Spectra of Large Block Matrices

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Outline

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2. Problem statement

3. Theorem

4. Sketch of the proof

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An example in wireless communication:

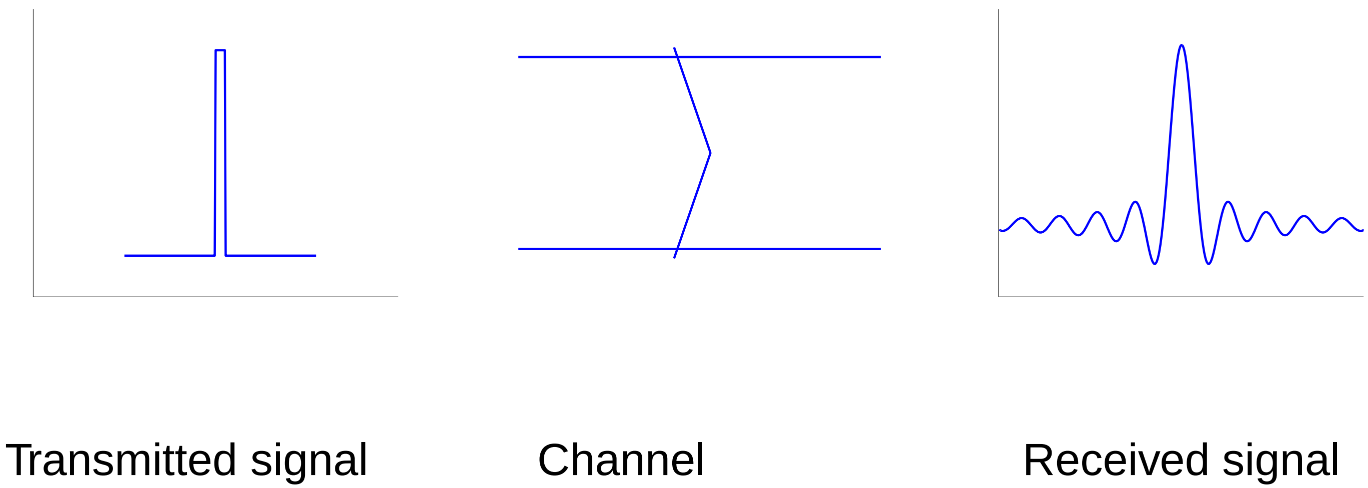
Multiple Input Multiple Output (MIMO) system with ISI

In a MIMO system:

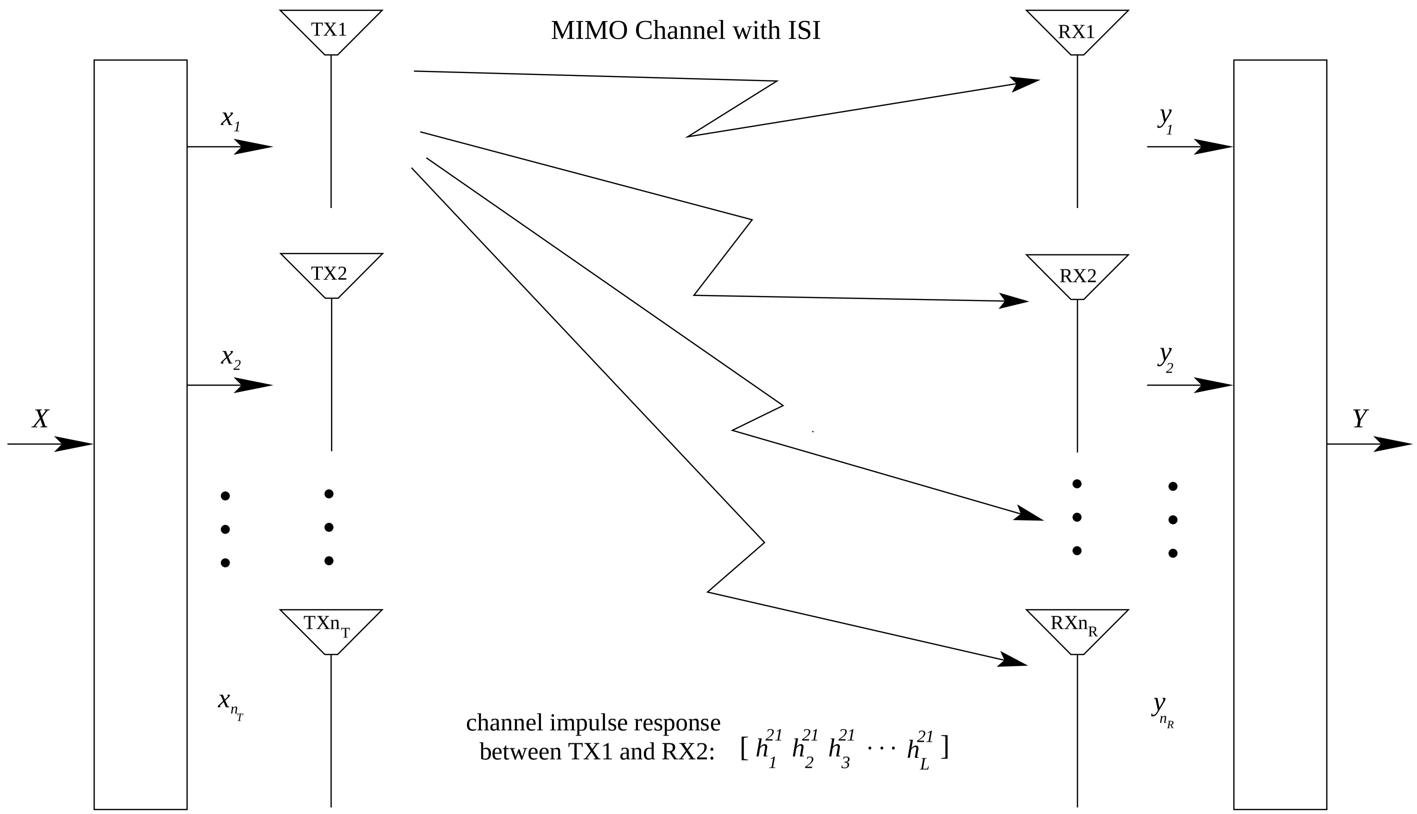
n_T transmitter antenna	n_R receiver antenna
$Y_n = [y_{1,n}, \cdots, y_{n_R,n}]^T$	received signal ($y_{i,n} = \sum_{j=1}^{n_T} y_{i,j,n}$)
$= HX_n + N_n$	
$H = (h_{i,j})_{i,j}$	channel matrix
$X_n = [x_{1,n}, \cdots, x_{n_T,n}]^T$	transmitted signal
N_n	noise signal

$h_{i,j}$ the channel effect on the signal transmitted from antenna j in the transmitter and received at antenna i in the receiver.

what happens in practice?



A more realistic model



Block diagram of a MIMO system with ISI

Considering Intersymbol-Interference (ISI) :

$y_{i,j,n} = \sum_{l=1}^{L} x_{j,n-l} h_l^{(ij)} + n_{i,j,n}$

$h_{ij} = \begin{bmatrix} h_1^{(ij)} & h_2^{(ij)} & \cdots & h_{L-1}^{(ij)} & h_L^{(ij)} \end{bmatrix}^T$

For a signal frame of K:

$$H = \begin{bmatrix} A_1 & A_2 & \cdots & A_L & \mathbf{0} & \mathbf{0} & & \cdots & \mathbf{0} \\ \mathbf{0} & A_1 & A_2 & \cdots & A_L & \mathbf{0} & & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_1 & A_2 & \cdots & A_L & \mathbf{0} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \cdots & & \ddots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & A_1 & A_2 & \cdots & A_L \end{bmatrix}$$

$A_l = (h_l^{(ij)})_{\substack{i=1,\cdots,n_R \\ j=1,\cdots,n_T}}$

But ...

the spectra of HH^* is the key factor in:

- Capacity of the channel,
- Structure of the receiver,

and the **trick** is to use:

$$\begin{bmatrix} \mathbf{0} & H \\ H^* & \mathbf{0} \end{bmatrix}$$

which has all the eigenvalues of HH^* plus some zeros!

Problem statement:

Selfadjoint block matrix:

$$X_N = \sum_{i,j=1}^{d} E_{ij} \otimes A^{(i,j)} \qquad d \text{ blocks of } N \times N \text{ matrices in each row}$$

where,

$$E_{ij} \ : \ d \times d \text{ Elementary matrix; entry } (i,j) = 1 \text{ the rest zero}$$

$$A^{(i,j)} = \left(a_{rp}^{(i,j)} \right)_{r,p=1,\ldots,N} \ : N \times N \text{ Gaussian random matrix}$$

$$A^{(i,j)} = A^{(j,i)*}$$

$$E\left(a_{rp}^{(i,j)} \right) = 0$$

$$E[a_{rp}^{(i,j)} \overline{a_{sq}^{(l,k)}}] = \frac{1}{n} \delta_{rs} \delta_{pq} \cdot \sigma(i,j;k,l) \quad (n = Nd)$$

Notations

- $\sigma = \Big(\sigma\,(i,j;k,l)\Big)_{i,j,k,l=1}^{d}$, covariance function
- $M_d\,(\mathbb{C})$: $d \times d$ matrices with complex entries
- $\eta : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ covariance mapping

$$[\eta(D)]_{ij} := \frac{1}{d} \sum_{k,l=1}^{d} \sigma(i,k;l,j) d_{kl} \quad \text{for } D = (d_{ij})_{i,j=1}^{d} \in M_d(\mathbb{C})$$

- $\mathrm{tr}_d\Big(D\Big) := \frac{1}{d} \sum_{i=1}^{d} [D]_{ii}$ normalized trace

Theorem

For $N \to \infty$, the $n \times n$ matrix X_N has a limiting eigenvalue distribution whose Cauchy transform $G(z)$ is determined by

$$G(z) = \mathrm{tr}_d(\mathcal{G}(z)),$$

where $\mathcal{G}(z)$ is an $M_d(\mathbb{C})$-valued analytic function on $\mathbb{C}^+$ uniquely determined by the facts that

1. $\lim_{|z|\to\infty, \Im(z)>0} z\mathcal{G}(z) = I_d$
2. $z\mathcal{G}(z) = I_d + \eta(\mathcal{G}(z)) \cdot \mathcal{G}(z) \quad \forall z \in \mathbb{C}^+$

Sketch of proof-1

- Wick formula: For Gaussian family of $x_1, x_2, \cdots, x_n$:

$$E\left[x_{i(1)} \cdots x_{i(k)}\right] = \sum_{\pi \in \mathcal{P}_2(k)} \prod_{(r,s)\in\pi} E\left[x_{i(r)} x_{i(s)}\right],$$

where $\mathcal{P}_2\left(k\right)$ is the set of all pairings of the set $\{1, \cdots, k\}$.

$$\Rightarrow \lim_{N\to\infty} E[\mathrm{tr}_n(X_N^m)] = \sum_{\pi \in NC_2(m)} \mathcal{K}_\pi,$$

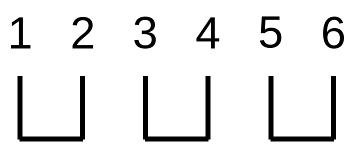
where

$$\mathcal{K}_\pi := \frac{1}{d^{m/2+1}} \sum_{i(1),\ldots,i(m)=1}^{d} \prod_{(p,q)\in\pi} \sigma\Big(i(p), i(p+1); i(q), i(q+1)\Big)$$

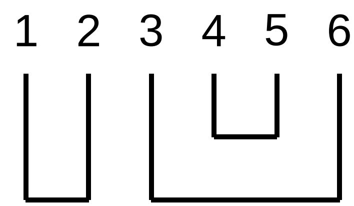
At the operator-valued level: $\mathcal{K}_\pi := \mathrm{tr}_d(\kappa_\pi), \kappa_\pi \in M_d(\mathbb{C})$

$$\Rightarrow \lim_{N\to\infty} E[\mathrm{tr}_n(X_N^m)] = \mathrm{tr}_d\Big\{\sum_{\pi\in NC_2(m)} \kappa_\pi\Big\}$$

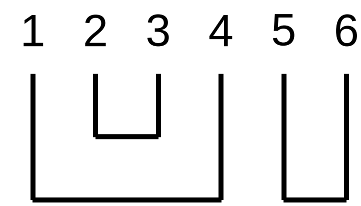
Non-crossing partions!



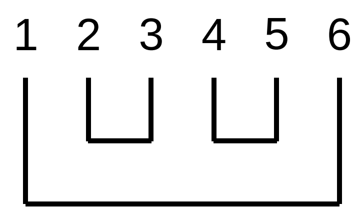
$\eta(I_d)\cdot\eta(I_d)\cdot\eta(I_d)$



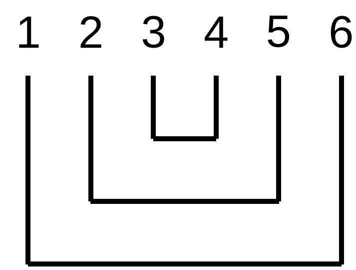
$\eta(I_d)\cdot\eta\Big(\eta(I_d)\Big)$



$\eta\Big(\eta(I_d)\Big)\cdot\eta(I_d)$



$\eta\Big(\eta(I_d)\cdot\eta(I_d)\Big)$



$\eta\Big(\eta\Big(\eta(I_d)\Big)\Big)$

Sketch of proof-2

- $\mathcal{E}(X^m) := \sum_{\pi \in NC_2(m)} \kappa_\pi$
  $\Rightarrow \lim_{N\to\infty} E[\mathrm{tr}_n(X_N^m)] = \mathrm{tr}_d(\mathcal{E}(X^m))$,
  X is called *“operator-valued semicircular element”*.
- $\mathcal{M}(z) = \sum_{m=0}^{\infty} \mathcal{E}[X^m] z^m$, generating power series

$$\Rightarrow \mathcal{M}(z) = I_d + z^2 \eta\Big(\mathcal{M}(z)\Big) \cdot \mathcal{M}(z)$$

- $\mathcal{G}(z) := \frac{1}{z}\mathcal{M}(1/z)$ operator-valued Cauchy transform

$$\Rightarrow z\mathcal{G}(z) = I_d + \eta(\mathcal{G}(z)) \cdot \mathcal{G}(z)$$

Example: Block Toeplitz matrices

$$X = \frac{1}{\sqrt{3N}} \begin{bmatrix} A & B & C \\ B & A & B \\ C & B & A \end{bmatrix}, \quad X = \frac{1}{\sqrt{4N}} \begin{bmatrix} A & B & C & D \\ B & A & B & C \\ C & B & A & B \\ D & C & B & A \end{bmatrix}$$

$$X = \frac{1}{\sqrt{5N}} \begin{bmatrix} A & B & C & D & E \\ B & A & B & C & D \\ C & B & A & B & C \\ D & C & B & A & B \\ E & D & C & B & A \end{bmatrix}$$

A, B, C, D, E selfadjoint Gaussian random matrices

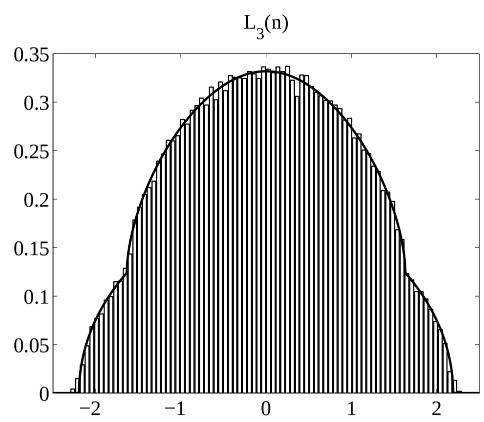
Step I : $\mathcal{G} = \begin{bmatrix} f & 0 & h \\ 0 & g & 0 \\ h & 0 & f \end{bmatrix}$

Step II : $\eta(\mathcal{G}) = \frac{1}{3} \begin{bmatrix} 2\,f + g & 0 & g + 2\,h \\ 0 & 2\,f + g + 2\,h & 0 \\ g + 2\,h & 0 & 2\,f + g \end{bmatrix}$

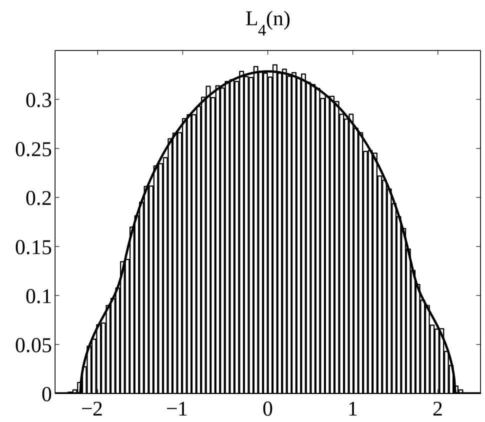
Step III : Solve the following system of equations:

$$z\mathcal{G}(z) = I_d + \eta(\mathcal{G}(z)) \cdot \mathcal{G}(z) \Rightarrow \begin{cases} zf &= 1 + \frac{g\,(f+h)+2\,\left(f^2+h^2\right)}{3} \\ zg &= 1 + \frac{g\,(g+2\,(f+h))}{3} \\ zh &= \frac{4\,f\,h+g\,(f+h)}{3} \end{cases}$$

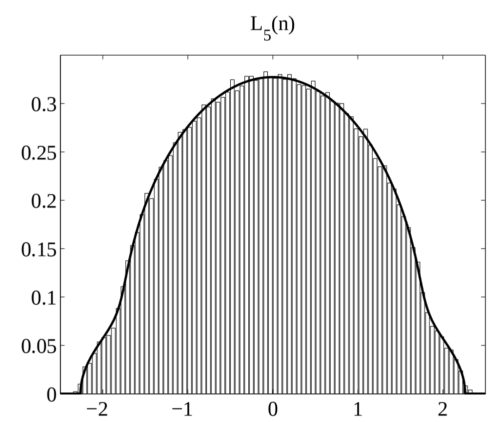
Step IV : Apply Stieltjes inversion formula



(a) 3×3



(b) 4×4



(c) 5×5

Block Toeplitz matrices spectra($N = 100$ for 100 realizations)

- as the number of blocks gets larger, the spectra goes to semicircular!

MIMO channel with ISI ($K = 4$ and $L = 4$)

Wishart type block matrices, like H^*H with

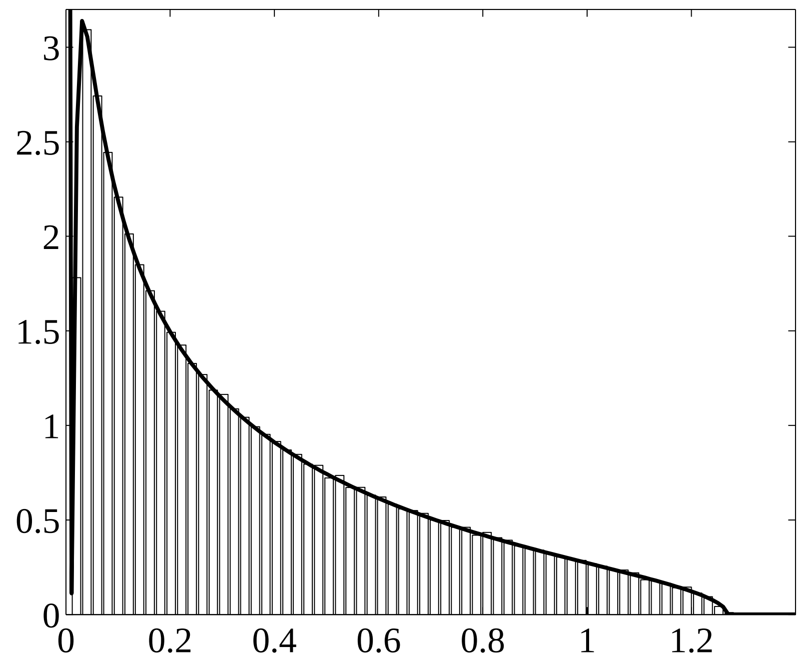
$$H = \begin{bmatrix} A & B & C & D & 0 & 0 & 0 \\ 0 & A & B & C & D & 0 & 0 \\ 0 & 0 & A & B & C & D & 0 \\ 0 & 0 & 0 & A & B & C & D \end{bmatrix}$$

A, B, C, D non-seldadjoint Gaussian random matrices.

we use :

$$\begin{bmatrix} 0 & H \\ H^* & 0 \end{bmatrix}$$

but we cancel the additional zeros!



Superimposed theoretical density of the eigenvalues of complex normal $H_n H_n^*/n$ for $K = 4$ and $L = 4$ over its histogram for $N = 100$, based on 100 realizations.