



Local regime

- **Limit of rescaled kernel**

$$\frac{1}{(cn)^\gamma} K_n\left(x^* + \frac{x}{(cn)^\gamma}, x^* + \frac{y}{(cn)^\gamma}\right)$$

  - **For x^* in the bulk, we take $\gamma = 1$, $c = \rho(x^*)$, and limit is the sine kernel** $\boxed{\dfrac{\sin \pi(x-y)}{\pi(x-y)}}$

Pastur, Shcherbina (1997), Bleher, Its (1999)

Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)

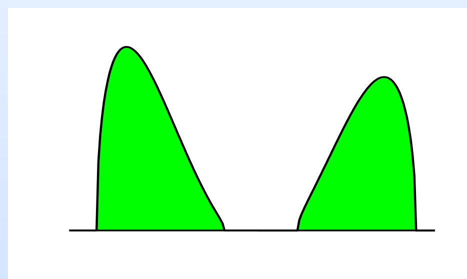
- **Scaling limit of kernel follows from detailed asymptotics of the orthogonal polynomials $P_{n,n}$ and $P_{n-1,n}$ as $n \to \infty$, which is available in the GUE case since then the orthogonal polynomials are Hermite polynomials.**
- **For more general cases a powerful new technique was developed: steepest descent analysis of Riemann-Hilbert problems**

Regular endpoint

- For a regular endpoint of the support, we take $\gamma = \frac{2}{3}$, and the scaling limit is the **Airy kernel**

$$\boxed{\frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x - y}}$$

- This always gives rise to the Tracy Widom distribution for the fluctations of the extreme eigenvalues.



Singular cases

Unitary ensemble

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM$$

Reference point x^*

- **Regular interior point:**
- **Regular endpoint:**
- **Singular case I:**
- **Singular case II:**
- **Singular case III:**

Singular cases

Unitary ensemble

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM$$

Reference point x^*

- **Regular interior point: sine kernel**
- **Regular endpoint: Airy kernel**
- **Singular case I:**
- **Singular case II:**
- **Singular case III:**

Singular cases

Unitary ensemble

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- **Singular case I: kernels built out of ψ functions associated with the Hastings-Mcleod solution of Painlevé II** **Claeys, AK (2006)**
- **Singular case II:**
- **Singular case III:**

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- **Singular case II: second member of Painlevé I hierarchy**

  Claeys, Vanlessen (in progress)
- **Singular case III:**

Singular cases

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- **Singular case III: ??**

Spectral singularity

Extra factor in random matrix model

$$\frac{1}{Z_n}|\det M|^{2\alpha} e^{-n \operatorname{Tr} V(M)} dM$$

The extra factor does not change the global behavior but it does change the local behavior around the reference point $x^* = 0$

- **Regular interior point:**
- **Regular endpoint:**
- **Singular case I:**
- **Singular case II:**
- **Singular case III:**

Spectral singularity

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- **Regular interior point: Bessel kernel** **AK, Vanlessen (2003)**
- **Regular endpoint:**
- **Singular case I:**
- **Singular case II:**
- **Singular case III:**

Spectral singularity

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- **Regular interior point: Bessel kernel** — **AK, Vanlessen (2003)**
- **Regular endpoint: general Painlevé II with parameter** $2\alpha + \frac{1}{2}$

$$\boxed{q'' = sq + 2q^3 - 2\alpha - \frac{1}{2}}$$

Its, AK, Östensson (in progress)

- **Singular case I:**
- **Singular case II:**
- **Singular case III:**

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- **Singular case I: Painlevé II with parameter α** — Claeys, AK, Vanlessen (arxiv 2005)
- **Singular case II:**
- **Singular case III:**

Spectral singularity

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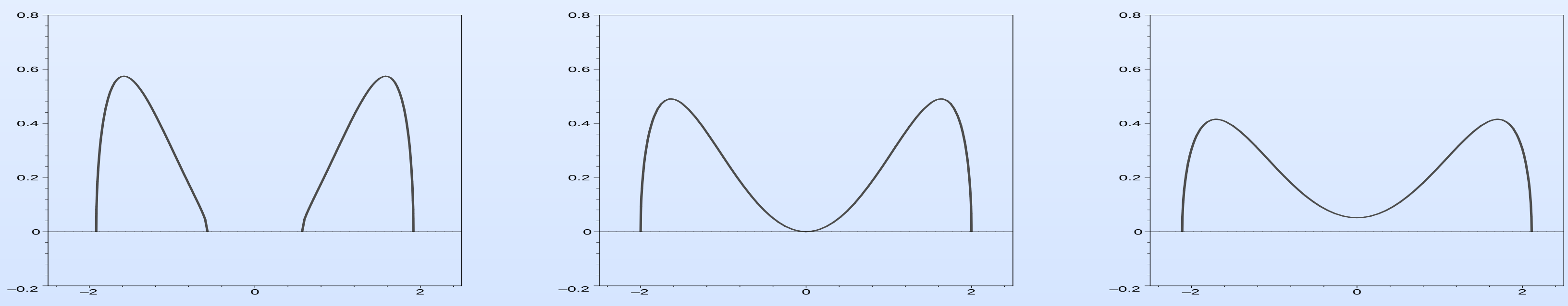
- **Singular case II: ??**
- **Singular case III: ??**

Singular case I

- Quartic external field $V(x) = \frac{1}{4}x^4 - x^2$ is simplest singular case I.
- Transition from two-interval to one-interval. If

$$V_t(x) = \frac{1}{t}\left(\frac{1}{4}x^4 - x^2\right)$$

then for $t < 1$: two intervals, and for $t > 1$: one interval



$t = 0.7$ $\qquad$ $t = 1$ $\qquad$ $t = 1.5$

- Consider singular case in double scaling limit where we rescale eigenvalues $\boxed{\lambda \mapsto (c_1 n)^{1/3}(\lambda - x^*)}$ and we let $t \to 1$ as $n \to \infty$ such that $\boxed{n^{2/3}(t-1) = c_2 s}$

Double scaling limit in singular case I

- **One-parameter family of limiting kernels, depending on s, but independent of V**

$$-\frac{\Phi_1(x;s)\Phi_2(y;s)-\Phi_2(x;s)\Phi_1(y;s)}{2\pi i(x-y)}$$

- **Φ_1 and Φ_2 satisfy a differential equation**

$$\frac{d}{dx}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=\begin{pmatrix}-4ix^2-i(s+2q^2) & 4xq+2ir\\ 4xq-2ir & 4ix^2+i(s+2q^2)\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}$$

with parameters s, q and r that are such that $q=q(s)$ satisfies Painlevé II:
$q''=sq+2q^3$ **and $r=r(s)=q'(s)$.**

for critical quartic V: Bleher, Its (2003)

for real analytic V: Claeys, AK (arxiv 2005)

for less smooth, even V: Shcherbina (arxiv 2006)

- **Our proof uses the fact that Φ_1 and Φ_2 solve a RH problem that can be used as a local parametrix in the steepest descent analysis.**