From random matrices to stochastic operators

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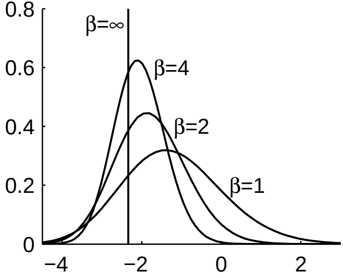
Acknowledgment

Work with/inspired by Alan Edelman and Ioana Dumitriu.

Abstract

Result 1. The *soft edge* (aka *Tracy-Widom*) distributions

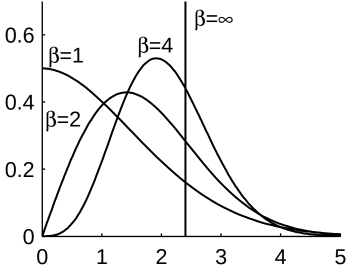


describe the eigenvalues of the *stochastic Airy operator*,

$$\left(-\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} B'\right) f = \lambda f, \qquad f(0) = f(\infty) = 0.$$

Abstract

Result 2. The *hard edge* distributions

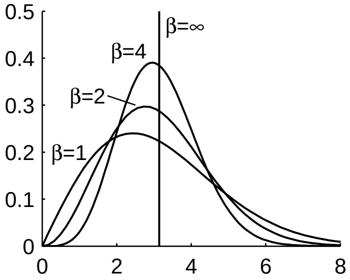


describe the singular values of the *stochastic Bessel operator*,

$$\left(-\frac{d}{dx}+\frac{2a+1}{2x}+\sqrt{\frac{2}{\beta}}\frac{1}{\sqrt{x}}B'\right)f=\sigma g, \qquad f(1)=0,\ g(0)=0.$$

Abstract

Result 3. The *bulk* (aka *universal spacing*) distributions



describe the eigenvalues of the *stochastic sine operator*,

$$\left(\left[\begin{array}{c|c} & -\frac{d}{dx} \\ \hline \frac{d}{dx} & \end{array} \right] + \text{noise} \right) \left[\begin{array}{c} f \\ \hline g \end{array} \right] = \lambda \left[\begin{array}{c} f \\ \hline g \end{array} \right], \qquad f(1) = 0,\ g(0) = 0.$$

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  $$\begin{aligned} &\text{classical random matrix models} \\ &= \text{finite difference schemes for SDOs} \end{aligned}$$

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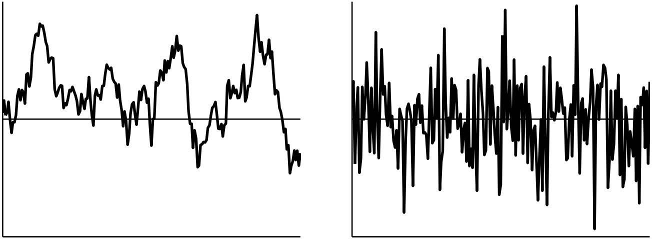
- Question: What is white noise? Does it cause problems?
  Answer:

Brownian motion $B(x)$ White noise $B'(x)$



By changing variables, we can express the SDOs in terms of Brownian motion instead of white noise.

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4. Wigner's original idea.
   Random operator $\leftrightarrow$ typical Hamiltonian.

The talk begins now.

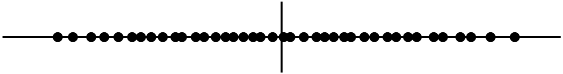
Background

1. Matrix models
   - Hermite (Gaussian)
   - Laguerre (Wishart)
   - Jacobi (MANOVA)
2. Local eigenvalue statistics
   - Soft edge
   - Hard edge
   - Bulk spacings
3. Sturm-Liouville problems
4. Finite differences

Matrix models

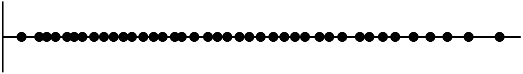
Hermite (Gaussian)

- Hermitian with independent Gaussian entries
- All *n* eigenvalues:



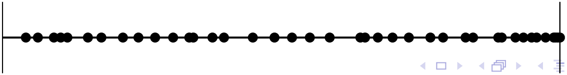
Laguerre (Wishart)

- Rectangular with independent Gaussian entries
- All *n* singular values:



Jacobi (MANOVA)

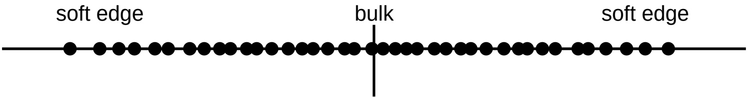
- Uniform distribution on unitary matrices
- All *n* CS values [S, Edelman]:



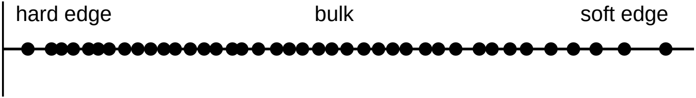
Local eigenvalue statistics

Different random matrix distributions exhibit the same local eigenvalue behavior.

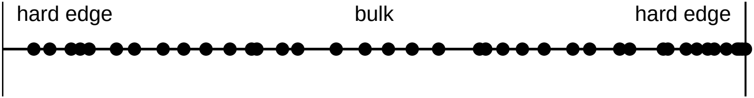
- Hermite



- Laguerre



- Jacobi



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Sturm-Liouville problems

The simplest Sturm-Liouville problem:

$$-\frac{d^2}{dx^2}f = \lambda f, \qquad f(0) = f(\pi) = 0.$$

Finite differences

Examples.

$$\frac{1}{h}D_1 = \frac{1}{h}\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix} \approx \frac{d}{dt}$$

$$\frac{1}{h^2}D_2 = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} \approx \frac{d^2}{dt^2}$$

How does a random matrix produce
a stochastic Sturm-Liouville problem?

Results

Classical random matrix models are finite difference schemes for SDOs.

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- Soft edge $\longleftrightarrow$ stochastic Airy op. $-\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} B'$
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Plan:

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   Hermite (scaled at right edge) $\rightarrow$ stochastic Airy operator

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Plan:

1. How were the SDOs discovered?
   Hermite (scaled at right edge) $\rightarrow$ stochastic Airy operator
2. What is white noise?
   Change of variables replaces B' with B.

Hermite $\to$ Airy

Hermite matrix model (rescaled)
= finite difference scheme for the stochastic Airy operator

Hermite → Airy

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$$\frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}G & G & G & \cdots & G \\ G & \sqrt{2}G & G & \cdots & G \\ G & G & \sqrt{2}G & \cdots & G \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G & G & G & \cdots & \sqrt{2}G \end{bmatrix}$$

↓ tridiagonalization ↓

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}G & \chi_{n-1} & & & \\ \chi_{n-1} & \sqrt{2}G & \chi_{n-2} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_2 & \sqrt{2}G & \chi_1 \\ & & & \chi_1 & \sqrt{2}G \end{bmatrix}$$

Hermite → Airy

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}G & \chi_{n-1} & & & \\ \chi_{n-1} & \sqrt{2}G & \chi_{n-2} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_2 & \sqrt{2}G & \chi_1 \\ & & & \chi_1 & \sqrt{2}G \end{bmatrix}$$

$\downarrow$ diagonal similarity $\downarrow$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}G & 1 & & & \\ \chi_{n-1}^2 & \sqrt{2}G & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_2^2 & \sqrt{2}G & 1 \\ & & & \chi_1^2 & \sqrt{2}G \end{bmatrix}$$

Hermite $\to$ Airy

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}G & 1 & & & \\ \chi^2_{n-1} & \sqrt{2}G & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \chi^2_2 & \sqrt{2}G & 1 \\ & & & \chi^2_1 & \sqrt{2}G \end{bmatrix}$$

$\downarrow$ translate, rescale, and rename $\downarrow$

$$-\frac{1}{h^2}D_2 + \mathrm{diag}_{-1}(x_1,\ldots,x_{n-1}) + \frac{2}{\sqrt{\beta}}\cdot\frac{1}{\sqrt{h}}\frac{1}{\sqrt{2}}\begin{bmatrix} G & & & & \\ \tilde{\chi}^2_{n-1} & G & & & \\ & \ddots & \ddots & & \\ & & \tilde{\chi}^2_2 & G & \\ & & & \tilde{\chi}^2_1 & G \end{bmatrix}$$

$h = n^{-1/3}$, $x_i = hi$, $E[\tilde{\chi}^2_r] = 0$, $\mathrm{Var}[\tilde{\chi}^2_r] \approx 1$.

Hermite → Airy

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↓ finite difference interpretation ↓

$$\mathcal{A}^\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} B'$$

This is the stochastic Airy operator.

$$\lambda_k(\mathcal{A}^\beta) \sim \text{soft edge}(\beta, k)$$

Hermite → Airy

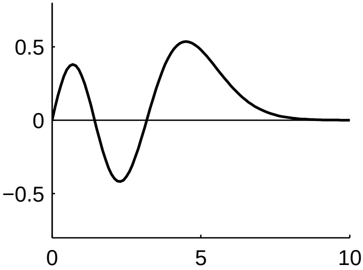
Classical Airy operator

$$\mathcal{A}^\infty = -\frac{d^2}{dx^2} + x$$

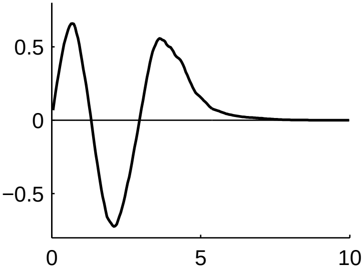
Stochastic Airy operator

$$\mathcal{A}^\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} B'$$

eigenfunction of $\mathcal{A}^\infty$



eigenvector of Hermite



Replacing white noise with Brownian motion

Stochastic Airy operator:

$$\mathcal{A}^\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} B'$$

B' is white noise, which is potentially problematic.
(Itô vs. Stratonovich, etc.)

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Without white noise:

$$\mathcal{A}^\beta = \phi \left(-\frac{d^2}{dx^2} - \frac{4}{\sqrt{\beta}} B \frac{d}{dx} + x - \frac{4}{\beta} B^2 \right) \phi^{-1}$$

$$\phi(x) = \exp \left(\frac{2}{\sqrt{\beta}} \int_0^x B(x') dx' \right)$$

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Lesson: The ith eigenfunction is $f_i \phi$, $f_i \in C^2$, $\phi \in C^1$.

Structure of eigenfunctions

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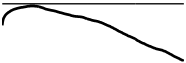
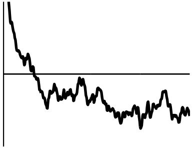
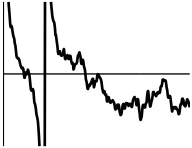
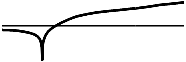
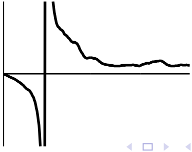
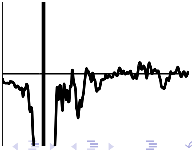
Hermite eigenvector	$\log\|v\|$	$\nabla \log\|v\|$	$\nabla^2 \log\|v\|$
1st			
2nd			
ratio			

More scaling limits

matrix model	scaling limit	behavior	stochastic operator
Hermite	left edge	soft	Airy
Hermite	right edge	soft	Airy
Hermite	center	bulk	sine
Laguerre	left edge	hard	Bessel
Laguerre	right edge	soft	Airy
Jacobi	left edge	hard	Bessel
Jacobi	right edge	hard	Bessel
Jacobi	center	bulk	sine

Stochastic Bessel operator

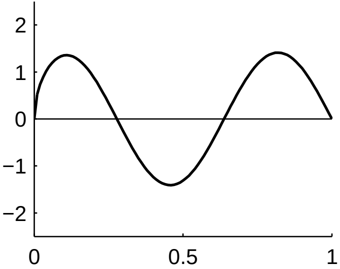
Classical Bessel operator

$$\mathcal{J}_a^\infty = -\frac{d}{dx} + \frac{2a+1}{2x}$$

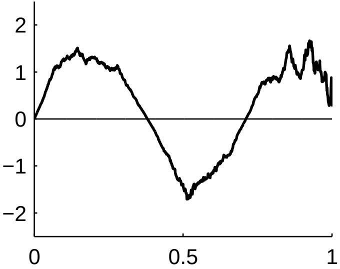
Stochastic Bessel operator

$$\mathcal{J}_a^\beta = -\frac{d}{dx} + \frac{2a+1}{2x} + \sqrt{\frac{2}{\beta}}\frac{1}{\sqrt{x}}B'$$

singular function of $\mathcal{J}_0^\infty$



singular vector of Jacobi



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Advantages:

1. Interesting eigenfunctions.
2. One-to-one correspondence
   local behavior (soft/hard/bulk) $\leftrightarrow$ operator (Airy/Bessel/sine)
3. General β.