On Powers of a Random Orthogonal Matrix

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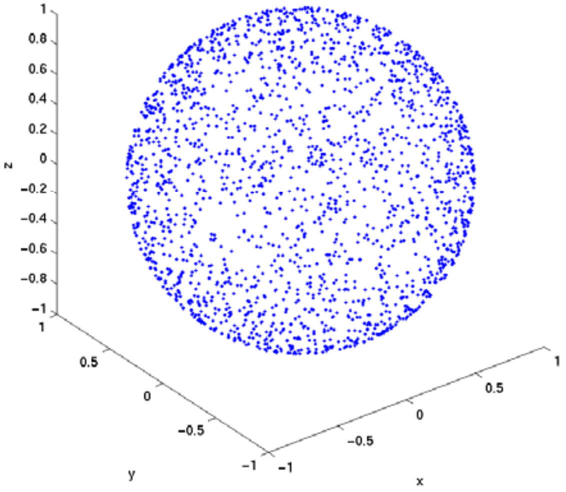
Outline

- Motivation
- Background
- Some new (we think …) distributional results

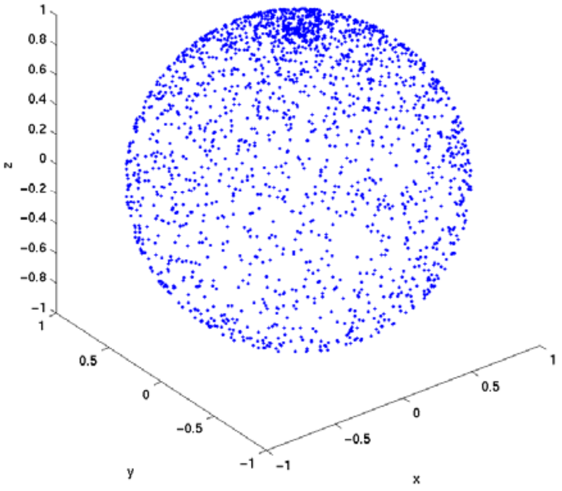
Motivation

- Talk by Tom Marzetta (Bell Labs) last October at SEA'05.
- Take a unit sphere (centered at the origin) in 3 dimensions, and transform the North Pole using a random (Haar) orthogonal matrix. Result: a point that is uniformly distributed on the sphere.
- Transform this point using the ***same*** orthogonal matrix. Result?

Orthogonal Transformations of the North Pole of a 3-dimensional Unit Sphere



Start at North Pole and apply random orthogonal transformation: uniformly random position



Apply the *same* transformation a second time.

Bias towards the northern hemisphere.

Some Background Material

- Let *O(p)* be the group of *p* x *p* orthogonal matrices *H* $(H'H = I_p)$.

- Let $\mathcal{D} \subset O(p)$. There is a unique probability measure μ on *O(p)* satisfying

$$\mu(\Gamma\mathcal{D}) = \mu(\mathcal{D}\Gamma) = \mu(\mathcal{D}), \text{ for all } \Gamma \in O(p).$$

- This is called the (invariant) Haar distribution on *O(p)* ; also the **uniform distribution** on *O(p)*.

Background (cont.)

To generate a uniformly distributed H:

- Take a p x p matrix of *iid* N(0, 1) variables).
- Now do Gram-Schmidt on the columns to get p orthonormal vectors.
- The p x p matrix H with these as columns then has the uniform distribution.

- Key fact: If H is uniform on $O(p)$ then H and PHQ have the same distribution, for all p x p orthogonal matrices P and Q.

Distributional Results

- Throughout this talk, we assume $p \geq 3$.
- On the unit *p*-sphere $S_p = \left\{ x \,\middle|\, x \in R^p,\ x'x = 1 \right\}$, let

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{(the "North Pole").}$$

- Let *H* be a uniform orthogonal *p* x *p* matrix.
- After *k* transformations by *H*, the North Pole is mapped to the (random) point $U_k = H^k x_0$ on S_{p}.

Distributional Results (cont.)

- We are interested in the first (i.e., the “north-pointing”) coordinate of U_k. This is the random variable

$$V_k = x_0' U_k = x_0' H^k x_0 .$$

- What can we say about the distribution of V_k? (Tom Marzetta’s simulations suggest that when p=3, $P(V_2 > 0) > \frac{1}{2}$.)

Distributional Results (cont.)

- Note that for each vector $x \in S_p$, $x'H^k x$ has the same distribution as $V_k = x_0'H^k x_0$.

Proof: For each $\Gamma \in O(p)$,

$$x'H^k x = (\Gamma x)'\underbrace{\Gamma H \Gamma' \cdots \Gamma H \Gamma'}_{k \text{ factors}}(\Gamma x)$$

Since $\Gamma H \Gamma'$ has the same distribution as H, it follows that $x'H^k x$ has the same distribution as $(\Gamma x)'H^k(\Gamma x)$. Now pick Γ so that $\Gamma x = x_0$.

Distributional Results (cont.)

- Objective of this talk: Describe the distributions of V_1 , V_2 , and V_3.
- Partition H as

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & H_{22} \end{pmatrix},$$

where

$$h_{11} \in (-1, 1), \quad h_{12} : 1 \times (p-1),$$
$$h_{21} : (p-1) \times 1, \quad H_{22} : (p-1) \times (p-1).$$

Distribution of V_1

Now, $V_1 = x_0' H x_0 = h_{11},$ and it is well-known that h_{11}^2 has a Beta distribution with parameters $\alpha = \frac{1}{2}$ and $\beta = \frac{p-1}{2}.$ Because h_{11} and $-h_{11}$ have the same distribution, it follows that the density function of h_{11} is

$$\boxed{f(t \mid p) = \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p-1}{2}\right)}\left(1-t^2\right)^{(p-3)/2}, \quad -1 < t < 1.}$$

Note: When $p=3$, h_{11} is uniform on (-1, 1), a result that dates back (at least) to Poincaré (1854-1912).

Distribution of V_2

We have

$$V_2 = x_0' H^2 x_0 = h_{11}^2 + h_{12} h_{21}.$$

Write this as

$$V_2 = h_{11}^2 + \left(1 - h_{11}^2\right) \frac{h_{12}}{\left(1 - h_{11}^2\right)^{1/2}} \frac{h_{21}}{\left(1 - h_{11}^2\right)^{1/2}}.$$

Distribution of V_2 (cont.)

$$V_2 = h_{11}^2 + \left(1 - h_{11}^2\right)\frac{h_{12}}{\left(1-h_{11}^2\right)^{1/2}}\frac{h_{21}}{\left(1-h_{11}^2\right)^{1/2}}.$$

Now, let Γ and Δ be fixed (for the moment) *p* x *p* orthogonal matrices of the form

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & \Gamma_1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 1 & 0 \\ 0 & \Delta_1 \end{pmatrix}.$$

where Γ_1 and Δ_1 are (*p*-1) x (*p*-1) orthogonal matrices.

Distribution of V_2 (cont.)

Then, because H and $\Gamma H \Delta$ have the same distribution, and

$$\Gamma H \Delta = \begin{pmatrix} 1 & 0 \\ 0 & \Gamma_1 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & H_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Delta_1 \end{pmatrix}$$

$$= \begin{pmatrix} h_{11} & h_{12}\Delta_1 \\ \Gamma_1 h_{21} & \Gamma_1 H_{22} \Delta_1 \end{pmatrix},$$

we have

$$V_2 \overset{d}{=} x_0' \Gamma H \Delta \Gamma H \Delta x_0 \overset{d}{=} h_{11}^2 + (1 - h_{11}^2) \frac{h_{12}}{\left(1 - h_{11}^2\right)^{1/2}} \Delta_1 \Gamma_1 \frac{h_{21}}{(1 - h_{11}^2)^{1/2}}.$$

Distribution of V_2 (cont.)

$$V_2 \overset{d}{=} h_{11}^2 + (1-h_{11}^2)\frac{h_{12}}{\left(1-h_{11}^2\right)^{1/2}}\Delta_1\Gamma_1\frac{h_{21}}{(1-h_{11}^2)^{1/2}}$$

Now, note that this holds for all fixed Δ_1 and Γ_1, so it also hold for *any* random Δ_1 and Γ_1 independent of H (because H and $\Gamma H\Delta$ have the same distribution). So, pick Δ_1 and Γ_1 to be independent uniform on $O(p\text{-}1)$, so that $\Delta_1\Gamma_1$ is uniform on $O(p\text{-}1)$ (independent of H).

Distribution of V_2 (cont.)

$$V_2 \overset{d}{=} h_{11}^2 + (1 - h_{11}^2) \frac{h_{12}}{\left(1 - h_{11}^2\right)^{1/2}} \Delta_1 \Gamma_1 \frac{h_{21}}{(1 - h_{11}^2)^{1/2}}$$

The next step involves conditioning on *H*, so that

$u = \dfrac{h_{12}}{\left(1 - h_{11}^2\right)^{1/2}}$ and $v = \dfrac{h_{21}}{\left(1 - h_{11}^2\right)^{1/2}}$ are fixed (as is h_{11}),

with $u, v \in S_{p-1}$.

Distribution of V_2 (cont.)

We now use the following result:

Lemma: If u and v are r x 1 fixed vectors of length 1, and Q is uniform on $O(r)$, then $u'Qv$ has the density function $f(\cdot|r)$.

Proof: Rotate both u and v into the first coordinate vector, and use the invariance property of the uniform distribution on $O(r)$. Then $u'Qv$ has the same distribution as the (1, 1) element of Q, which is $f(\cdot|r)$.

Distribution of V_2 (cont.)

It then follows that, conditional on *H*,

$$\frac{h_{12}}{\left(1-h_{11}^2\right)^{1/2}}\Delta_1\Gamma_1\frac{h_{21}}{(1-h_{11}^2)^{1/2}}$$

has the density $f\left(\cdot\,|\,p-1\right)$. Since this does not depend on *H*, it is also the unconditional distribution. Thus we have:

Distribution of V_2 (cont.)

Theorem 1

$$V_2 \overset{d}{=} T + (1-T)Y$$

where T and Y are independent, $T \sim \mathrm{Beta}\left(\tfrac{1}{2}, \tfrac{p-1}{2}\right)$ and Y has density function

$$f_Y(y) \equiv f\left(y \,\middle|\, p-1\right)$$

$$= \frac{\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p}{2}-1\right)}\left(1-y^2\right)^{(p-4)/2}, \quad -1 < y < 1.$$

Distribution of V_2 (cont.)

From this, it follows easily that

$$P(V_2 > 0) > \tfrac{1}{2}, \text{ and } P(V_2 > 0) \to \frac{1}{2} \text{ as } p \to \infty.$$

p	3	4	5	10	20	50	100	500
$P(V_2 > 0)$	.71	.68	.66	.62	.59	.56	.54	.52

Distribution of V_2 (cont.)

- The distribution of V_2 just given is also a consequence of the following result, whose proof requires (we think) a substantial amount of group invariance theory.

Theorem 2: If

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & H_{22} \end{pmatrix}$$

is uniform, then

$$h_{11}, \quad W_1 = \frac{h_{21}}{\left(1 - h_{11}^2\right)^{1/2}}, \quad W_2 = \frac{h_{12}'}{\left(1 - h_{11}^2\right)^{1/2}}$$

are independent, with W_1 and W_2 being uniformly distributed on $S_{p\text{-}1}$ and h_{11} having the density $f\left(\cdot \mid p\right)$.

Distribution of V_2 (cont.)

- In terms of $h_{11}, W_1, W_2,$

$$V_2 = h_{11}^2 + \left(1 - h_{11}^2\right) W_2' W_1.$$

Lemma: $Y \equiv W_2' W_1$ has density $f\left(\cdot \mid p-1\right).$

Proof: The conditional distribution of Y given W_2 has density $f\left(\cdot \mid p-1\right)$ by the orthogonal invariance of the distribution of W_1.

Distribution of V_3

- The description of the distribution of V_1 involves a single random variable (namely h_{11}) with density $f(\cdot | p)$.
- Our description of the distribution of V_2 involves two independent random variables: h_{11} with density $f(\cdot | p)$ and Y with density $f(\cdot | p-1)$.
- Claim: The distribution of V_3 may be described using three independent random variables: h_{11} with density $f(\cdot | p)$, Y with density $f(\cdot | p-1)$, and Z with density $f(\cdot | p-2)$.

$$\boxed{V_3 = x_0' H^3 x_0 = h_{11}^3 + 2h_{11}h_{12}h_{21} + h_{12}H_{22}h_{21}}$$

Distribution of V_3 (cont.)

Theorem 3

V_3 has the same distribution as

$$h_{11}^3 + 2h_{11}\left(1-h_{11}^2\right)Y + \left(1-h_{11}^2\right)\left[-h_{11}Y^2 + \left(1-Y^2\right)Z\right],$$

where h_{11}, Y and Z are independent with respective density functions $f\left(\cdot\,|\,p\right)$, $f\left(\cdot\,|\,p-1\right)$, and $f\left(\cdot\,|\,p-2\right)$.

Note: When p=3, $f\left(\cdot\,|\,1\right)$ is not a density. In this case, its interpretation is as a discrete distribution, taking the values 1 and –1 with probability ½ each.

Distribution of V_3 (cont.)

- The distribution of V_3 is symmetric; i.e., V_3 and -V_3 have the same distribution. This is true of V_k for any odd integer k:

$$V_k = x_0' H^k x_0$$

$$-V_k = x_0' \left(-H\right)^k x_0 \overset{d}{=} x_0' H^k x_0 = V_k .$$

Open Problems & Conjectures

1. What are the distributions of V_k, for $k \geq 4$?
2. Conjecture: V_2 is stochastically bigger than V_1: i.e.,

   $$P(V_2 > t) \geq P(V_1 > t), \text{ for all } t \in (-1,1).$$

   (We know this holds for $t = 0$.)
3. Conjecture: V_3 is has “more mass” near +1 and –1 than V_1. One possibility is that V_3^2 is stochastically bigger than V_1^2.