Fluctuations of ASEP and RMT

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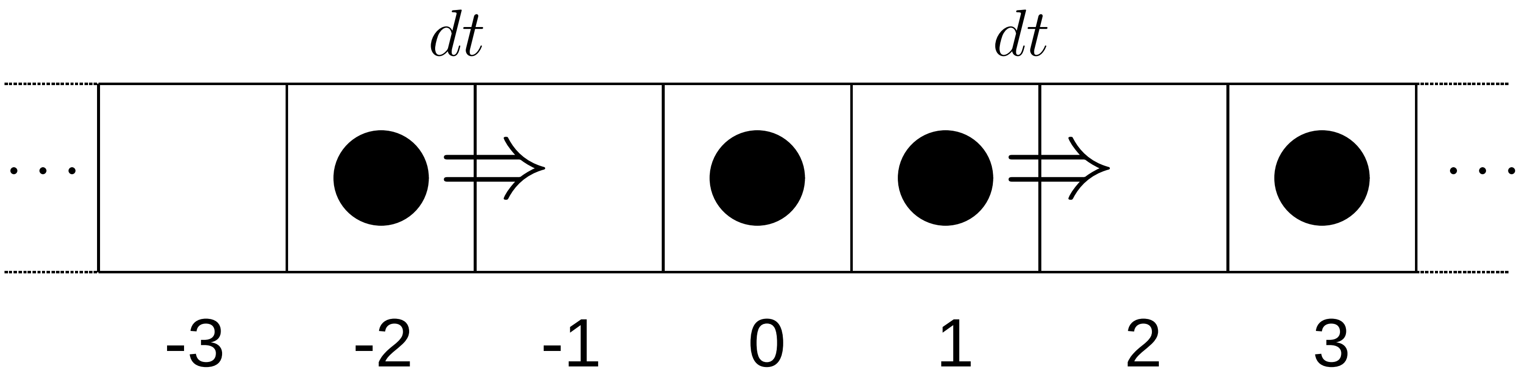
T. Sasamoto Chiba University

(based on collaborations with

T. Nagao, T. Imamura)

1. What is ASEP ?

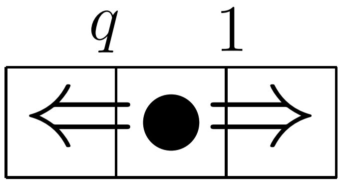
ASEP $=$ asymmetric simple exclusion process



- Each site is occupied or empty.
- During short time dt each particle tries to hop to the right neighboring site w.p. dt.
- If the target site is occupied, the hopping doesn't occur (exclusion).

Various versions

- Continuous time totally ASEP on the infinite lattice
- Discrete time (sequential update, sublattice update, parallel update,...)
- Asymmetry

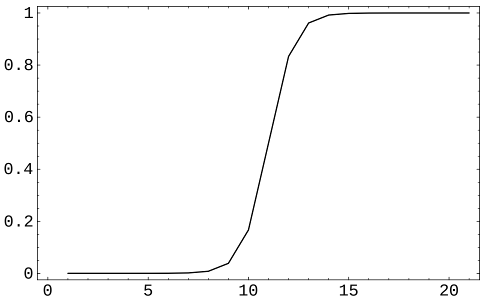


totally asymmetric($q = 0$)
partially asymmetric($0 < q < 1$)
symmetric($q = 1$)

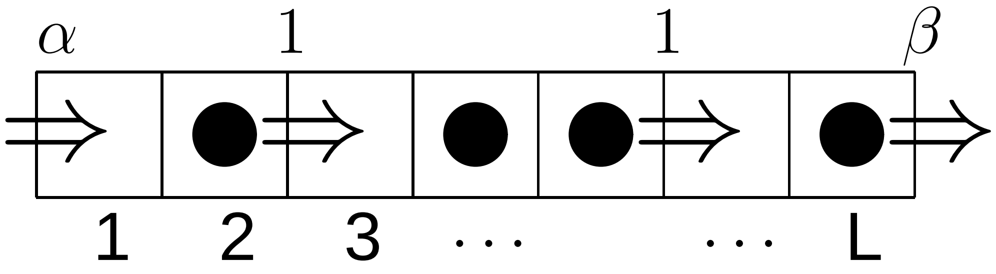
- Boundary conditions (periodic, closed, open, semi-infinite)

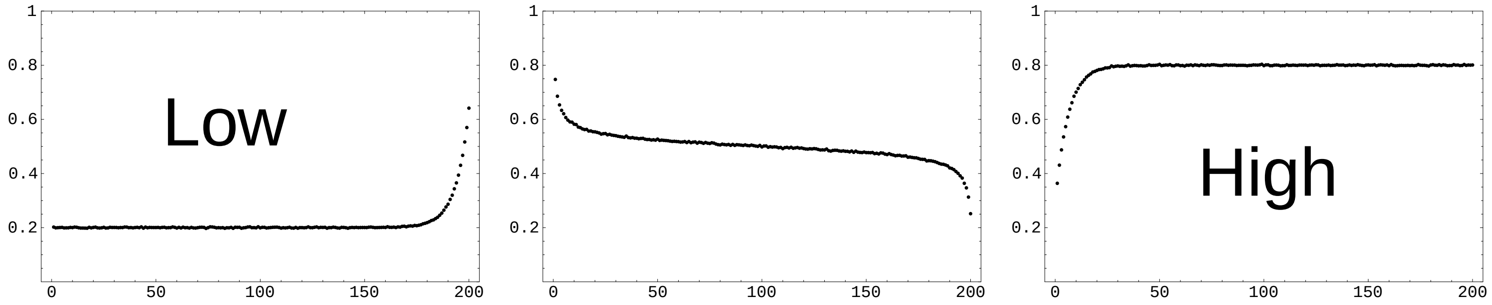
Interesting Phenomena

- Shock wave
    - nonlinearity
    - effects of noise



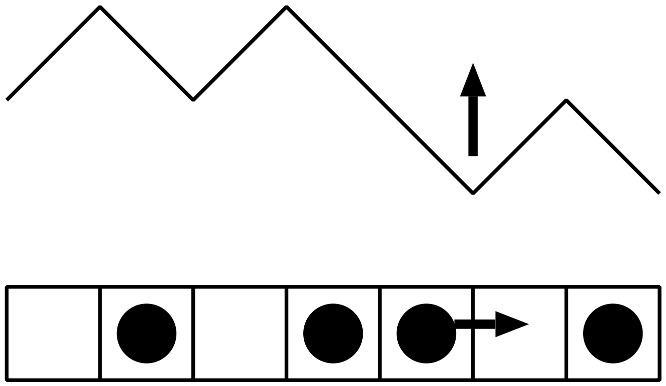
- Boundary induced phase transition





Applications

- Traffic flow
- Ionic conductor
- Ribosome on mRNA
- Molecular motor
- Surface growth



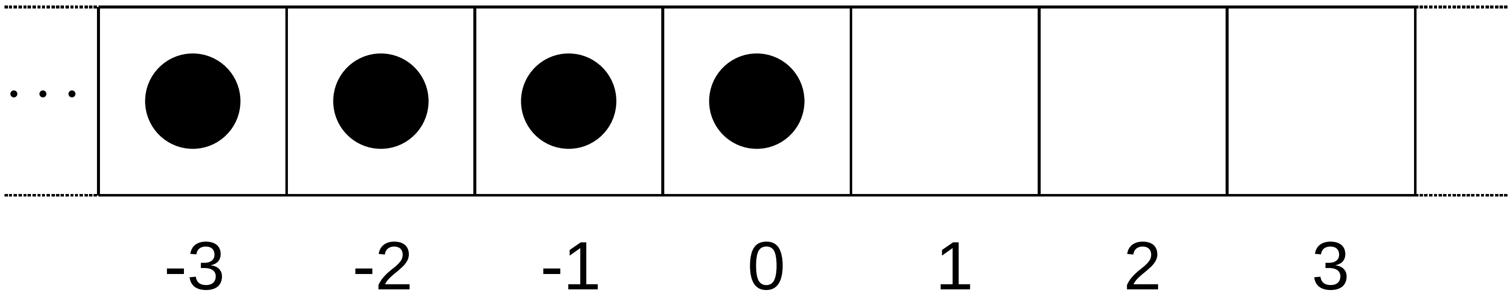
Related Fields

- Probability...Interacting stochastic systems
- Nonequilibrium statistical physics
- Integrable systems
    - q-orthogonal polynomials
    - Random matrix theory

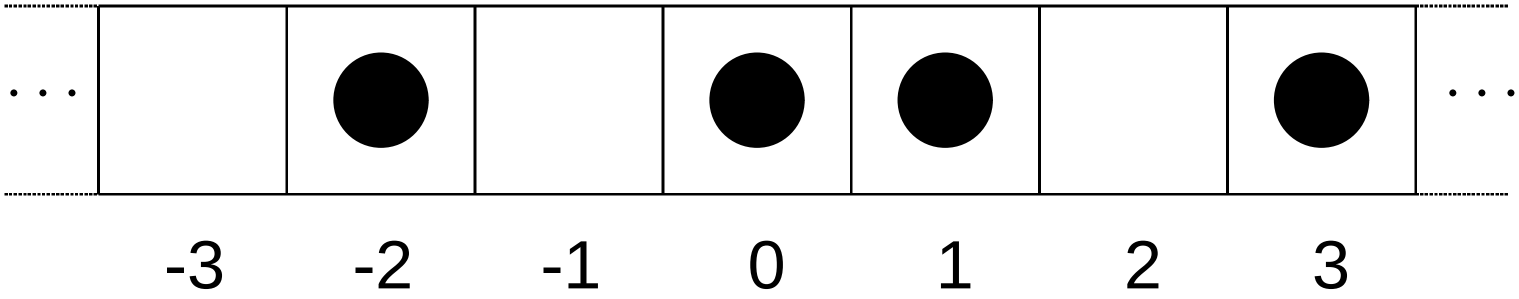
2. Current Fluctuation

Totally ASEP on infinite lattice

Step initial condition ($t=0$)



At time t



$N(t)$:$\#$ of particles which passed the origin until t

$= \int_0^t J(s)ds$: Integrated current

Long time behavior

- As $t \to \infty$, the system goes to Bernoulli with density 1/2.

$\cdots$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\cdots$
	-3	-2	-1	0	1	2	3	

Average $\lim_{t\to\infty} \frac{\langle N(t) \rangle}{t} = \frac{1}{4}$ (only ⊡☐ contributes)

- What about fluctuation? If Gaussian

$$\lim_{t\to\infty} \text{Prob}\left[N(t) \le \frac{t}{4} + c\xi\sqrt{t} \right] \overset{?}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\eta^2/2} d\eta$$

Not True!

Johansson 2000

$$\boxed{\lim_{t\to\infty} \text{Prob}\left[\frac{\frac{t}{4} - N(t)}{2^{-4/3} t^{1/3}} < s\right] = F_2(s)}$$

where $F_2(s)$ is the GUE TW distribution;

$$F_2(s) = \det(1 - K_2 \chi_s)$$

$$K_2(x,y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}$$

current of ASEP $\sim$ largest e.v. of RM

Step $\Leftrightarrow$ GUE

Key formula; connection to Laguerre UE

$$\text{Prob}[N(t) > N] = \frac{1}{Z_N} \int_{[0,t]^N} \prod_{1 \le j < k \le N} (x_j - x_k)^2 \prod_{j=1}^{N} e^{-x_j} dx^N$$

- Waiting time table of the discrete TASEP is mapped to the matrices with integer entries.
- Mapping to the pair of semi-standard Young tableaux by RSK
- Use a formula of Schur functions.

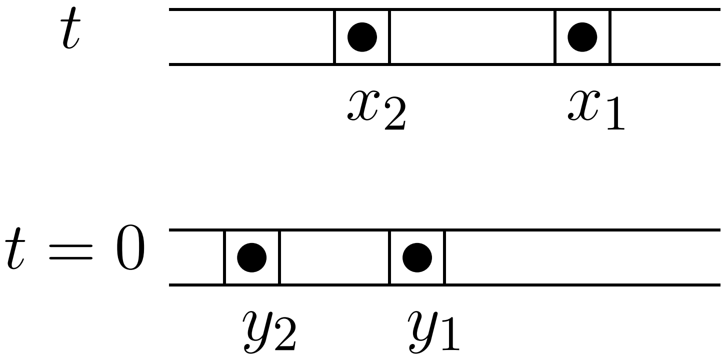
Transition Probability

Schütz 1997

Probability $G(x_1,\cdots,x_N;t)(=G(x_1,\cdots,x_N;t|y_1,\cdots,y_N;0))$ that N particles starting from $y_1,y_2,\cdots,y_N$ ($y_N<\cdots<y_1$) are on $x_1,x_2,\cdots,x_N$ ($x_N<\cdots<x_1$) at time t

$$G(x_1,x_2,\cdots,x_N;t)=\det[F_{k-j}(x_{N-k+1}-y_{N-j+1};t)]_{j,k=1,\cdots,N}$$

One particle



$$G(x;t)=F_0(x-y;t)=\frac{t^{x-y}}{(x-y)!}e^{-t}$$

Two particles

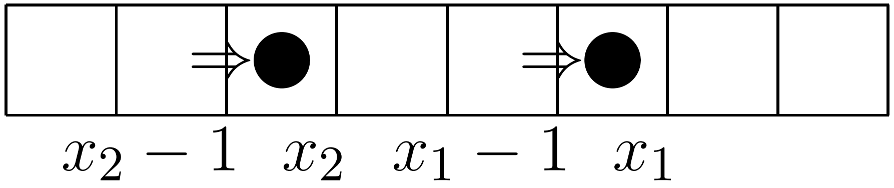
$$G(x_1,x_2;t)=\begin{vmatrix}F_0(x_2-y_2;t) & F_1(x_1-y_2;t)\\ F_{-1}(x_2-y_1;t) & F_0(x_1-y_1;t)\end{vmatrix}$$

$N = 2$ Case

Check $G(x_1, x_2, t|y_1, y_2; 0)$ satisfies the equation

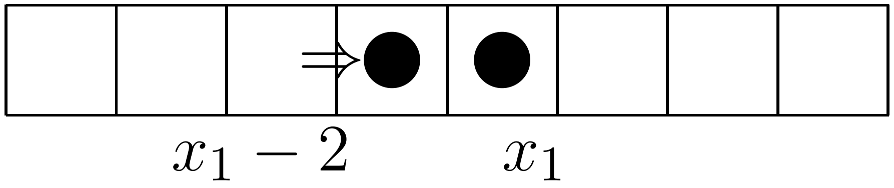
$$\frac{d}{dt}G(x_1, x_2; t) = G(x_1 - 1, x_2; t) + G(x_1, x_2 - 1; t) - 2G(x_1, x_2; t)$$

when $x_1 - x_2 \geq 2$,



$$\frac{d}{dt}G(x_1, x_1 - 1; t) = G(x_1, x_1 - 2; t) - G(x_1, x_1 - 1; t)$$

when $x_2 = x_1 - 1$,
and the initial condition



$$G(x_1, x_2; t = 0) = \delta_{x_1 y_1}\delta_{x_2 y_2}$$

Some properties of $F_n(x;t)$

Definition of $F_n(x;t)$

$$F_n(x;t) = e^{-t}\frac{t^x}{x!}\sum_{k=0}^{\infty}(-1)^k\frac{(n)_k}{(x+1)_k}\frac{t^k}{k!}$$

Properties of $F_n(x;t)$

$$\frac{d}{dt}F_n(x;t) = F_n(x-1;t) - F_n(x;t)$$

$$\int_0^t dtF_{n-1}(x-1;t) = F_n(x;t)$$

$$F_n(x;t) = \sum_{x_1=x}^{\infty} F_{n-1}(x_1;t)$$

$$F_{n-1}(x;t) = F_n(x;t) - F_n(x+1;t)$$

Another root to the Key formula

Nagao and S, Nucl. Phys. B **699** (2004) 487–502.

Observation

$$\text{Prob}[N(t) > N] = \text{Prob}[X_N(t) \geq 1]$$

where $X_N(t)$ is the position of the N-th particle from the right at time t when the system starts from the step initial condition.

- Each particle cannot affect the particles on its right
- RHS is equal to

$$\sum_{1 \leq x_N < \cdots < x_2 < x_1} G(x_1, \cdots, x_N; t | y_1 = 0, \cdots, y_N = -N+1; 0)$$

$N = 2$ Case

$$\begin{aligned}
\text{Prob}[N(t) > N] &= \text{Prob}[X_2(t) \geq 1] \\
&= \sum_{1 \leq x_2 < x_1} G(x_1, x_2; t | y_1 = 0, y_2 = -1; 0) \\
&= \begin{vmatrix} F_1(2;t) & F_2(3;t) \\ F_0(1;t) & F_1(2;t) \end{vmatrix} \\
&= \int_0^t dt_2 \int_0^t ds \begin{vmatrix} F_0(1;t_2) & F_1(2;s) \\ F_{-1}(0;t_2) & F_0(1;s) \end{vmatrix} \\
&= \int_0^t dt_2 \int_0^t ds \int_0^s dt_1 \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1) \\ F_{-1}(0;t_2) & F_{-1}(0;t_1) \end{vmatrix}
\end{aligned}$$

$N = 2$ Case

$$= \int_0^t dt_2 \int_0^t dt_1 (t - t_1) \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1) \\ F_{-1}(0;t_2) & F_{-1}(0;t_1) \end{vmatrix}$$

$$= \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 (t_2 - t_1) \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1) \\ F_{-1}(0;t_2) & F_{-1}(0;t_1) \end{vmatrix}$$

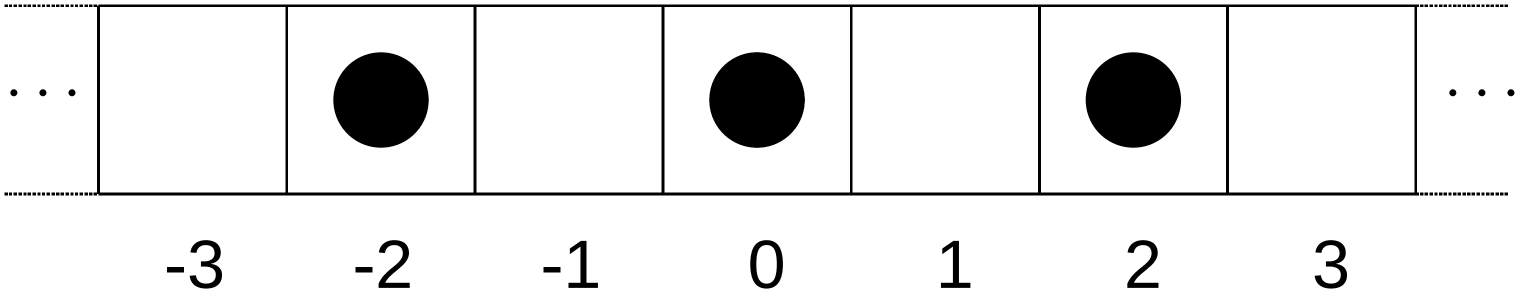
$$= \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 (t_2 - t_1) \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1) \\ F_0(0;t_2) & F_0(0;t_1) \end{vmatrix}$$

$$= \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 (t_2 - t_1) \begin{vmatrix} t_2 e^{-t_2} & t_1 e^{-t_1} \\ e^{-t_2} & e^{-t_1} \end{vmatrix}$$

$$= \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 (t_2 - t_1)^2 e^{-t_1 - t_2}$$

Generalizable to general N! $\rightarrow$ Key formula

Alternating initial condition



Baik Rains, Prähofer Spohn 2000-2001

$$\boxed{\lim_{t\to\infty} \text{Prob}\left[A_1(0) = \frac{\frac{t}{4} - N(t)}{2^{-4/3}t^{1/3}} < s\right] = F_1(s)}$$

- $F_1(s)$: GOE TW distribution

Alternating $\Leftrightarrow$ GOE

- Also possible to obtain via another root

3. Joint distribution for Step

- $N(x,t)$: Integrated current at position x and at time t.
- Joint distribution of $N(x_1,t)$ and $N(x_2,t)$.
- Joint distribution $\sim$ spatial correlation of surface.

Prähofer Spohn, Johansson 2002-2005

$A_2(0), A_2(\tau)$: scaled currents at 2pts for step

$$\text{Prob}[A_2(0) < s_1, A_2(\tau) < s_2] = \det(1 - K_2\chi_{s_1,s_2})$$

$$K_2 = \begin{cases} -\int_{-\infty}^{0} d\lambda e^{-\lambda(\tau_1-\tau_2)}\text{Ai}(\xi_1+\lambda)\text{Ai}(\xi_2+\lambda) & \tau_1 < \tau_2 \\ \int_0^{\infty} d\lambda e^{-\lambda(\tau_1-\tau_2)}\text{Ai}(\xi_1+\lambda)\text{Ai}(\xi_2+\lambda) & \tau_1 \geq \tau_2 \end{cases}$$

Step $\Leftrightarrow$ tGUE ($\beta = 2$ Dyson BM)

Joint distribution for alternating

Two problems

- How to compute ?
- Relation to tGOE($\beta = 1$ Dyson BM) ?

	step	alternating
1pt	GUE	GOE
2pt	tGUE	tGOE?

The results (S. J. Phys. A 38(2005)L549-L556.)

- Compute correlation by reinterpreting the transition probability
- Joint distribution is again Fredholm determinant
- Relation to tGOE still unknown

Reinterpretation of G

Two particles

$$G(x_1, x_2; t) = \begin{vmatrix} F_0(x_2 - y_2; t) & F_1(x_1 - y_2; t) \\ F_{-1}(x_2 - y_1; t) & F_0(x_1 - y_1; t) \end{vmatrix}$$

$$= \sum_{x_2^2 (> x_1^2)} \begin{vmatrix} \phi(x_1^1, x_1^2) & \phi(x_1^1, x_2^2) \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \psi_0^{(2)}(x_1^2) & \psi_0^{(2)}(x_2^2) \\ \psi_1^{(2)}(x_1^2) & \psi_1^{(2)}(x_2^2) \end{vmatrix}$$

where $x_1^1 = x_1, x_1^2 = x_2$ and

$$\phi(x_1, x_2) = 0(x_1 > x_2), \quad -1(x_1 \le x_2)$$

$$\psi_0^{(2)}(x) = -F_{-1}(x - y_1; t)$$

$$\psi_1^{(2)}(x) = F_0(x - y_2; t)$$

Auxiliary weight

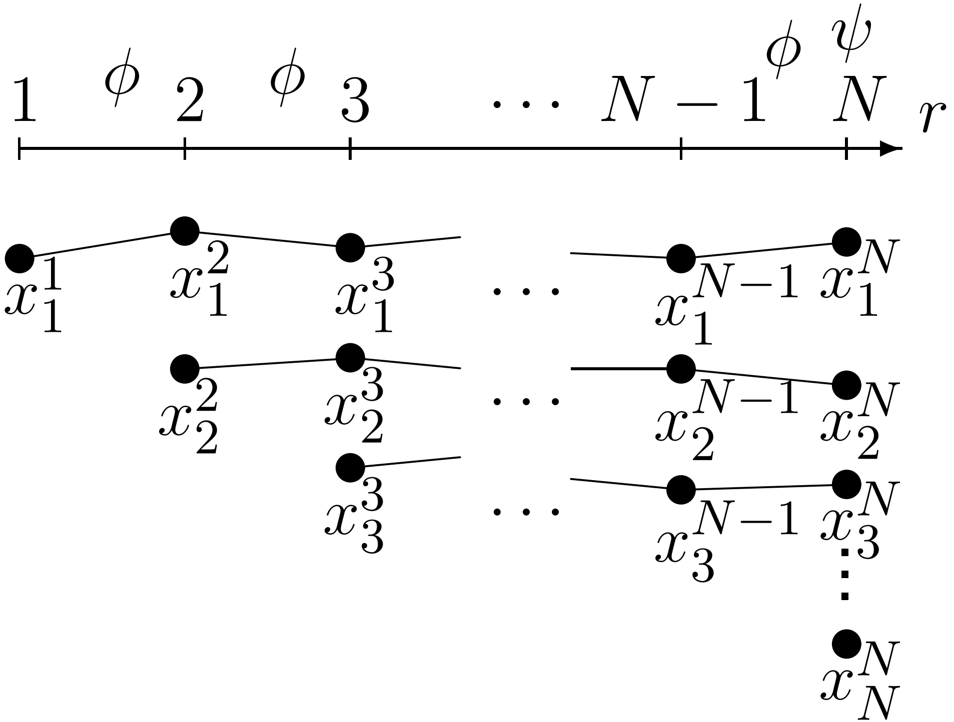
$$\prod_{r=1}^{N-1} \det[\phi(x_j^r, x_k^{r+1})]_{j,k=1}^{r+1} \cdot \det[\psi_j^{(N)}(x_{k+1}^N)]_{j,k=0}^{N-1}$$

$$= \begin{vmatrix} \phi(x_1^1, x_1^2) & \phi(x_1^1, x_2^2) \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \phi(x_1^2, x_1^3) & \phi(x_1^2, x_2^3) & \phi(x_1^2, x_3^3) \\ \phi(x_2^2, x_1^3) & \phi(x_2^2, x_2^3) & \phi(x_2^2, x_3^3) \\ 1 & 1 & 1 \end{vmatrix} \cdots$$

$$\times \begin{vmatrix} \psi_0^{(N)}(x_1^N) & \psi_0^{(N)}(x_2^N) & \cdots & \psi_0^{(N)}(x_N^N) \\ \psi_1^{(N)}(x_1^N) & \psi_1^{(N)}(x_2^N) & \cdots & \psi_1^{(N)}(x_N^N) \\ \vdots & \vdots & & \vdots \\ \psi_{N-1}^{(N)}(x_1^N) & \psi_1^{(N)}(x_{N-1}^N) & \cdots & \psi_{N-1}^{(N)}(x_N^N) \end{vmatrix}$$

where $\psi_j^{(r)}(x) = (-1)^{r-1-j} F_{-r+1+j}(x - y_{j+1}; t)$

Vicious walk interpretation



Let $\mathbb{P}$ denote the corresponding measure. We have

$$\boxed{G(x_1,\cdots,x_N;t)=\mathbb{P}[x_1^r=x_r\ (r=1,\cdots,N)]}$$

TASEP particle configuration $=$ dynamics of the 1st walker

Joint distribution for Alternating

$A_1(0), A_1(\tau)$: scaled currents at 2pts for alternating

$$\text{Prob}[A_1(0) < s_1, A_1(\tau) < s_2] = \det(1 - K_1\chi_{s_1,s_2})$$

$$K_1(\tau_1, \xi_1; \tau_2, \xi_2) = \tilde{K}_1(\tau_1, \xi_1; \tau_2, \xi_2) - \Phi_1(\tau_1, \xi_1; \tau_2, \xi_2)$$

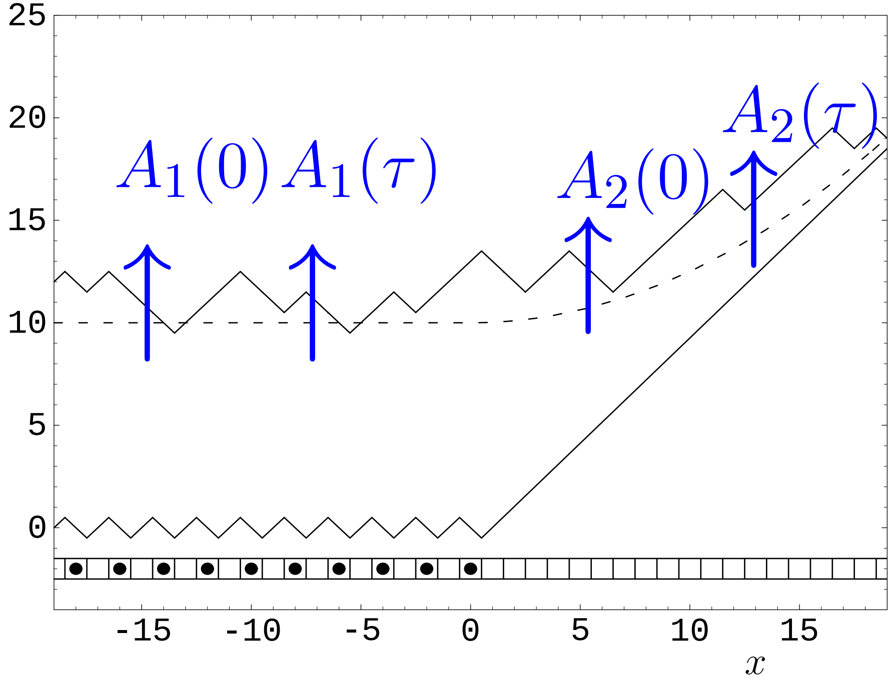
where

$$\tilde{K}_1(\tau_1, \xi_1; \tau_2, \xi_2) = \frac{1}{2} e^{\frac{(\tau_2-\tau_1)(\xi_1+\xi_2)}{4} + \frac{(\tau_2-\tau_1)^3}{12}} \text{Ai}\left(\frac{\xi_1+\xi_2}{2} + \frac{(\tau_2-\tau_1)^2}{4} \right)$$

$$\Phi_1(\tau_1, \xi_1; \tau_2, \xi_2) = \begin{cases} \frac{1}{\sqrt{8\pi(\tau_2-\tau_1)}} \exp\left[-\frac{(\xi_2-\xi_1)^2}{8(\tau_2-\tau_1)} \right] & \tau_1 < \tau_2 \\ 0 & \tau_1 \geq \tau_2 \end{cases}$$

Alternating $\overset{?}{\Leftrightarrow}$ tGOE ($\beta = 1$ Dyson BM)

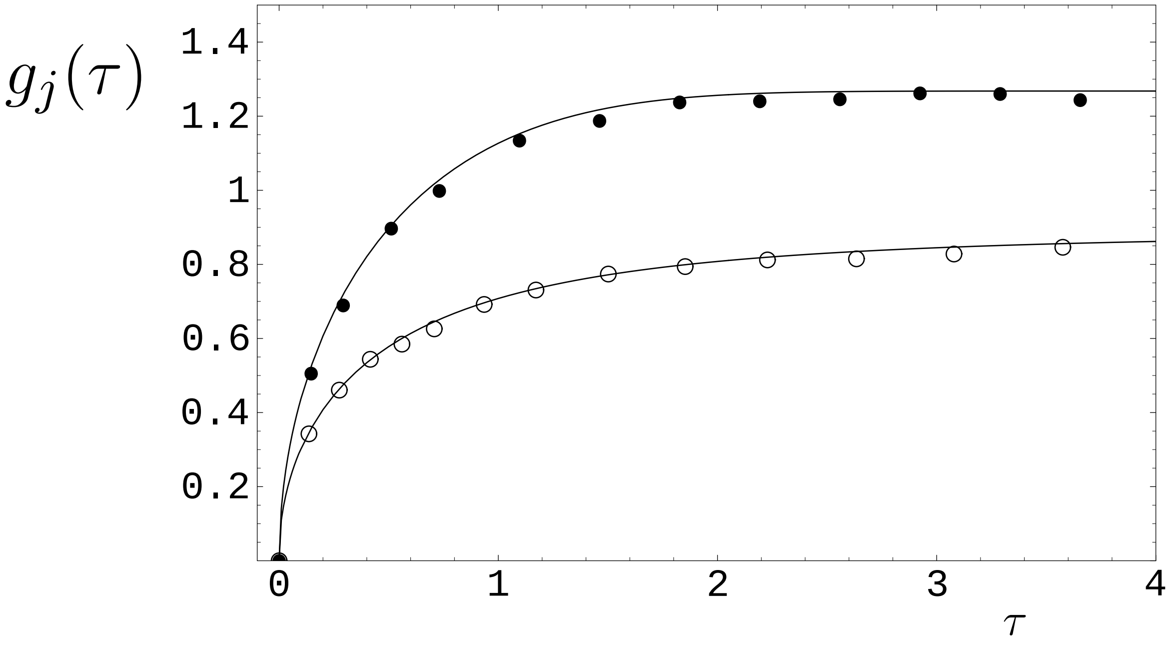
”Half alternating” initial condition



- Each particle cannot affect the particles on its right
- Same as alternating case deep inside the negative ($x < 0$) region

2pt correlation

$$g_j(\tau) = \sqrt{\frac{\langle (A_j(\tau) - A_j(0))^2 \rangle}{2}} \quad (j = 1, 2)$$



4. Conclusion

Summary

- ASEP is an interesting and important model
- Fluctuations of ASEP is often described by RMT
- Spatial correlation for the alternating case

Future Problems

- Two-time correlation
- Finite geometry
- Partially ASEP