β-ensembles, Stochastic Airy Spectrum, and a Diffusion

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GUE ($\beta = 2$), the workhorse ensemble

Let M be an $n \times n$ Hermitian matrix comprised of unit complex Gaussians, independent save for the symmetry condition:

$$M_{ij} = \overline{M_{ji}} \sim \mathcal{N}(0, 1/4) + \sqrt{-1}\mathcal{N}(0, 1/4)$$

and $M_{ii} \sim \mathcal{N}(0, 1/2)$.

Equivalently, M is distributed according to

$$dP(M) = \frac{1}{Z} e^{-\mathrm{tr} M^2} dM$$

where dM indicates Lebesgue measure and $Z < \infty$ is a normalizer. This better explains the name.

Integrability

For spectral questions, GUE is integrable in the sense that the full joint density of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ may be computed:

$$\begin{aligned} P(\lambda_1, \lambda_2, \ldots, \lambda_n) &= \frac{1}{Z_n} \prod_{j<k} |\lambda_j - \lambda_k|^2 \exp\Big[- \sum_{k=1}^{n} \lambda_k^2\Big] \\ &= \frac{1}{n!} \det\Big[K_n(\lambda_j, \lambda_k)\Big]_{1\le j,k\le n}. \end{aligned}$$

Here, $K_n(\lambda, \mu)$ is the (explicit) kernel of the projection onto the span of the first Hermite polynomials in $L^2(\mathbb{R}, e^{-\lambda^2} d\lambda)$.

Allows for very delicate local fluctuation results.

Globally: The density of satisfies

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{\lambda_k/\sqrt{n}}(x)\rightarrow\frac{1}{\pi}\sqrt{2-x^2},$$

the semi-circle law.

Locally: The fluctuations of λ_{max} about its endpoint are described by the result of Tracy-Widom

$$\begin{aligned}\lim_{n\rightarrow\infty}P\left(\frac{1}{\sqrt{2}}n^{1/6}(\lambda_{max}-\sqrt{2n})\leq\lambda\right)\\ = \ \exp\left(-\int_{\lambda}^{\infty}(s-\lambda)q^2(s)\,ds\right),\end{aligned}$$

where q solves $q''=sq+2q^3$ (Painlevé II) subject to $q(s)\sim Ai(s)$ as $s\rightarrow+\infty$.

Gaussian Orthogonal and symplectic ensembles

If take iid real or quaternion Gaussian entries, have again integrable ensembles with joint eigenvalue density:

$$P(\lambda_1, \ldots, \lambda_n) = \prod_{j<k} |\lambda_j - \lambda_k|^{(1 \text{ or } 4)} \times \frac{1}{Z} \exp\Big[-\sum_{k=1}^{n} \lambda_k^2\Big].$$

With the same order fluctuations, the (scaled) largest eigenvalues have limiting distributions

$$F_1(\lambda) = \exp\Big(-\frac{1}{2}\int_\lambda^\infty (s-\lambda) q^2(s)\, ds - \frac{1}{2}\int_\lambda^\infty q(s)\, ds\Big)$$

and, with $\lambda' = 2^{2/3}\lambda$,

$$F_4(\lambda) = \exp\Big(-\frac{1}{2}\int_{\lambda'}^\infty (s-\lambda') q^2(s)\, ds\Big) \cosh\Big(\int_{\lambda'}^\infty q(s)\, ds\Big).$$

General Beta Ensemble

On physical grounds it is interesting to consider the family of ensembles on real points:

$$P_\beta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} e^{-\beta \sum_{k=1}^{n} \lambda_k^2} \prod_{k<j} |\lambda_j - \lambda_k|^\beta$$

for which G(O/U/S)E occur at $\beta = 1$,2, and 4. Here, beta plays the roll of inverse temperature in a one-dimensional caricature of a coulomb gas.

While apparently not integrable, Dumutriu-Edelman (2002) discovered that there is a family of matrix models: the eigenvalues of

$$H_n^\beta = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2}g_1 & \chi_{\beta(n-1)} & & & \\ \chi_{\beta(n-1)} & \sqrt{2}g_2 & \chi_{\beta(n-2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{\beta 2} & \sqrt{2}g_{n-1} & \chi_\beta \\ & & & \chi_\beta & \sqrt{2}g_n \end{bmatrix},$$

have joint density P_β. Here $g_1, g_2, \ldots$ are independent unit Gaussians and each χ_r stands for the square-root of an independent χ^2 with parameter r.

Stochastic Airy operator

In the edge scaling, one looks at the family of matrices:

$$\tilde{H}_n^\beta = \sqrt{\frac{2}{\beta}}\, n^{1/6}(H_n^\beta - \sqrt{2\beta n}\, I).$$

Edelman-Sutton present a convincing argument that $\tilde{H}_n^\beta$ should have the *continuum limit*

$$\frac{d^2}{dx^2} - x + \frac{2}{\sqrt{\beta}} b'(x),$$

in which b' a White Noise. This random Schrödinger operator is termed the "Stochastic Airy" equation.

Main result

Set

$$H = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x)$$

with tied boundary conditions at $x = 0$. The presence of b' means that the eigenvalue problem $H\psi = \lambda\psi$ reads

$$d\psi'(x) = (2/\sqrt{\beta})\psi(x)db(x) + (x - \lambda)\psi(x)dx$$

and this can be solved for $\psi \in C^{3/2-}$.

Theorem *With* $\lambda_{\beta,1} \geq \lambda_{\beta,2} \geq \cdots$ *the ordered eigenvalues of the* β-ensemble H_n^β, *and* $\Lambda_0 \leq \Lambda_1 \cdots \leq$ *the spectral points of* H, *it is the case that, for any finite* k, *the family*

$$\left\{\sqrt{\frac{2}{\beta}} n^{1/6}(\lambda_{\beta,\ell} - \sqrt{2\beta n})\right\}_{\ell=1,\ldots,k}$$

converges in distribution as $n \to \infty$ to

$$\{-\Lambda_0, -\Lambda_1, \cdots, -\Lambda_{k-1}\}.$$

Of course, part of this result is that H has an almost surely finite ground state eigenvalue Λ_0, as well as well defined higher eigenvalues $\Lambda_1, \Lambda_2, \ldots$.

Remark Balínt Virág has proved this independently.

The advantages:

- Ties the Tracy-Widom distribution to, for people of certain backgrounds, a much more basic model - a 1-D Schrödinger operator with particulary simple random potential.

- Allows for a proof of the scaling limit that is entirely probabilistic.

- The real point: gives a characterization of Tracy-Widom in terms of the explosion/non-explosion of a Markov process.

The Diffusion

The above result provides a new expression of the “Tracy-Widom laws” (at all beta) in terms of the explosion probability of a one dimensional diffusion.

Return to the eigenvalue problem

$$H\psi(x) = -\frac{d^2}{dx^2}\psi(x) + q(x)\psi(x) = \lambda\psi(x)$$

with $q(x)$ the (scaled) White Noise $(2/\sqrt{\beta})b'(x)$ plus linear restoring force, now viewed with Dirichlet conditions on the finite range $0 \leq x \leq L$.

The classical *Ricatti transformation*,

$$p(x) = \psi'(x)/\psi(x),$$

turns the above problem into one of first order:

$$q(x)\Big(= \frac{2}{\sqrt{\beta}} b'(x) + x\Big) = \lambda + p'(x) + p^2(x).$$

Re-written as in

$$dp(x) = \frac{2}{\sqrt{\beta}} db(x) + (x - \lambda - p^2(x))dx,$$

one would like to interpret this as saying that

$$x \mapsto p(x) = (\log \psi)'(x)$$

performs the indicated diffusion.

Ricatti done right

Take the sine-like solution of $H\psi = \lambda\psi$. That is, the unique solution $\psi_0(x) = \psi_0(x,\lambda)$ of the *initial value* problem with $\psi_0(0) = 0$, $\psi_0'(0) = 1$.

Find eigenvalues by “shooting”: the event that the minimum Dirichlet eigenvalue $\Lambda_0(L)$ lies to the right of a given λ is the event that $\psi_0(x)$ has no root before $x \le L$.

And since $\psi_0(x)$ solves an initial value problem, the corresponding $p(x,\lambda)$ is Markov:

$$dp(x) = (2/\sqrt{\beta})db(x) + (x - \lambda - p^2(x))dx$$

may be read as an Itô equation.

Further, the event that $\psi_0(x)$ has a root before $x = L$ is the event that p begun at $\psi'(0,\lambda)/\psi(0,\lambda) = +\infty$ at $x = 0$ fails to explode down to $-\infty$ before $x = L$. Thus,

$$\begin{aligned} P(\Lambda_0 > \lambda) &= P(\psi_0(\cdot,\lambda) \text{ never vanishes}) \\ &= P_{+\infty}\Big(p(\cdot,\lambda) \text{ does not explode}\Big), \end{aligned}$$

upon taking $L \uparrow \infty$.

The Upshot

Thorem *With* $x \mapsto p(x) = p(x, \lambda)$ *the above motion,* $P_\bullet$ *the measure on paths induced by* p *begun at* $p(0) = \bullet$ *and* $\mathfrak{m}(\lambda, \beta)$ *its passage time to* $-\infty$:

$$\lim_{n \to \infty} P\Big(\sqrt{\frac{2}{\beta}} n^{1/6} (\lambda_{\beta,1} - \sqrt{2\beta n}) \le \lambda \Big) = P_{+\infty}\Big(\mathfrak{m}(-\lambda, \beta) = +\infty \Big).$$

Higher Eigenvalues

Tied to the chance that $p(x, \lambda)$ explodes multiple times. For example:

If the second eigenvalue $\Lambda_1(L) > \lambda$, it must be that $\psi_0(x, \lambda)$ has *at most* one root in $[0, L]$.

At the place $\mathfrak{m}$ where $\psi_0(\mathfrak{m}, \lambda) = 0$, the process $p = -\infty$, but after ψ_0 passes below the axis, p will reappear at $+\infty$. Here it begins afresh, but now with a new value of the spectral parameter $\lambda' = \lambda + \mathfrak{m}$.

Corollary: *With* $F_{\beta,k}(\lambda)$ *the limiting distribution function of the (scaled)* k*-th largest eigenvalue, it holds*

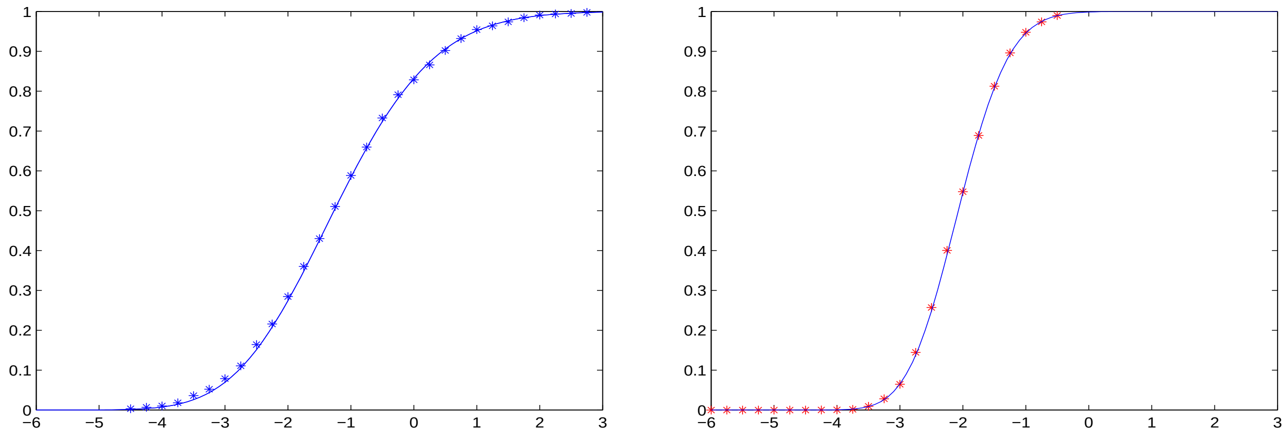
$$\begin{aligned} &F_{\beta,k}(\lambda) \\ &\quad = \sum_{\ell=1}^{k} \int_{\mathbb{R}_+^\ell} P_{+\infty}\Big(\mathfrak{m}(-\lambda,\beta) \in dx_1\Big) \\ &\qquad\qquad \times P_{+\infty}\Big(\mathfrak{m}(-\lambda+x_1,\beta) \in dx_2\Big) \cdots \\ &\qquad\qquad \times P_{+\infty}\Big(\mathfrak{m}(-\lambda+x_1+\cdots+x_{\ell-1},\beta) = +\infty\Big). \end{aligned}$$

Each term $D_{\beta,k}$ *in the sum satisfies the recurrence:*

$$D_{\beta,k}(\lambda) = \int_0^\infty D_{\beta,k-1}(\lambda - x)\, P_{+\infty}(\mathfrak{m}(-\lambda,\beta) \in dx),$$

where $D_{\beta,0}(\lambda) = F_{\beta,1}(\lambda)$.

Tracy-Widom distributions at $\beta = 1$ and 4 compared with the probability that $p(x, \lambda)$ does not explode:



The simulated $p(x)$ is started at 10^5 at $x = 0$ and survival of the path up to $x = 10$ proved sufficient to test non-explosion. Once it enters the parabola $p^2(x) \leq x - \lambda$, explosion is hard!

Painlevé? Cannot even give direct proofs of the known identities at $\beta = 1, 2, 4$! Easy to conclude the continuity in β.

Tail Universality Consider the Schrödinger operator

$$-\frac{d^2}{dx^2} + b'(x)$$

with periodic boundary conditions on $[0, 1]$. With J. Ramirez we prove

$$P(\Lambda_0 = \lambda) = e^{-\frac{1}{2}\lambda^2(1+o(1))} \text{ for } \lambda \to +\infty,$$

$$P(\Lambda_0 = \lambda) = e^{-\frac{8}{3}|\lambda|^{3/2}(1+o(1))} \text{ for } \lambda \to -\infty.$$

Left tail has same shape as TW distribution. Is this a general "White Noise" phenomena?

Idea of the proof (for the gound state)

Step 1 The noise in the matrix operator H_n^β is coupled to the BM b in H and the proof is made almost surely.

For each diagonal g_k we write

$$n^{-1/6} g_k = b^\sharp((k+1)n^{-1/3}) - b^\sharp(kn^{-1/3}) := \Delta_k^n(b^\sharp)$$

with a standard BM $b^\sharp$,

For the off-diagonal χ's we bring in a new BM $b^\flat$ and introduce a family of independent $b^\flat$- stopping times $\mathfrak{m}_{n,k}$ for $k \le n$ and each $n = 1, 2, \ldots$ such that

$$\sqrt{2} n^{-1/6} (\chi_{\beta(n-k)} - E\chi_{\beta(n-k)}) \\ \sim b^\flat (\sum_{\ell=1}^{k} \mathfrak{m}_{n,\ell}) - b^\flat (\sum_{\ell=1}^{k-1} \mathfrak{m}_{n,\ell}) := \Delta_k^n (b^\flat).$$

This is Skorohod's imbedding.

Then we set $b(x) = \frac{1}{\sqrt{2}} b^\sharp (x) + \frac{1}{\sqrt{2}} b^\flat (x)$.

Step 2 is an energy estimate. Putting $2/\sqrt{\beta} = 1$:

$$\begin{aligned}\langle v, -\tilde{H}_n^\beta v\rangle &= n^{2/3}\sum_{k=0}^{n}(v_{k+1}-v_k)^2\\ &\quad +n^{1/6}\sum_{k=0}^{n}(\sqrt{n}-E\chi_{(n-k)})v_k v_{k+1}\\ &\quad +n^{1/3}\sum_{k=0}^{n}(\Delta_k^n(b^\sharp)v_k^2+\Delta_k^n(b^\flat)v_k v_{k+1}),\end{aligned}$$

and in the continuum version,

$$\langle f, Hf\rangle = \int_0^\infty (f')^2 + \int_0^\infty xf^2 + \int_0^\infty f^2 db.$$

Each of these is to be bounded below with probability one.

A bound of the form: with $\int_0^\infty f^2 = 1$,

$$\left| \int_0^\infty f^2 db \right| \leq \frac{1}{2} \int_0^\infty |f'|^2 + \frac{1}{2} \int_0^\infty x f^2 + C(b),$$

and $C(b)$ a.s finite gives a bound below on Λ_0 and control of $\int_0^\infty (f')^2 + \int_0^\infty x f^2$ over any minimizing sequence.

A similar estimate in the discrete problem(s) reads:

$$\begin{aligned} \langle v, -\tilde{H}_n^\beta v \rangle \geq & \frac{1}{12} \sum_{k=1}^n n^{2/3} (v_{k+1} - v_k)^2 \\ & + \frac{1}{12} \sum_{k=1}^n k n^{-1/3} v_k^2 - C'(b^\sharp, b^\flat). \end{aligned}$$

with an a.s. finite C' *independent of* n.

Step 3 The previous step shows that $\{\lambda_{n,1}\}$ is a.s. bounded. Thus there is a subsequence n' over which

$$\lambda_{n',1} \to \lambda^*$$

with probability one. With $\{v_n\}$ the corresponding eigenfunctions the energy estimate provides enough control to allow the choice of a further subsequence over which: with any test function ϕ:

$$\langle \phi, -\tilde{H}_n^\beta v_n \rangle = \langle -\tilde{H}_n^\beta \phi, v_n \rangle = \lambda_n \langle \phi, v_n \rangle$$

goes over into

$$\langle H\phi, v^* \rangle = \lambda^* \langle \phi, v^* \rangle$$

with $v^* \in L^2 \cap H^1_{loc}$. And so, (λ^*, v^*) is an eigenvalue/ eigenvector pair with $\lambda^* \geq \Lambda_0$.

Step 4 The identification is made by taking f_0, the continuum ground state, discretizing it on the scale $n^{-1/3}$ to produce f_0^n and showing by hand

$$\lim_{n\to\infty} \frac{\langle f_0^n, -\tilde{H}_n^\beta f_0^n \rangle}{\langle f_0^n, f_0^n \rangle} = \langle f_0, H f_0 \rangle = \Lambda_0 \quad a.s.$$

(This requires regularity and decay of f_0).

It follows that $\lambda^* \le \limsup_{n\to\infty} \lambda_{n,1} \le \Lambda_0$.