On the Signal-to-Interference-Ratio of CDMA Systems in Wireless Communications

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I. Results. *Direct-sequence code-division multiple-access* (or DS-CDMA) network, used in wireless communications:

Each user has assigned to it a vector of high dimension, called a *signature sequence*, used to transmit data.

Suppose:

1. K users and L receive antennas.
2. N is the dimension of the signature sequences, $s_k \in \mathbb{C}^N$ is the signature sequence assigned to user k.
3. At a particular instant of time $X_k \in \mathbb{R}$ is the value transmitted by user k.
4. $T_k \in \mathbb{R}^+$ is user k's transmit power.
5. $\gamma_k(\ell)$ fading channel gain from user k to antenna ℓ.
6. X_k's are independent standardized random variables.

7. $W(\ell) \in \mathbb{C}^N$ noise associated with transmission to antenna ℓ, entries $W_i(\ell)$ i.i.d. across i and ℓ, mean zero and

$$\mathsf{E}|W_i(\ell)|^2 = \sigma^2.$$

Then the data recorded at antenna ℓ is modeled by

$$Y(\ell) = \sum_{k=1}^{K} X_k \sqrt{T_k}\gamma_k(\ell)s_k + W(\ell).$$

Let $Y = [Y(1)^T, \ldots, Y(L)^T]^T \in \mathbb{C}^{NL}$.

Goal: Capture the transmitted X_k for each user in a linear fashion, that is, by taking the inner product of Y with an appropriate vector $c_k \in \mathbb{C}^{NL}$ (the linear receiver for user k). For user 1, $\hat{X}_1 = c_1^* Y$ is the estimate of transmitted X_1

The output ***signal-to-interference ratio***

$$\frac{|c_1^* \hat{s}_1|^2}{\sigma^2 \|c_1\|^2 + \sum_{k=2}^{K} |c_1^* \hat{s}_k|^2}$$

associated with user 1, is typically used as a measure for evaluating the performance of the linear receiver, where

$$\hat{s}_k = \sqrt{T_k}[\gamma_k(1)s_k^T, \ldots, \gamma_k(\ell)s_k^T]^T .$$

Fact: The choice of c_1 which minimizes $\mathsf{E}(\hat{X} - X)^2$ (the ***minimum mean-square error***) also maximizes user 1's signal-to-interference ratio, the latter taking the value

$$\mathbf{SIR}_1 = \hat{s}_1^* \left(\sum_{k=2}^{K} \hat{s}_k \hat{s}_k^* + \sigma^2 I \right)^{-1} \hat{s}_1 ,$$

where I is the $NL \times NL$ identity matrix.

Properties of $\mathbf{SIR}_1$ and their dependency on the $\gamma_k(\ell)$'s, T_k's, σ^2, L, N, and K, when the latter two values are large, are explored by proving limiting results, as N and K approach infinity with their ratio approaching a positive constant, under the assumption that the s_k's are randomly generated (which is usually done in practice). They are independent i.i.d. random vectors containing i.i.d. mean zero, variance $1/N$ entries, independent of the $\gamma_k(\ell)$'s and T_k's.

THEOREM. *Let* $\{s_{ij} : i,j = 1,2,\ldots\}$ *be a doubly infinite array of i.i.d. complex random variables with* $\mathsf{E}s_{11} = 0$, $\mathsf{E}|s_{11}|^2 = 1$. *Define for* $k = 1,2,\ldots,K$ $s_k = s_k(N) = (s_{1\,k}, s_{2\,k}, \ldots, s_{N\,k})^T$. *We assume* $K = K(N)$ *and* $K/N \to c > 0$ *as* $N \to \infty$. *For each* N *let* $\gamma_k(\ell) = \gamma_k^N(\ell) \in \mathbb{C}$, $T_k = T_k^N \in \mathbb{R}^+$, $k = 1,\ldots,K$, $\ell = 1,\ldots,L$ *be random variables, independent of* $s_1,\ldots,s_K$. *Let for each* N *and* k

$$\alpha_k = \alpha_k^N = \sqrt{T_k}(\gamma_k(1),\ldots,\gamma_k(L))^T.$$

Assume almost surely, the empirical distribution of $\alpha_1,\ldots,\alpha_K$ *weakly converges to a probability distribution* H *in* $\mathbb{C}^L$.

Let $\beta_k = \beta_k(N) = \sqrt{T_k}(\gamma_k(1)s_k^T,\ldots,\gamma_k(L)s_k^T)^T$, *and*

$$C = C(N) = \frac{1}{N}\sum_{k=2}^{K}\beta_k\beta_k^*.$$

Define

$$\mathrm{SIR}_1 = \frac{1}{N}\beta_1^*(C + \sigma^2 I)^{-1}\beta_1.$$

then, with probability one

$$\lim_{N\to\infty} \mathrm{SIR}_1 = T_1 \sum_{\ell,\ell'=1}^{L} \bar{\gamma}_1(\ell)\gamma_1(\ell')a_{\ell,\ell'}$$

where the $L \times L$ matrix $A = (a_{\ell,\ell'})$ is nonrandom, Hermitian positive definite, and is the unique Hermitian positive definite matrix satisfying

$$A = \left(c\,\mathsf{E}\frac{\alpha\alpha^*}{1+\alpha^* A\alpha} + \sigma^2 I_L\right)^{-1} \tag{1.1}$$

where $\alpha \in \mathbb{C}^L$ has distribution H and I_L is the $L \times L$ identity matrix.

Clearly $\mathbf{SIR}_1$ defined in this theorem is the same as the one initially introduced, the only difference in notation being the removal of the scaling by $1/\sqrt{N}$ in the definition of the s_k's.

Importance: Arbitrary scenarios can be analyzed and compared. In applications the empirical distribution of the α_k's would typically be used for H, the matrix A thereby satisfying

$$A = \left(\frac{1}{N} \sum_{k=2}^{K} \frac{\alpha_k \alpha_k^*}{1 + \alpha_k^* A \alpha_k} + \sigma^2 I_L \right)^{-1} .$$

Although there appears to be no explicit solution to (1.1), the paper shows that A can be computed numerically by iteration of the right side of (1.1), provided the eigenvalues of the initial choice in the iteration lie in a certain closed interal in $(0, \infty)$.

Fact: Let α_ℓ denote the ℓ^{th} entry of the random vector α having distribution H. If the $\gamma_k(\ell)$'s are independent and circularly symmetric, with angles independent of the T_k's, then $\sqrt{T_k}\gamma_k(\ell)$ and $\sqrt{T_k}\gamma_k(\ell')$ are uncorrelated for $\ell \neq \ell'$. It follows that for $\ell \neq \ell'$ and positive $a_1, \ldots, a_L$

$$\mathsf{E}\frac{\alpha_\ell \bar{\alpha}_{\ell'}}{1 + \sum_{\underline{\ell}} a_{\underline{\ell}} |\alpha_{\underline{\ell}}|^2} = 0 \tag{1.2}$$

COROLLARY 1. *Under the conditions in the Theorem and (1.2), the limiting $A = diag(a_1, \ldots, a_L)$ where the a_ℓ's are positive satisfying*

$$a_\ell = \frac{1}{c\,\mathsf{E}\frac{|\alpha_\ell|^2}{1+\sum_{\underline{\ell}} a_{\underline{\ell}} |\alpha_{\underline{\ell}}|^2} + \sigma^2} \tag{1.3}$$

COROLLARY 2. *Suppose the conditions in the Theorem are met except, for the limiting behavior of the α_k's, it is only known that:*

1. *the empirical distribution of*

$$T_k(|\gamma_k(1)|^2, \dots, |\gamma_k(L)|^2)^T \quad 2 \le k \le K \tag{1.4}$$

converges almost surely in distribution to a probability distribution G in $\mathbb{R}^L$, and

2. *for $\ell \neq \ell'$ and positive $a_1, \dots, a_L$*

$$\frac{1}{K-1}\sum_{k=2}^{K}\frac{T_k\gamma_k(\ell)\bar{\gamma}_k(\ell')}{1+\sum_{\underline{\ell}} a_{\underline{\ell}} T_k|\gamma_k(\underline{\ell})|^2} \to 0$$

almost surely, as $N \to \infty$.

Let $(\delta_1, \dots, \delta_L)^T \in \mathbb{R}^L$ denote a random vector having distribution G. Then the conclusions of the Theorem and Corollary 1 hold, with each $|\alpha_\ell|^2$ in (1.3) replaced by δ_ℓ.

Remarks:

1. The assumption of the s_{ij} coming from a doubly infinite array can be replaced with $s_{ij} = s_{ij}(N)$, $1 \le i \le N$, $1 \le j \le K$, with no dependency assumptions for different N, provided $\mathsf{E}|s_{11}|^4 < \infty$.

2. The Theorem only provides limiting properties of the signal-to-interference ratio with respect to one user.

Let

$$C_k = \frac{1}{N}\left(\sum_{j=1}^{K} \beta_j \beta_j^* - \beta_k \beta_k^*\right).$$

Then

$$\mathbf{SIR}_k \equiv \frac{1}{N}\beta_k^*(C_k + \sigma^2 I)^{-1}\beta_k = \frac{1}{N}\sum_{\ell,\ell'} \bar{\alpha}_k(\ell)\alpha_k(\ell')s_k^*(C_k + \sigma^2 I)^{-1}_{\ell,\ell'} s_k$$

represents user k's best signal-to-interference ratio.

If $\mathsf{E}|s_{11}|^4 < \infty$, or if the doubly infinite array assumption is dropped, $\mathsf{E}|s_{11}|^6 < \infty$, then

$$\max_{k \le K} |N^{-1} s_k^* (C_k + \sigma^2 I)^{-1}_{\ell,\ell'} s_k - a_{\ell,\ell'}| \to 0 \quad a.s.$$

as $N \to \infty$.

2. Basic tools. For any rectangular matrix X, **vec**X denotes the column vector consisting of stacking the columns of X on top of each other, first column on top, last on bottom.

Lemma 2.1. *Let $\sigma^2 > 0$, B, A $n \times n$ matrices with B Hermitian non-negative definite, and $x \in \mathbb{C}^n$. Then*

$$\mathrm{tr}\,((B+xx^*+\sigma^2 I)^{-1} - (B+\sigma^2 I)^{-1})A| = \left| \frac{x^*(B+\sigma^2 I)^{-1} A (B+\sigma^2 I)^{-1} x}{1 + x^*(B+\sigma^2 I)^{-1} x} \right|$$

$$\leq \frac{\|A\|}{\sigma^2}.$$

Lemma 2.2. *For any matrix A $N \times K$ and $\sigma^2 > 0$*

$$(AA^* + \sigma^2 I_N)^{-1} = \sigma^{-2}(I_N - A(A^*A + \sigma^2 I_K)^{-1} A^*).$$

Lemma 2.3. *Suppose $A_1, \ldots, A_L$ are $N \times K$, and $\sigma^2 > 0$. Define the ℓ, ℓ' block of the $NL \times NL$ matrix A by $A_{\ell,\ell'} = A_\ell A^*_{\ell'}$, and,* *splitting* $(A+\sigma^2 I)^{-1}$ *into* L^2 $N \times N$ *matrices, let* $(A+\sigma^2 I)^{-1}_{\ell,\ell'}$ *denote its* ℓ, ℓ' *block. Then*

$$(A+\sigma^2 I)^{-1}_{\ell,\ell'} = \sigma^{-2}\left(\delta_{\ell,\ell'} I_N - A_\ell \left(\sum_{\underline{\ell}} A^*_{\underline{\ell}} A_{\underline{\ell}} + \sigma^2 I_K\right)^{-1} A^*_{\ell'}\right).$$

LEMMA 2.4 *Given* $A_1, ..., A_L$ *are* $N \times K$ *and* $z_1, \ldots, z_\ell \in \mathbb{C}$ *with* $\sum_\ell |z_\ell|^2 = 1$

$$\left(\sum_\ell A_\ell z_\ell\right)\left(\sum_\ell A^*_\ell \hat{z}_\ell\right) \preceq \sum_\ell A_\ell A^*_\ell,$$

where "$\preceq$*" represents the partial ordering on Hermitian nonnegative definite matrices.*

Lemma 2.5 *For* $A_1, ..., A_L$, A, σ^2 *in Lemma 2.3, the* $L \times L$ *matrix* $(\mathrm{tr}\,(A+\sigma^2 I)^{-1}_{\ell,\ell'})$ *is positive definite with smallest eigenvalue bounded below by*

$$\mathrm{tr}\left(\sum_\ell A_\ell A^*_\ell + \sigma^2 I_N\right)^{-1}.$$

For $A = (a_{ij})$ $m \times n$ and B $p \times q$, the Kronecker product of A and B, denoted by $A \otimes B$, is the $mp \times nq$ matrix, expressed in blocks of $p \times q$ matrices, the i, j block being $a_{ij}B$. We will need the following, which is Lemma 4.2.10 of Horn and Johnson (1991)

LEMMA 2.6. (Lemma 4.2.10 of Horn and Johnson (1991)) *For A $m \times n$, B $p \times q$, C $n \times k$ and D $q \times r$ we have*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

LEMMA 2.7, SCHAUDER FIXED POINT THEOREM (Nirenberg (1961)). *If $\mathcal{A}$ is a convex, compact subset of a Banach space $\mathcal{X}$ and $g : \mathcal{A} \to \mathcal{A}$ is continuous, then g has a fixed point in $\mathcal{A}$.*

LEMMA 2.8. *Let $A = (a_{ij}) = (a_1, \ldots, a_n)$ $(m \times n)$ and B $(h \times g)$ be two random matrices, the entries having bounded second moments.*

Then

$$\|\mathsf{E}A \otimes B\| \leq \min\left(\sqrt{\|\mathsf{E}AA^*\|\,\|\mathsf{E}B^*B\|}, \sqrt{\|\mathsf{E}A^*A\|\,\|\mathsf{E}BB^*\|}\right).$$

LEMMA 2.9. *For $X = (X_1, \dots, X_n)^T$ i.i.d. standardized entries, C $n \times n$, we have for any $p \geq 2$*

$$\mathsf{E}|X^*CX - \operatorname{tr} C|^p \leq K_p\left(\left(\mathsf{E}|X_1|^4 \operatorname{tr} CC^*\right)^{p/2} + \mathsf{E}|X_1|^{2p} \operatorname{tr}(CC^*)^{p/2}\right)$$

where the constant K_p does not depend on n, C, nor on the distribution of X_1.

3. Sketch of proof of Theorem 1.1. Write $\sqrt{T_k}\gamma_k(\ell)$ as $\alpha_k(\ell)$. It is first shown that, after truncating and centralizing the entries of s_1, from Lemma 2.9, with probability one

$$|\mathbf{SIR}_1 - N^{-1}\sum_{\ell,\ell'}\bar{\alpha}_1(\ell)\alpha_1(\ell')\mathrm{tr}\,(C+\sigma^2 I)^{-1}_{\ell,\ell'}| \to 0.$$

After additional truncation and centralization steps can assume for each N:

1. $s_{nk} = s_{nk}(N)$, $1 \le n \le N$, $2 \le k \le K$, i.i.d. standardized random variables with $|s_{nk}| \le \log N$.
2. $\max_{2\le k\le K,\ell} |\alpha_k(\ell)|^2 \le \log N$.

Define $C_{(k)} = C - (1/N)\beta_k\beta_k^*$.

Define the $L \times L$ matrix $\underline{B} = (\underline{b}_{\ell,\ell'})$ with

$$\underline{b}_{\ell,\ell'} = \frac{1}{N}\sum_{k=2}^{K}\frac{\bar{\alpha}_k(\ell')\alpha_k(\ell)}{1+\frac{1}{N}\beta_k^*(C_{(k)}+\sigma^2 I)^{-1}\beta_k},$$

and define the $NL \times NL$ matrix B in terms of the Kronecker product: $B = \underline{B} \otimes I_N$. We have $(B+\sigma^2 I)^{-1} = (\underline{B}+\sigma^2 I_L)^{-1} \otimes I_N$. Denote the ℓ, ℓ' entry of $(\underline{B}+\sigma^2 I_L)^{-1}$ by $\hat{b}_{\ell,\ell'}$.

Let $I_{\ell',\ell}$ denote the $NL \times NL$ matrix consisting of the $N \times N$ identity matrix in the ℓ', ℓ block, zeros elsewhere. We write

$$C + \sigma^2 I - (B + \sigma^2 I) = \frac{1}{N} \sum_{k=2}^{K} \beta_k \beta_k^* - B.$$

Taking inverses on each side, we have

$$(B+\sigma^2 I)^{-1} - (C+\sigma^2 I)^{-1}$$

$$= \frac{1}{N} \sum_{k=2}^{K} (B+\sigma^2 I)^{-1} \beta_k \beta_k^* (C+\sigma^2 I)^{-1} - (B+\sigma^2 I)^{-1} B (C+\sigma^2 I)^{-1}$$

$$= \frac{1}{N} \sum_{k=2}^{K} \frac{(B+\sigma^2 I)^{-1} \beta_k \beta_k^* (C_{(k)}+\sigma^2 I)^{-1}}{1 + (1/N) \beta_k^* (C_{(k)}+\sigma^2 I)^{-1} \beta_k} - (B+\sigma^2 I)^{-1} B (C+\sigma^2 I)^{-1}.$$

Multiplying on the right by $I_{\ell',\ell}$, taking traces, and dividing by N we get

$$N^{-1}\mathrm{tr}\,(B+\sigma^2 I)^{-1}_{\ell,\ell'} - N^{-1}\mathrm{tr}\,(C+\sigma^2 I)^{-1}_{\ell,\ell'}$$

$$= \frac{1}{N}\sum_{k=2}^{K} \frac{\frac{1}{N}\beta_k^*(C_{(k)}+\sigma^2 I)^{-1} I_{\ell',\ell}(B+\sigma^2 I)^{-1}\beta_k}{1+\frac{1}{N}\beta_k^*(C_{(k)}+\sigma^2 I)^{-1}\beta_k}$$

$$-N^{-1}\mathrm{tr}\,B(C+\sigma^2 I)^{-1} I_{\ell',\ell}(B+\sigma^2 I)^{-1}$$

$$= \frac{1}{N}\sum_{k=2}^{K} \frac{\sum_{\underline{\ell},\underline{\ell}'} \bar{\alpha}_k(\underline{\ell})\alpha_k(\underline{\ell}')\frac{1}{N}s_k^*[(C_{(k)}+\sigma^2 I)^{-1} I_{\ell',\ell}(B+\sigma^2 I)^{-1}]_{\underline{\ell},\underline{\ell}'} s_k}{1+\frac{1}{N}\beta_k^*(C_{(k)}+\sigma^2 I)^{-1}\beta_k}$$

$$-\frac{1}{N}\mathrm{tr}\sum_{\underline{\ell},\underline{\ell}'} B_{\underline{\ell}',\underline{\ell}}[(C+\sigma^2 I)^{-1} I_{\ell',\ell}(B+\sigma^2 I)^{-1}]_{\underline{\ell},\underline{\ell}'}$$

$$= \sum_{\underline{\ell},\underline{\ell}'} \frac{1}{N} \Bigg[\sum_{k=2}^{K} \frac{1}{N} \frac{s_k^* [(C_{(k)} + \sigma^2 I)^{-1} I_{\ell',\ell} (B + \sigma^2 I)^{-1}]_{\underline{\ell},\underline{\ell}'} s_k \bar{\alpha}_k(\underline{\ell}) \alpha_k(\underline{\ell}')}{1 + \frac{1}{N} \beta_k^* (C_{(k)} + \sigma^2 I)^{-1} \beta_k}$$

$$- \mathrm{tr}\, B_{\underline{\ell}',\underline{\ell}} [(C + \sigma^2 I)^{-1} I_{\ell',\ell} (B + \sigma^2 I)^{-1}]_{\underline{\ell},\underline{\ell}'} \Bigg]$$

$$= \sum_{\underline{\ell},\underline{\ell}'} \frac{1}{N} \Bigg[\sum_{k=2}^{K} \frac{1}{N} \frac{s_k^* (C_{(k)} + \sigma^2 I)^{-1}_{\underline{\ell},\ell'} (B + \sigma^2 I)^{-1}_{\ell,\underline{\ell}'} s_k \bar{\alpha}_k(\underline{\ell}) \alpha_k(\underline{\ell}')}{1 + \frac{1}{N} \beta_k^* (C_{(k)} + \sigma^2 I)^{-1} \beta_k}$$

$$- \mathrm{tr}\, B_{\underline{\ell}',\underline{\ell}} [(C + \sigma^2 I)^{-1}_{\underline{\ell},\ell'} (B + \sigma^2 I)^{-1}_{\ell,\underline{\ell}'} \Bigg]$$

$$= \sum_{\underline{\ell},\underline{\ell}'} \frac{1}{N} \sum_{k=2}^{K} \frac{\hat{b}_{\ell,\underline{\ell}'} \bar{\alpha}_k(\underline{\ell}) \alpha_k(\underline{\ell}') N^{-1} (s_k^* (C_{(k)} + \sigma^2 I)^{-1}_{\underline{\ell},\ell'} s_k - \mathrm{tr}\, (C + \sigma^2 I)^{-1}_{\underline{\ell},\ell'})}{1 + \frac{1}{N} \beta_k^* (C_{(k)} + \sigma^2 I)^{-1} \beta_k}.$$

Noticing that

$$N^{-1}\mathrm{tr}\,(B+\sigma^2 I)^{-1}_{\ell,\ell'} = \hat{b}_{\ell,\ell'},$$

we get from Lemmas 2.1 and 2.9

$$|\hat{b}_{\ell,\ell'} - N^{-1}\mathrm{tr}\,(C+\sigma^2 I)^{-1}_{\ell,\ell'}| \to 0 \quad a.s.$$

Again, from these lemmas

$$\left| \underline{b}_{\ell,\ell'} - \frac{1}{N}\sum_{k=2}^{K} \frac{\bar{\alpha}_k(\ell')\alpha_k(\ell)}{1+\sum_{\underline{\ell},\underline{\ell}'} \bar{\alpha}_k(\underline{\ell})\alpha_k(\underline{\ell}')N^{-1}\mathrm{tr}\,(C+\sigma^2 I)^{-1}_{\underline{\ell},\underline{\ell}'}} \right| \to 0 \quad a.s.$$

4. Proof of uniqueness. Suppose A and $\widetilde{A}$ are two different $L \times L$ Hermitian positive definite matrices satisfying (1.1). Then

$$A - \widetilde{A} = c\,\mathsf{E}\frac{A\alpha\alpha^*\widetilde{A}\alpha^*(A-\widetilde{A})\alpha}{(1+\alpha^*A\alpha)(1+\alpha^*\widetilde{A}\alpha)}.$$

Multiplying $A^{-1/2}$ on the left and $\widetilde{A}^{-1/2}$ on the right we obtain

$$A^{1/2}\widetilde{A}^{-1/2} - A^{-1/2}\widetilde{A}^{1/2} = c\,\mathsf{E}\frac{A^{1/2}\alpha\alpha^*\widetilde{A}^{1/2}\alpha^*(A-\widetilde{A})\alpha}{(1+\alpha^*A\alpha)(1+\alpha^*\widetilde{A}\alpha)}$$

$$= c\,\mathsf{E}\frac{\eta\widetilde{\eta}^*\eta^*(A^{1/2}\widetilde{A}^{-1/2} - A^{-1/2}\widetilde{A}^{1/2})\widetilde{\eta}}{(1+\alpha^*A\alpha)(1+\alpha^*\widetilde{A}\alpha)},$$

where $\eta = A^{1/2}\alpha$ and $\widetilde{\eta} = \widetilde{A}^{1/2}\alpha$. Write

$$\mu = \mathbf{vec}(A^{1/2}\widetilde{A}^{-1/2} - A^{-1/2}\widetilde{A}^{1/2}).$$

With the aid of the Kronecker product we can write the above equation as

$$\mu = c\,\mathsf{E}\frac{(\overline{\widetilde{\eta}}\otimes\eta)(\widetilde{\eta}^T\otimes\eta^*)}{(1+\alpha^*A\alpha)(1+\alpha^*\widetilde{A}\alpha)}\mu. \tag{4.1}$$

Using Lemma 2.6 we have

$$c\,\mathsf{E}\frac{(\overline{\widetilde{\eta}}\otimes\eta)(\widetilde{\eta}^T\otimes\eta^*)}{(1+\alpha^*A\alpha)(1+\alpha^*\widetilde{A}\alpha)} = c\,\mathsf{E}\left[\frac{\overline{\widetilde{\eta}\widetilde{\eta}^*}}{1+\alpha^*\widetilde{A}\alpha}\otimes\frac{\eta\eta^*}{1+\alpha^*\widetilde{A}\alpha}\right]$$

and, since $\mu\neq 0$, this matrix has an eigenvalue equal to 1. By Lemma 2.8 its largest squared eigenvalue cannot be greater than

$$\left\|c\,\mathsf{E}\left(\frac{\widetilde{\eta}\widetilde{\eta}^*}{1+\alpha^*\widetilde{A}\alpha}\right)^2\right\|\left\|c\,\mathsf{E}\left(\frac{\eta\eta^*}{1+\alpha^*A\alpha}\right)^2\right\|.$$

We have

$$c\,\mathsf{E}\left(\frac{\eta\eta^*}{1+\alpha^*A\alpha}\right)^2 = c\,\mathsf{E}\frac{A^{1/2}\alpha\alpha^*A^{1/2}\alpha^*A\alpha}{(1+\alpha^*A\alpha)^2},$$

and since

$$\frac{A^{1/2}\alpha\alpha^* A^{1/2}}{1+\alpha^* A\alpha} - \frac{A^{1/2}\alpha\alpha^* A^{1/2}\alpha^* A\alpha}{(1+\alpha^* A\alpha)^2}$$

is non-negative definite we have

$$c\,\mathsf{E}\left(\frac{\eta\eta^*}{1+\alpha^* A\alpha}\right)^2 \preceq c\,\mathsf{E}\frac{A^{1/2}\alpha\alpha^* A^{1/2}}{1+\alpha^* A\alpha} = A^{1/2}(A^{-1} - \sigma^2 I_L)A^{1/2}$$

$$= I_L - \sigma^2 A,$$

the eigenvalues of which must all be less than one. The same result applies for the other matrix involving $\widetilde{A}$. Therefore, the matrix in (4.1) cannot have an eiqenvalue equal to one, a contradiction. So we conclude that there is only one Hermitian positive definite solution to (1.1).