2. A model of an unreliable machine

Introduction

In this chapter we analyze the following stochastic system: a single machine produces parts at a deterministic rate but is subject to random failures. When the machine fails, it is completely inoperable until it is repaired. Hence, at any time the machine is in one of two states: working or failed. We assume that the times between failures are i.i.d. exponential with mean time between failures (MTBF) equal to $1/\lambda$ and that times to repair are i.i.d. exponential with mean time to repair (MTTR) equal to $1/\mu$. We assume operation-dependent failures (Gershwin, 1994); that is, the machine can not fail while it is under repair or idle.

Literature review

Reliability has been a topic of active research with origins dating back to the turn of the century (according to Nahmias, 1989). See Shaked and Shanthikumar (1990) for a recent survey of the field. The analysis of a single unit with two operating states is perhaps the simplest problem in the study of the reliability. Within this problem subclass, the case of i.i.d. exponential failure and repair times is the most tractable, and has received virtually independent attention in a variety of fields. The problem has also been studied in the telecommunications literature as the *asymmetric random telegraph signal*, in radioactive physics as a type II counter problem (Bharucha-Reid, 1960), in the engineering literature in the analysis of the output of a resistance-capacitance (RC) filter driven by a random binary process (Munford, 1986), and in the biology literature in the analysis of channels in the nerve membrane (see FitzHugh, 1983), among others. The problem can also be analyzed using many different methodologies; for example, the problem can be viewed as a simple case of a Markov Process, or as an alternating renewal process.

Barlow and Hunter (1961) provide an excellent summary of the known results at that time, based largely on Laplace transform and renewal theory. They derive the transform of the expected number of failures and repairs in [0, T), the asymptotic number of failures and repairs, and the transform of the distribution of the number of failures in [0, T). If one knows the expected number of failures and repairs in [0, T) then the transient or asymptotic *availability coefficient* (the probability that the unit is functioning at a given time) is given by their difference. For the case of exponential repairs and failures, they give closed form expressions for the availability coefficient and the distribution of the number of failures in [0, T). Classic texts such as Barlow and Proschan (1965) and Gnedenko et al. (1969) on reliability, and Cox (1962) on renewal theory derive many of these results.

Barlow and Hunter (1961) also give an expression for the distribution of downtime over [0, T) as an infinite series of the n-fold convolutions of the failure and repair distributions with themselves. They also express the result as an integral in the case of exponential failure and repair times, and for general failure and repair times give the asymptotic distribution as T approaches infinity. These results are all due to Takács (1957a, 1957b, 1959). For exponential failures and exponential repairs, Gnedenko et al. (1969) express the distribution of total operating time over a fixed period [0, T) as a double infinite series.

Lie et al. (1977) give a comprehensive although now somewhat outdated survey and classification of availability models. More recently, Baxter (1985) presents a critical review of the literature on the availability of two-state unit modeled as an alternating renewal process. This paper, in conjunction with Baxter (1981), seem to be the most complete and recent summary of important results, and also “fill some gaps in the theory". Unique to this paper is a review and extension of the results on waiting times (i.e., the distribution of time until a repair greater than a certain length occurs) and on the alternating renewal process where each repair time is correlated with the previous failure time. Baxter also reviews and extends the theory on point availability and average availability, and criticizes Barlow and Proschan (1965) for their "uncritical" application of the asymptotic approximation for the distribution of availability, citing simulation studies which show that passage to the limit can be extremely slow.

Other important contributions are numerous and are scattered over a variety of works. Baxter (1985) finds expressions for the average availability of an alternating renewal process, and gives the simple result in the case of exponential failure and repair times. The average availability over (0, T] can be used to find the average uptime over (0, T] simply by multiplying by T, and the average repair time by subtracting the average uptime from T. Martz (1971) develops a method which can be used to find the distribution of the average availability over n failure and repair cycles for any failure and repair distributions for which the n-fold convolutions are known. For exponential failures and exponential repairs, FitzHugh (1983) finds both the density and the Laplace transform of the number of failure / repair cycles over a fixed period [0, T). He also cites expressions for the autocorrelation and spectral density of the process, which have been derived in the biology and physics literature. For the general case of alternating renewal processes, Mortensen (1990) finds the Laplace transform for the density of the availability coefficient at time T and the asymptotic autocorrelation of the availability coefficient.

Feller (1971), Brouwers (1986) and Kim and Alden (1992) independently derived the density of time to produce a fixed lot size on a machine operating at a constant speed with exponential failure and repair times. This is equivalent to the density of time until the total uptime reaches some constant T. These authors express the result as a modified Bessel function, which has important theoretical and practical implications, and are a considerable improvement over the previous result of Gnedenko et al. The latter two works also give a simple expression for the variance as a function of T, which is a trivial result if one recognizes that the process can be viewed as a Compound Poisson process (Ross, 1983).

There has also been considerable work, both exact and approximate, on other failure and repair distributions. For example, Kabak (1969) analyzes the exponential failure and constant repair time problem, finds the average availability over (0, T], and develops an approximation for the variance. Takács (1951) uses his general methods to find the distribution of repair time over [0, T), with the machine either starting in a known state or starting in steady-state. For the case of constant repair times and the family of Weibull failure distributions (which include the exponential), Dickey (1991) derives a double series for the availability coefficient at time T and renewal function (expected number of failures in [0, T)).

In addition, numerical results for general failure and repair distributions can be obtained by the method of Cléroux and McConalogue (1976) and McConalogue (1978, 1981). This method numerically evaluates convolution integrals, so many of the above results can (at least in principle) be obtained by algorithm for the general case. When the Laplace transforms of the failure and repair distributions are explicitly known, they could instead be numerically inverted. See Baxter (1981a, 1981b) for a further discussion. Laplace transform inversion has received considerable attention from many authors; see Krylov and Skoblya (1969), for a survey of the classic methods. The last two decades have seen a variety of more powerful and sophisticated methods using both new and old techniques, such as Fourier series approximation (Crump, 1976, De

Hoog, et al. 1982, Piessens and Huysmans, 1984, Abate and Whitt, 1992), continued fraction expansion (Grundy, 1977), contour integration (Murli and Rizzardi, 1990), and expansion in Laguerre polynomials (Garbow et al. 1988). Many of these codes are available via *netlib* (Dongarra and Grosse, 1987).

Lastly, we note that some work on more complex systems could be used to analyze our (relatively) more simple system. For example, Sericola (1990) develops a closed-form solution for the transient distribution of total time spent in a subset of states of a homogenous Markov process over a fixed period of time [0, T). Our two state Markov process is the most trivial problem to which this method could be applied.

In the sections that follow we explore many of these same question as those described above, sometimes from new perspectives and obtaining some new results. When our results duplicate those of previous works, references will be given in context. In whole, this chapter will present a unified treatment of the results that we will need in other parts of our work.

Notation, summary of key results, and overview of this chapter

Density functions will be denoted by a lowercase letter (r), cumulative distribution functions by an uppercase letter (R) and random variables by a bold capital letter (**R**). The Laplace transform of a function g(t) will be denoted by g*(s). We will also use $\mathcal{L}$ { } and $\mathcal{L}^{-1}${ } to denote the Laplace transform and inverse Laplace transform of the expression in brackets. We will use the symbol ☆ to denote the convolution operator, and the symbol ⇔ to represent that the expression on the left is the Laplace transform of the expression on the right. Pr{ } will denote the probability of the event in brackets. When we wish to write the probability that a continuous random variable **X** (a, a+dx), we will write dens{ **X** = a }, since Pr{ **X** = a } = 0.

λ will denote the failure rate when the machine is working, and μ will denote the repair rate when the machine is failed. Let $\alpha(\cdot)$ be an indicator function, where $\alpha(\tau) = 0$ if the machine is failed at time τ, and $\alpha(\tau) = 1$ if it is working at time τ.

We will now give an overview of the remainder of this chapter. This overview will also serve to introduce much of the important notation that we will use. As we proceed, we will list some of the key results of this chapter. Many of the equation numbers for these key results are also given, in parentheses.

The purpose of the next seven sections will be to characterize the number of parts produced over a fixed time interval, and the quantity of time required to produce a fixed number of parts. Although it is *numbers of parts* that we are concerned with, we will often derive expressions in terms of machine time. It is important to keep in mind that machine time can be converted to parts by simply multiplying by the production rate, which is assumed to be constant when the machine is working.

The purpose of Sections 2.1 - 2.5 is to derive the PDF, CDF and Laplace transform of the number of parts produced over a fixed time interval, and of the quantity of time required to produce a fixed number of parts. The models in Chapters 3 will require a probabilistic description of the number of parts produced over a fixed time interval. The simulations in Chapter 4 will require the quantity of time required to produce a fixed number of parts. We will see throughout the development that the number of parts produced over a fixed time interval and the quantity of time required to produce a fixed number of parts are very closely related.

We now give a summary of the key results of each section of this chapter. The focus of Section 2.1 will be to derive the probability density function for the uptime of an unreliable machine over an interval of length T. If the machine is working at time 0, we will denote this density as this as this as $f(t; T \mid \alpha(0) = 1)$ and abbreviate it as $f(t; T \mid 1)$. We will show that

$$(3) \quad f(t; T \mid 1) = \left[\lambda\mu t \, \frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + \lambda \; I_0\left(2\sqrt{x}\right) \right] e^{-\lambda t - \mu(T-t)} + u_0(T-t)\, e^{-\lambda T}, \qquad 0 \le t \le T$$

where I_0 and I_1 are modified Bessel functions of orders zero and one, $x = \lambda \, \mu \, t \, (T-t)$, and $u_0(z)$ is the unit impulse (Dirac delta) function which is zero everywhere except for an impulse of mass one at z. We derive similar expressions for the cases where the machine is initially failed (4) or in steady state (5). We then derive the probability density function of uptime over an interval of length T conditional on the initial machine state and the machine state T time units later. In our notation this is $f(t; T \mid \alpha(0) = a, \alpha(T) = b)$, and we will abbreviate this as $f(t; T \mid ab)$. The results for the four possible combinations of beginning and ending machine states are given in (6) - (9). We believe that these results have not appeared in the literature. Unfortunately, these PDFs are not easily integrated, so we will need to obtain the CDF and the Laplace transform by other methods.

In Section 2.2 we derive the Laplace transform of the density function (3). The result is

$$(17) \quad f^*(s, \; T \mid 1) = \frac{(\lambda + \mu - s)T}{2} \; \frac{\mathbf{sinh}\, y}{y \; e^h} + \frac{\mathbf{cosh}\, y}{e^h},$$

where $y = \sqrt{\lambda^2 + 2\lambda\mu + \mu^2 + 2\lambda s - 2\mu s + s^2} \;\; T / 2$ and $h = (\lambda + \mu + s)\, T / 2$.

We believe that this result is new. We then use this expression to find the transient mean (18) and variance (25). The asymptotic mean and variance follow easily. We derive similar results for the cases where the machine is initially failed or in steady state. Lastly, we derive the transform, mean and variance for the cases where the machine where the initial and terminal machine states are given.

In Section 2.3 we turn our attention to characterizing the time to produce a fixed lot of parts. Since the quantity of machine uptime required to produce a fixed size lot is deterministic, we focus on the probability density function for downtime incurred while producing a fixed size lot. We will denote this PDF by r(t; b | 1) where $b = \lambda q / p$, q is size of the lot to be produced, and p is the production rate of the machine when it is not failed. We will show that the Laplace transform of this PDF is

$$(32)\quad r^*(s;\ b\,|\,1) \;=\; \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right)$$

which, when inverted, is

$$(33)\quad r(t;b\,|1) \;=\; u_0(t)\mathbf{exp}(-b) \;+\; \mu\ b\ \mathbf{exp}(-\mu t - b)\ I_1\!\left(2\sqrt{\mu b t}\right)\ (\mu b t)^{-\frac{1}{2}},\ \ t \ge 0.$$

Although these results have been derived by Kim and Alden (1992) and others, the approach that we present is different, and although not new, provides insight into a more general problem. We also derive the transform and density for the case where the machine is initially failed.

Sections 2.4 and 2.5 derive expressions for the cumulative distributions of Sections 2.1 and 2.3. In Section 2.4 we show that

$$\text{(36)} \quad R(t; b \mid 1) \;=\; \mathbf{exp}(-\mu t - b) \sum_{v=0}^{\infty} \left(\frac{\mu t}{b}\right)^{\frac{v}{2}} I_v\left(2\sqrt{\mu b t}\right), \quad t \geq 0,$$

and a similar expression for R(t; b | 0), where $I_v(z)$ is the modified Bessel function of order v. We believe these results to be new, but have been independently derived by Kim (1994, unpublished).

In Section 2.5 we describe an equivalence between R(t; b | 1) from Section 2.4 and F(t; T | 1), the CDF corresponding to the density of Section 2.1. Using this equivalence we easily conclude that

$$\text{(39)} \quad F(t; T \mid 1) = \begin{cases} 0 & t = 0 \\ 1 - e^{-\mu(T-t)-\lambda t} \displaystyle\sum_{v=0}^{\infty} \left(\frac{\mu(T-t)}{\lambda t}\right)^{\frac{v}{2}} I_v\left(2\sqrt{\mu \;\lambda\; t\; (T-t)}\right) & 0 < t < T \\ 1 & t \geq T. \end{cases}$$

We obtain a similar result for the case where the machine is initially failed.

The remainder of Section 2.5 focuses on the more difficult case where we are given both the initial machine state and the machine state at some future point in time. This is useful in the dynamic decision making context described in Chapter 3, where the decision made at some future point in time may depend on the state of the machine.

We first derive the probability that the downtime while producing a batch of size q is at most t, given that the machine starts working and is also working at time t + q / p, where p is the production speed of the machine. In our notation, this probability is R(t; λ q / p | α(0) = 1, α(t+q / p) = 1), which we abbreviate as R(t; b | 11). The result is

$$(41)\quad R(t;b \mid 11) = \frac{\mu\, R(t;\ b \mid 1) + \lambda\, e^{-b-\mu t} \sum_{v=0}^{\infty} (-1)^v \left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_v\left(2\sqrt{\mu b t}\right)}{\mu + \lambda\, e^{-(\lambda+\mu)(t+q/p)}}.$$

We derive expressions for the other three cases (00, 01, 10) as well (42)-(44). An equivalence is between R(t; b | 11) and F(t; T | 11) is described and used to conclude that

$$(45)\quad F(t;T \mid 11) = 1 - \frac{\mu\left(1 - F(t;\ T \mid 1)\right) + \lambda\, e^{-\lambda t - \mu(T-t)} \sum_{v=0}^{\infty} (-1)^v \left(\frac{\lambda(T-t)}{\mu t}\right)^{v/2} I_v\left(2\sqrt{\mu\lambda t(T-t)}\right)}{\mu + \lambda\, e^{-(\lambda+\mu)T}}.$$

Similar expressions are derived for the other three cases (46)-(48). We then find

$$(49)\quad R^*(s;b \mid 10) = \sum_{n=0}^{\infty} \mathbf{exp}\left(-(\lambda+\mu)n\frac{q}{p} - b + b\frac{\mu}{s + \mu + (\lambda+\mu)\, n}\right) \times \left[\frac{1}{s + (\lambda+\mu)\, n} - \frac{1}{s + (\lambda+\mu)\,(n+1)}\right]$$

and similar expressions for the other three cases (50)-(52). We also show that

$$(53)\quad L\{R(t;b \mid 11)\, P_{11}(t+q/p)\} = \frac{1}{\lambda+\mu}\, \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right)\left[\frac{\mu}{s} + \frac{\lambda}{s + (\lambda+\mu)}\right],$$

where $P_{11}(T) = \Pr\{\alpha(T) = 1 \mid \alpha(0) = 1\}$. The left-hand side can be interpreted as $\Pr\{\text{downtime} \le t \ \&\ \alpha(t+q/p) = 1 \mid \alpha(0) = 1\}$. The advantage of (53) is that it does not contain an infinite series like the one in (49). We also obtain similar results for the other three cases. Lastly, equation (53) also allows us to write

$$(57)\quad F(q/p;\, t+q/p \mid 11) =$$

$$1 - \frac{1}{P_{11}\left(t + q/p\right)} \; L^{-1}\left\{ \frac{1}{\lambda+\mu}\, \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu} \right) \left[\frac{\mu}{s} + \frac{\lambda}{s + (\lambda+\mu)} \right] \right\}.$$

We believe all of these results to be new.

Section 2.6 explores the transient effects of initial machine conditions on the mean and variance of uptime over a fixed period of time. In Section 2.7 we investigate the accuracy of using a normal distribution to approximate the distribution of parts produced over a fixed period of time. Our results will confirm those cited by Baxter (1985), namely, that under certain conditions the normal distribution can be a poor approximation even after long time intervals. In Section 2.8 we develop exact and approximate methods for obtaining the distribution of time to produce multiple batches on a single machine, and show how that distribution is equivalent to another distribution of interest. Lastly, in an appendix to this chapter we discuss our experience testing two different Laplace transform inversion algorithms that we use in our empirical work in subsequent chapters.

2.1 Distribution of parts over a fixed period of time

In this section we analyze the uptime of a machine over the period of time [0, T) when interarrivals of failures and repairs are exponentially distributed with means λ and μ, respectively. This is equivalent to analyzing the number of parts produced if the uptime is scaled by the processing speed p.

Machine initially working

We will denote the state of the machine at any point in time $\alpha(\cdot)$, where $\alpha(\tau) = 0$ if the machine is failed at time τ, and $\alpha(\tau) = 1$ if it is working at time τ. Let $f(t; T \mid \alpha(0) = 1)$ be the PDF of uptime over [0, T) conditional on the machine working at time 0. We will abbreviate this as $f(t; T \mid 1)$. Let $h(t; T \mid 1)$ denote the PDF of downtime over [0, T) conditional on the machine working at time 0. Note that

$$f(t; T \mid 1) = h(T-t; T \mid 1),$$

since downtime = T – uptime. We will use this relation when we derive expressions for the PDF of uptime by characterizing the amount of downtime.

The PDF $f(t; T \mid 1)$ has both a continuous and a discrete component. The discrete component is an impulse at T that corresponds to the probability that the machine does not fail over the entire interval of length T. This is the probability that the time of the first arrival in a Poisson process of rate λ is greater than T, so that

$$(1) \quad f(T; T \mid 1) = u_0(T-t)\, e^{-\lambda T}.$$

where $u_0(z)$ is the unit impulse (Dirac delta) function, that is, a function that is zero everywhere except for an impulse of mass one at z.

The continuous component corresponds to the density of uptime for $0 \le t < T$. For $0 \le t < T$ we can write

$$f(t; T \mid 1) = h(T\text{–}t; T \mid 1) = \sum_{n=1}^{\infty} \text{dens}\{ \text{ n failures comprising T-t units of downtime } \}$$

$$= \sum_{n=1}^{\infty} \text{dens}\{ \text{ (n failures) and (T–t units of downtime \& machine working at time T) } \}$$
$$+ \sum_{n=1}^{\infty} \text{dens}\{ \text{ (n failures) and (T–t units of downtime \& machine failed at time T) } \}.$$

In order for the machine to be working at time T when there are n failures, the n^{th} repair must occur after T-t units of downtime and there must be n failures in the t units of uptime. Note that these events are independent once we have fixed t, the amount of uptime. Similarly, in order for the machine to be failed at time T when there are n failures, the n^{th} failure must occur after t units of uptime and there must be n-1 repairs in the T-t units of downtime (the n^{th} repair has not yet occurred). These two events are also independent once we have fixed t. Therefore,

$$f(t; T \mid 1) = \sum_{n=1}^{\infty} \Pr\{ \text{ n failures in t time units } \} \text{ dens}\{ \text{ } n^{th} \text{ repair occurs at time T–t } \}$$
$$+ \sum_{n=1}^{\infty} \Pr\{ \text{ n–1 repairs in (T–t) units } \} \text{ dens}\{ \text{ } n^{th} \text{ failure occurs at time t } \}$$

$$= \sum_{n=1}^{\infty} \Pr\{ \text{ n arrivals in Poisson process at rate } \lambda \text{ t } \} \text{ dens}\{ \text{ time of the } n^{th} \text{ arrival in Poisson process of rate } \mu \text{ is T–t } \}$$

$$+ \sum_{n=1}^{\infty} \text{Pr}\{ \text{ n–1 arrivals in Poisson process at rate } \mu \text{ (T–t) } \} \text{ dens}\{ \text{ time of the } n^{th} \text{ arrival in Poisson process of rate } \lambda \text{ is t } \}$$

for $0 \le t < T$. Substituting the Poisson PMF and Erlang density (Ross, 1983) we obtain

$$\sum_{n=1}^{\infty} \frac{(\lambda t)^n \; e^{-\lambda t}}{n!} \; \frac{\mu^n \; (T-t)^{n-1} \; e^{-\mu(T-t)}}{(n-1)!} + \frac{(\mu(T-t))^{n-1} \; e^{-\mu(T-t)}}{(n-1)!} \; \frac{\lambda^n \; t^{n-1} \; e^{-\lambda t}}{(n-1)!}$$

$$= e^{-\lambda t - \mu(T-t)} \sum_{n=1}^{\infty} \left[\lambda\mu t \frac{(\lambda\mu t(T-t))^{n-1}}{(n-1)! \; n!} + \lambda \frac{(\lambda\mu t(T-t))^{n-1}}{(n-1)! \; (n-1)!} \right].$$

Letting $x = \lambda \mu t (T–t)$,

$$f(t; T \mid 1) = e^{-\lambda t - \mu(T-t)} \sum_{n=1}^{\infty} \left[\lambda\mu t \frac{x^{n-1}}{(n-1)! \; n!} + \lambda \frac{x^{n-1}}{(n-1)! \; (n-1)!} \right], \qquad 0 \le t < T.$$

Each of the two terms in the brackets can be written as terms of modified Bessel functions (of order 1 and 0, respectively),

$$(2) \quad f(t; T \mid 1) = \left[\lambda\mu t \; \frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + \lambda \; I_0\left(2\sqrt{x}\right) \right] e^{-\lambda t - \mu(T-t)}, \qquad 0 \le t < T$$

where

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k! \; k!}, \quad I_1(z) = \frac{z}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k! \; (k+1)!}.$$

The modified Bessel functions I_0 and I_1 can be computed numerically using a variety of methods. For example, Press et al. (1989) present a polynomial approximation based on Abramowitz and Stegun (1964). More sophisticated methods have been developed by many authors, including Sookne (1973), Cody (1983), and Boisvert and Saunders (1992). Codes are also provided in most commercial numerical libraries, although many excellent codes are in the public domain and are available via *netlib* (Dongarra and Grosse, 1987).

To complete the derivation of f(t; T | 1), we add the continuous and discrete components (1) and (2). In total,

$$\text{(3)}\quad f(t;T \mid 1) = \left[\lambda\mu t \, \frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + \lambda \; I_0\left(2\sqrt{x}\right) \right] e^{-\lambda t - \mu(T-t)} + u_0(T-t)\, e^{-\lambda T}, \qquad 0 \le t \le T.$$

Machine initially failed

Using an analogous argument to the one above, we could derive f(t; T | 0), the PDF of uptime over [0, T) conditional on the machine being failed at time 0. However, with a few simple observations we can more easily obtain the result. First, simply note that the PDF of downtime over [0, T) conditional on the machine being failed at time 0 is described by the same stochastic process as f(t; T | 1), with the roles of λ and μ reversed, that is,

$$h(t; T, \lambda, \mu \mid 0) = f(t; T, \mu, \lambda \mid 1),$$

Therefore,

$$h(t;T \mid 0) = \left[\lambda\mu t \, \frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + \mu \; I_0\left(2\sqrt{x}\right) \right] e^{-\mu t - \lambda(T-t)} + u_0(T-t)\, e^{-\mu T}, \qquad 0 \le t \le T.$$

Since downtime = T – uptime, the PDFs of uptime and downtime are simply mirror images of one another, thus

(4) $f(t; T \mid 0) = h(T{-}t; T \mid 0) =$

$$\left[\lambda\mu(T-t)\,\frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + \mu\, I_0\left(2\sqrt{x}\right)\right] e^{-\mu(T-t)-\lambda t} + u_0(t)\, e^{-\mu T}, \qquad 0 \le t \le T.$$

Further, we can conclude that the density of *downtime* conditional on the machine working at time 0 is

$h(t; T \mid 1) = f(T{-}t; T \mid 1) =$

$$\left[\lambda\mu(T-t)\,\frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + \lambda\, I_0\left(2\sqrt{x}\right)\right] e^{-\lambda(T-t)-\mu t} + u_0(t)\, e^{-\lambda T}, \qquad 0 \le t \le T.$$

Machine initially in steady state

The PDF of uptime with the initial state of the machine randomized (i.e., starting in steady state) can be written as

Pr{ machine initially working } $f(t; T \mid 1)$ + Pr{ machine initially failed } $f(t; T \mid 0)$.

Since the steady-state probability that the machine starts out working is $\mu/(\lambda+\mu)$ and failed is $\lambda/(\lambda+\mu)$ (Ross, 1983), $f(t; T)$ can now be seen to equal

$$f(t; T) = \left[\frac{\mu}{\lambda+\mu}\left(\lambda\mu t\,\frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + \lambda\, I_0\left(2\sqrt{x}\right)\right) + \frac{\lambda}{\lambda+\mu}\left(\lambda\mu(T-t)\,\frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + \mu\, I_0\left(2\sqrt{x}\right)\right)\right] e^{-\lambda t-\mu(T-t)} +$$

$$\frac{\mu}{\lambda+\mu}\, u_0(T-t)\, e^{-\lambda T} + \frac{\lambda}{\lambda+\mu}\, u_0(t)\, e^{-\mu T}, \qquad 0 \le t \le T.$$

which, after simplification, is equal to

$$\text{(5)} \quad f(t; T) = \frac{\lambda\mu}{\lambda+\mu}\left(\left(\mu t + \lambda(T-t)\right) \frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + 2\ I_0\left(2\sqrt{x}\right)\right) e^{-\lambda t-\mu(T-t)} +$$
$$\frac{\mu}{\lambda+\mu}\, u_0(T-t)\, e^{-\lambda T} + \frac{\lambda}{\lambda+\mu}\, u_0(t)\, e^{-\mu T}, \qquad 0 \le t \le T.$$

Density with known starting and terminal machine states

Lastly, we wish to compute $f(t; T \mid \alpha(0) = a, \alpha(T) = b)$, the PDF of uptime over [0, T) conditional on the machine being in state a at time 0 and state b at time T. We will abbreviate this as $f(t; T \mid ab)$. From the development above, the results follow immediately, for example,

$$\text{dens}\{t \text{ units of uptime in } [0, T) \text{ \& machine failed at } T \mid 1\} = \lambda\ I_0\left(2\sqrt{x}\right)\ e^{-\lambda t-\mu(T-t)},$$

so that from the law of conditional probability,

$$\text{dens}\{\ t \text{ units of uptime in } [0, T) \text{ \& machine failed at } T \mid 1\ \} =$$
$$\text{dens}\{\ t \text{ units of uptime in } [0, T) \mid 10\ \}\ P_{10}(T)$$

where $P_{10}(T)$ is the probability that the machine is failed at time T given that it is working at time 0. Therefore,

$$\text{(6)} \quad f(t; T \mid 10) = \frac{1}{P_{10}(T)}\ \lambda\ I_0\left(2\sqrt{x}\right)\ e^{-\lambda t-\mu(T-t)}, \qquad 0 \le t \le T.$$

By similar reasoning,

$$(7)\quad f(\,t;T \mid 11\,) \;=\; \frac{1}{P_{11}(T)}\left[\lambda\mu t\;\frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}}\;e^{-\lambda t-\mu(T-t)} + u_0(T-t)e^{-\lambda t}\right], \qquad 0 \le t \le T,$$

$$(8)\quad f(\,t;T \mid 01\,) \;=\; \frac{1}{P_{01}(T)}\;\mu\;I_0\left(2\sqrt{x}\right)\;e^{-\lambda t-\mu(T-t)}, \qquad 0 \le t \le T,$$

$$(9)\quad f(\,t;T \mid 00\,) \;=\; \frac{1}{P_{00}(T)}\left[\lambda\mu(T-t)\;\frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}}\;e^{-\lambda t-\mu(T-t)} + u_0(t)e^{-\mu T}\right], \qquad 0 \le t \le T.$$

The four probabilities $P_{00}(T)$, $P_{01}(T)$, $P_{10}(T)$ and $P_{11}(T)$ are well-known and are derived classic texts on reliability, such as Barlow and Proschan (1965). They are

$$(10)\quad P_{10}(T) \;=\; \frac{\lambda}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)T}\right),$$

$$(11)\quad P_{11}(T) \;=\; 1-P_{10}(T) \;=\; \frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)T},$$

$$(12)\quad P_{01}(T) \;=\; \frac{\mu}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)T}\right),$$

$$(13)\quad P_{00}(T) \;=\; 1-P_{01}(T) \;=\; \frac{\lambda}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)T}.$$

Note that it is now easy to see that $f(t;T \mid 10) = f(t;T \mid 01)$, as a result of the reversibility of the process.

As an aside, it is interesting to note that through this derivation we can write new expressions for certain integrals of modified Bessel functions. For example, since a probability density function must integrate to one,

$$\int_0^T f(\,t;T \mid 10\,)\,dt \;=\; 1,$$

so that

$$\int_0^T \frac{1}{P_{10}(T)} \ \lambda \ I_0\left(2\sqrt{\lambda\mu t(T-t)}\right) \ e^{-\lambda t-\mu(T-t)} \ dt \ = \ 1.$$

Rearranging this equation, we obtain

$$\int_0^T I_0\left(2\sqrt{\lambda\mu t(T-t)}\right) \ e^{-\lambda t-\mu(T-t)} \ dt \ = \ \frac{P_{10}(T)}{\lambda}$$

Substituting $P_{10}(T)$, it follows that

$$\int_0^T I_0\left(2\sqrt{\lambda\mu t(T-t)}\right) \ e^{-\lambda t-\mu(T-t)} \ dt \ = \ \frac{1}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)T}\right)$$

and

$$\int_0^T I_0\left(2\sqrt{\lambda\mu t(T-t)}\right) \ e^{-(\lambda-\mu)t} \ dt \ = \ \frac{1}{\lambda+\mu}\left(e^{\mu T}-e^{-\lambda T}\right).$$

2.2 Laplace transform and moments of uptime over a fixed time interval

In the development that follows we will compute the Laplace transform of the uptime distribution over the period [0, T), conditional on the machine working at time 0,

$$\mathcal{L}\,\{\, f(t;\, T \mid 1)\,\} = f^*(s;\, T \mid 1) = \int_{t=0}^{T} e^{-st}\; f(t;\; T \mid 1)\; dt,$$

and the first and second moments of the distribution. We will also derive results assuming the machine is initially failed or in steady state. From these results, we will also show how to easily obtain results for the case when both the initial and terminal machine states are known.

Laplace transform of uptime

We have shown in the previous section that $f(t;\, T \mid 1)$ is

$$f(t;\, T \mid 1) = e^{-\lambda t-\mu(T-t)} \sum_{n=1}^{\infty} \left[\lambda\mu t \frac{\left(\lambda\mu t(T-t)\right)^{n-1}}{(n-1)!\; n!} + \lambda \frac{\left(\lambda\mu t(T-t)\right)^{n-1}}{(n-1)!\;(n-1)!} \right] + u_0(T-t)\, e^{-\lambda T}.$$

We will find the Laplace transform of f by breaking it into three separate functions $f_1 + f_2 + f_3$, finding the Laplace transform of each of the three functions, and then summing to obtain the overall transform of f. In particular, define

$$f_1(t,\, T) = e^{-\lambda t-\mu(T-t)} \sum_{n=1}^{\infty} \lambda\mu t \frac{\left(\lambda\mu t(T-t)\right)^{n-1}}{(n-1)!\; n!} = \sum_{n=1}^{\infty} \frac{\lambda^n t^n e^{-\lambda t}}{n!} \frac{\mu^n (T-t)^{n-1} e^{-\mu(T-t)}}{(n-1)!}$$

$$f_2(t,\, T) = e^{-\lambda t-\mu(T-t)} \sum_{n=1}^{\infty} \lambda \frac{\left(\lambda\mu t(T-t)\right)^{n-1}}{(n-1)!\;(n-1)!} = \sum_{n=1}^{\infty} \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \frac{\mu^{n-1}(T-t)^{n-1} e^{-\mu(T-t)}}{(n-1)!}$$

$$f_3(t,\, T) = u_0(T-t)\, e^{-\lambda T}.$$

Further, we will find the transforms of f_1 and f_2 by finding the transform of each term in the infinite series. To derive the transforms, we require the following

Lemma. If g(t), h(T–t) are two functions and f(t,T) = g(t) h(T–t) then the Laplace transform of f(t,T), denoted by f*(s, T), is given by the inverse Laplace transform (with respect to r and introducing the variable T) of g*(r+s) h*(r).

Proof. First, by definition of a Laplace transform,

$$L\{ f(t,T) \} = f^*(s, T) = \int_{t=0}^{T} g(t)\ h(T-t)\ e^{-st}\ dt\ .$$

Now note that

$$f^*(s, T) = \left\{g(t)\ e^{-st}\right\} \ \star\ \left\{h(t)\right\}$$

where ☆ represents the convolution operator and the parameter of convolution is T. Treating T as a variable and taking the transform of both sides with respect to T and introducing the variable r, we obtain

$$f^{**}(s, r) = g^*(r+s)\ h^*(r).$$

Taking the inverse transform of both sides with respect to r and reintroducing the variable T produces the desired result. ❑

To use the lemma, we write the n^{th} term of $f_1(t, T)$ as $g_1(t)\ h_1(T–t)$, where

$$g_1(t) = \frac{\lambda^n t^n e^{-\lambda t}}{n!}$$

and

$$h_1(T–t) = \frac{\mu^n (T-t)^{n-1} e^{-\mu(T-t)}}{(n-1)!}.$$

We also write the n^{th} term of $f_2(t, T)$ as $g_2(t)\, h_2(T–t)$ where

$$g_2(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$

and

$$h_2(T–t) = \frac{\mu^{n-1} (T-t)^{n-1} e^{-\mu(T-t)}}{(n-1)!}.$$

Note that

$$g_1^*(s) = \frac{\lambda^n}{(\lambda+s)^{n+1}},\ h_1^*(s) = \frac{\mu^n}{(\mu+s)^n},\ g_2^*(s) = \frac{\lambda^n}{(\lambda+s)^n},\ \text{and } h_2^*(s) = \frac{\mu^{n-1}}{(\mu+s)^n}.$$

Given these definitions and the above lemma, in order to find the Laplace transform of $f_1(t, T)$ we must find

$$L\{ f_1(t,T) \} = L^{-1}\{ \sum_{n=1}^{\infty} \frac{\lambda^n \mu^n}{(\lambda+s+r)^{n+1} (\mu+r)^n} \}$$

and in order to find the Laplace transform of $f_2(t, T)$ we must find

$$L\{ f_2(t,T) \} = L^{-1}\{ \sum_{n=1}^{\infty} \frac{\lambda^n \mu^{n-1}}{(\lambda+s+r)^n (\mu+r)^n} \}$$

where the inverse transforms are with respect to the variable r and introduce the variable T.

We will first focus on finding the transform of $f_2(t, T)$. In the discussion that follows (_._._) refers to an equation in Abramowitz and Stegun (1964). The first term for n = 1 is trivial. By (29.3.12),

$$L^{-1}\left\{ \frac{\lambda}{(\lambda + s + r)(\mu + r)} \right\} = \lambda \left(\frac{e^{-\mu T} - e^{-(\lambda + s)T}}{\lambda + s - \mu} \right).$$

For $n \geq 2$, we have from (29.3.50) that

$$L^{-1}\left\{ \frac{\lambda^n \mu^{n-1}}{(\lambda + s + r)^n (\mu + r)^n} \right\} = \sqrt{\pi}\ \frac{(\lambda\mu)^n}{\mu\ (n-1)!} \left(\frac{T}{\lambda + s - \mu} \right)^{n-\frac{1}{2}} e^{-\frac{1}{2}(\lambda + s + \mu)T}\ I_{n-\frac{1}{2}}\left(\tfrac{1}{2}(\lambda + s - \mu)T \right)$$

where $I_{n-1/2}$ is a modified spherical Bessel function. We now desire an expression for the sum of these terms from n = 2 to infinity. Writing this sum, and then rearranging the terms and reindexing to begin at n = 1, we obtain

$$\sum_{n=2}^{\infty} L^{-1}\left\{ \frac{\lambda^n \mu^{n-1}}{(\lambda + s + r)^n (\mu + r)^n} \right\} =$$
$$\lambda T\ e^{-\frac{1}{2}(\lambda + s + \mu)T} \sum_{n=1}^{\infty} \sqrt{\frac{\pi}{2}} \left(\tfrac{1}{2}(\lambda + s - \mu)T \right)^{-\frac{1}{2}} \frac{1}{n!} \left(\frac{\lambda\mu T}{\lambda + s - \mu} \right)^n I_{n+\frac{1}{2}}\left(\tfrac{1}{2}(\lambda + s - \mu)T \right).$$

Infinite sums of modified spherical Bessel functions are governed by the generating function (10.2.31),

$$\sum_{n=0}^{\infty} \sqrt{\frac{\pi}{2}}\ z^{-\frac{1}{2}}\ \frac{t^n}{n!}\ I_{n-\frac{1}{2}}(z) = \frac{\cosh \sqrt{z^2 + 2zt}}{z}.$$

To obtain a new identity that suits our purposes we take the partial derivative of both sides with respect to t and reindex to obtain

$$\sum_{n=0}^{\infty} \sqrt{\frac{\pi}{2}}\ z^{-\frac{1}{2}}\ \frac{t^n}{n!}\ I_{n+\frac{1}{2}}(z) \ = \ \frac{\mathbf{sinh}\sqrt{z^2 + 2zt}}{\sqrt{z^2 + 2zt}},$$

where $\mathbf{cosh}\ z = (\mathbf{exp}(z) + \mathbf{exp}(-z))/2$ and $\mathbf{sinh}\ z = (\mathbf{exp}(z) - \mathbf{exp}(-z))/2$. With this new identity we can rewrite our infinite series as

$$\sum_{n=2}^{\infty} L^{-1}\{\ \frac{\lambda^n \mu^{n-1}}{(\lambda + s + r)^n (\mu + r)^n}\ \} =$$

$$\lambda T\ e^{-\frac{1}{2}(\lambda + s + \mu)T} \left[\frac{\mathbf{sinh}\sqrt{\left(\frac{1}{2}(\lambda + s - \mu)T\right)^2 + \lambda\mu T^2}}{\sqrt{\left(\frac{1}{2}(\lambda + s - \mu)T\right)^2 + \lambda\mu T^2}} - \frac{\mathbf{sinh}\left(\frac{1}{2}(\lambda + s - \mu)T\right)}{\left(\frac{1}{2}(\lambda + s - \mu)T\right)} \right].$$

The second term in the square brackets arises because we were missing the zeroth term of the generating function and thus subtracted $\sqrt{\pi/2z}\ I_{1/2}(z)$, which by (10.2.13) is just $\mathbf{sinh}(z) / z$. Adding this expression with the term for n = 1 and simplifying, we obtain the Laplace transform of $f_2(t, T)$,

(14) $$L\{ f_2(t,T) \} = \lambda T\ \frac{\mathbf{sinh}\ y}{y\ e^h},$$

where $y = \sqrt{\lambda^2 + 2\lambda\mu + \mu^2 + 2\lambda s - 2\mu s + s^2}\ T / 2$ and $h = (\lambda + \mu + s)\ T / 2$. This completes the subproblem we have focused on.

Recall that the second part of the problem is to compute the Laplace transform of $f_1(t, T)$,

$$L\{ f_1(t,T) \} = \sum_{n=1}^{\infty} L^{-1}\{\ \frac{\lambda^n \mu^n}{(\lambda + s + r)^{n+1} (\mu + r)^n}\ \}.$$

This turns out to be somewhat more cumbersome but is similar to the development above. Here the first term is

$$L^{-1}\left\{ \frac{\lambda\mu}{(\lambda+s+r)^2(\mu+r)} \right\} = \lambda\mu\; e^{-(\lambda+s)T} \int_0^T \frac{1-e^{-(\mu-\lambda)\tau}}{\mu-\lambda}\, d\tau$$

$$= \lambda\mu\; e^{-(\lambda+s)T} \left[\frac{-1}{(\lambda+s-\mu)^2} + \frac{e^{(\lambda+s-\mu)T} - (\lambda+s-\mu)T}{(\lambda+s-\mu)^2} \right].$$

For $n \geq 2$, we proceed in three steps, first finding

$$L^{-1}\left\{ \frac{\lambda^n\mu^n}{r\; r^n\; (\mu-\lambda-s+r)^n} \right\} =$$

$$\int_{\tau=0}^{\tau=T} \sqrt{\pi}\, \frac{(\lambda\mu)^n}{(n-1)!} \left(\frac{\tau}{\lambda+s-\mu} \right)^{n-\frac{1}{2}} e^{\frac{1}{2}(\lambda+s-\mu)\tau}\; I_{n-\frac{1}{2}}\left(\tfrac{1}{2}(\lambda+s-\mu)\tau\right)\, d\tau\,.$$

To evaluate this integral we first rewrite it as

$$\sqrt{\pi}\, \frac{(\lambda\mu)^n}{(n-1)!} \left(\frac{2}{\lambda+s-\mu} \right)^{2n} \left(\frac{1}{2} \right)^{n-\frac{1}{2}} \int_{u=0}^{u=w} u^{n-\frac{1}{2}}\; e^u\; I_{n-\frac{1}{2}}(u)\, du$$

where $u = \frac{1}{2}(\lambda+s-\mu)\tau$ and $w = \frac{1}{2}(\lambda+s-\mu)T$. Note that $du = \frac{1}{2}(\lambda+s-\mu)d\tau$. Finally, applying (11.3.12), and simplifying we obtain

$$\frac{\sqrt{\pi}}{2}\, \frac{(\lambda\mu)^n}{n!} \left(\frac{1}{\lambda+s-\mu} \right)^{n-\frac{1}{2}} T^{n+\frac{1}{2}}\; e^{\frac{1}{2}(\lambda+s-\mu)T} \left[I_{n-\frac{1}{2}}\left(\tfrac{1}{2}(\lambda+s-\mu)T\right) - I_{n+\frac{1}{2}}\left(\tfrac{1}{2}(\lambda+s-\mu)T\right) \right].$$

The second step is to use (29.2.12) to substitute r+λ+s for r to obtain

$$\mathcal{L}^{-1}\left\{ \frac{\lambda^n \mu^n}{(\lambda + s + r)^{n+1}\,(\mu + r)^n} \right\} =$$

$$\frac{\sqrt{\pi}}{2}\ \frac{(\lambda\mu)^n}{n!}\left(\frac{1}{\lambda + s - \mu}\right)^{n-\frac{1}{2}} T^{n+\frac{1}{2}}\ e^{-\frac{1}{2}(\lambda+s+\mu)T}\left[I_{n-\frac{1}{2}}\left(\tfrac{1}{2}(\lambda + s - \mu)T\right) - I_{n+\frac{1}{2}}\left(\tfrac{1}{2}(\lambda + s - \mu)T\right)\right].$$

The third step, as before, is to evaluate the sum from n=2 to infinity. This produces two infinite series, the first given by

$$w\ e^{-\frac{1}{2}(\lambda+s+\mu)T} \sum_{n=2}^{\infty} I_{n-\frac{1}{2}}(w)\ \sqrt{\frac{\pi}{2w}}\ \frac{v^n}{n!}$$

where $w = ((\lambda+s-\mu)T)/2$ and $v = \lambda\,\mu\,T\ /\ (\lambda+s-\mu)$. Applying the generating function (10.2.31) we obtain

$$e^{-\frac{1}{2}(\lambda+s+\mu)T}\left(\mathbf{cosh}\left(\sqrt{w^2 + 2wv}\right) - \mathbf{cosh}(w) - v\ \mathbf{sinh}(w)\right)$$

where we note that the generating function sum starts at zero so we have, using (10.2.13) and (10.2.14), subtracted off $\sqrt{\pi/2w}\ I_{-1/2}(w) = \mathbf{cosh}(w)/w$ and $v\ \sqrt{\pi/2w}\ I_{1/2}(w) = v\ \mathbf{sinh}(w)/w$. The second series can be written as

$$-w\ e^{-\frac{1}{2}(\lambda+s+\mu)T} \sum_{n=2}^{\infty} I_{n+\frac{1}{2}}(w)\ \sqrt{\frac{\pi}{2w}}\ \frac{v^n}{n!}.$$

Using our modified generating function and subtracting off the missing terms using (10.2.13) and (10.2.14) as before,

$$-w\ e^{-\frac{1}{2}(\lambda+s+\mu)T}\left(\frac{\mathbf{sinh}\sqrt{w^2 + 2wv}}{\sqrt{w^2 + 2wv}} - \frac{\mathbf{sinh}\ w}{w} - v\left(\frac{\mathbf{cosh}\ w}{w} - \frac{\mathbf{sinh}\ w}{w^2}\right)\right).$$

Combining these results we can, after a considerable amount of algebra, obtain a simpler expression for the Laplace transform of $f_1(t, T)$,

$$(15)\quad \mathcal{L}\{ f_1(t,T) \} = \frac{\mathbf{cosh}\, y}{e^h} - w\,\frac{\mathbf{sinh}\, y}{y\ e^h} - e^{-(\lambda+s)T}.$$

This completes the computation of the Laplace transform of $f_1(t, T)$ and $f_2(t, T)$. The only remaining step is to compute the Laplace transform of the impulse term $f_3(t, T)$, which is simply

$$(16)\quad \mathcal{L}\{ f_3(t,T) \} = e^{-(\lambda+s)T}.$$

Combining the three transforms (14), (15), and (16), we finally obtain

$$(17)\quad f^*(s,\ T\ |\ 1) = \frac{(\lambda+\mu-s)T}{2}\ \frac{\mathbf{sinh}\, y}{y\ e^h} + \frac{\mathbf{cosh}\, y}{e^h},$$

or, in terms of exponentials,

$$f^*(s,\ T\ |\ 1) = \frac{(\lambda+\mu-s)T}{2}\ \frac{e^{y-h} - e^{-(y+h)}}{2y} + \frac{e^{y-h} + e^{-(y+h)}}{2},$$

where $y = \sqrt{\lambda^2 + 2\lambda\mu + \mu^2 + 2\lambda s - 2\mu s + s^2}\ T\ /\ 2$ and $h = (\lambda + \mu + s)\ T\ /\ 2$. Note that $y = h$ when $s = 0$, so that the above expression is easily seen to equal 1 at $s = 0$. This is an important check because the density $f(t;\ T\ |\ 1)$ must integrate to 1.

Further, we observe that by definition, the Laplace transform of $f(t;\ T\,|\,10)\ P_{10}(T)$ is the Laplace transform of $f_2(t, T)$ and the Laplace transform of $f(t;\ T\,|\,11)\ P_{11}(T)$ is the Laplace transform of $f_1(t, T) + f_3(t, T)$, all of which have been derived above. As we would expect, the Laplace transform of $f_2(t, T)$ equals $P_{10}(T)$ and the Laplace transform of $f_1(t, T) + f_3(t, T)$ equals $P_{11}(T)$ at s = 0.

Mean uptime

The mean is obtained by taking the derivative of the transform with respect to s and then setting s equal to zero and negating the result. The process is straightforward although cumbersome, and after much simplification yields

$$(18)\quad E[f \mid 1] = \frac{\mu}{\lambda+\mu} T + \frac{\lambda}{(\lambda+\mu)^2}\left(1 - e^{-(\lambda+\mu)T}\right).$$

This expression agrees with the result of Barlow and Proschan (1965). Also, this can be easily derived from the general Laplace transform result of Takács (1957a). As a simple check we note that for small T, $E[f \mid 1] = T + O(T^2)$; at T = 0 the above expression is zero; and as T approaches infinity, $E[f \mid 1] / T$ approaches $\mu/(\lambda+\mu)$, as we would expect.

Up to this point we have supposed that the machine is working at time 0. To derive expressions where the machine is *failed* at time zero, we will be working with h(t; T), the PDF for *downtime* over an interval of length T. To find the mean downtime over an interval of length T when the machine is failed at time 0, we simply reverse the roles of λ and μ to obtain

$$E[h \mid 0] = \frac{\lambda}{\lambda+\mu} T + \frac{\mu}{(\lambda+\mu)^2}\left(1 - e^{-(\lambda+\mu)T}\right).$$

Since $E[f \mid 0] + E[h \mid 0] = T$, we simply subtract the above expression from T to obtain

$$(19)\quad E[f \mid 0] = \frac{\mu}{\lambda+\mu}T - \frac{\mu}{(\lambda+\mu)^2}\left(1 - e^{-(\lambda+\mu)T}\right).$$

Since we have found the transform of f(t; T | 10) and f(t; T | 11), we can also derive E[f | 10] and E[f | 11] by taking the derivative of the transform with respect to s and then setting s equal to zero and negating the result. After simplification, we obtain

$$(20)\quad E[f \mid 10] = \left(\frac{\lambda}{\lambda+\mu}\right)^2 \frac{T}{1-e^{(\lambda+\mu)T}} + \left(\frac{\mu}{\lambda+\mu}\right)^2 \frac{T}{1-e^{-(\lambda+\mu)T}} + \frac{\lambda\mu T + \lambda - \mu}{(\lambda+\mu)^2},$$

$$(21)\quad E[f \mid 11] = \frac{2\lambda\mu}{(\lambda+\mu)^2}\left(\frac{1}{\mu+\lambda e^{-(\lambda+\mu)T}} - \frac{1}{\lambda+\mu e^{(\lambda+\mu)T}}\right) + \frac{T}{(\lambda+\mu)^2}\left(\frac{\lambda^3}{\lambda+\mu e^{(\lambda+\mu)T}} + \frac{\mu^3}{\mu+\lambda e^{-(\lambda+\mu)T}}\right) + \frac{\lambda\mu T}{(\lambda+\mu)^2},$$

and not surprisingly, we can also show that

$$(22)\quad E[f \mid 01] = E[f \mid 10],$$

$$(23)\quad E[f \mid 00] = T - \frac{2\lambda\mu}{(\lambda+\mu)^2}\left(\frac{1}{\lambda+\mu e^{-(\lambda+\mu)T}} - \frac{1}{\mu+\lambda e^{(\lambda+\mu)T}}\right) - \frac{T}{(\lambda+\mu)^2}\left(\frac{\mu^3}{\mu+\lambda e^{(\lambda+\mu)T}} + \frac{\lambda^3}{\lambda+\mu e^{-(\lambda+\mu)T}}\right) - \frac{\lambda\mu T}{(\lambda+\mu)^2}.$$

These expressions divided by T approach $\mu/(\lambda+\mu)$ as T approaches infinity, and approach zero as T approaches zero, as we would expect.

Laplace transform with machine initially failed or in steady state

Using the same observations as above we can easily find the Laplace transform of f(t; T | 0). We reverse the roles of λ and μ, add T to t (which corresponds to multiplying the transform by **exp**(s T)) and then negate t (which corresponds to negating the transform variable s). The result is

$$(24)\quad f^*(s,\ T\ |\ 0) = h\ \frac{e^{y-h} - e^{-(y+h)}}{2y} + \frac{e^{y-h} + e^{-(y+h)}}{2}.$$

As a final check on our work, let us examine the case where the machine state at time 0 is unknown, i.e., the machine starts out in steady-state. Then the probability that the machine starts out working is $\mu/(\lambda+\mu)$ and failed is $\lambda/(\lambda+\mu)$. Thus, the expected time the machine is working over an interval of length T is

$$\frac{\mu}{\lambda+\mu} E[f\,|1] + \frac{\lambda}{\lambda+\mu} E[f\,|0]$$

Substituting the above expressions, we obtain

$$\frac{\mu}{\lambda+\mu}\left[\frac{\mu}{\lambda+\mu}T + \frac{\lambda}{(\lambda+\mu)^2}\left(1 - e^{-(\lambda+\mu)T}\right)\right] + \frac{\lambda}{\lambda+\mu}\left[\frac{\mu}{\lambda+\mu}T - \frac{\mu}{(\lambda+\mu)^2}\left(1 - e^{-(\lambda+\mu)T}\right)\right]$$

which is, after simplification,

$$E[f] = T\,\mu/(\lambda+\mu),$$

as we would expect from the theory of alternating renewal processes. In fact, it is well known that this result holds for any repair and failure distributions (Ross, 1983).

Variance of uptime

To derive an expression for the variance of uptime over an interval of length T when the machine is working at time 0, we first find the second moment by taking the second derivative of the transform (17) with respect to s and then setting s = 0, which, after simplification, gives

$$\frac{2\lambda^2 - 4\lambda\mu}{(\lambda+\mu)^4}\left(1 - e^{-(\lambda+\mu)T}\right) + \frac{4\lambda\mu}{(\lambda+\mu)^3}T - \frac{2\lambda^2}{(\lambda+\mu)^3}Te^{-(\lambda+\mu)T} + \frac{\mu^2}{(\lambda+\mu)^2}T^2.$$

To find the variance we square our expression for the mean and subtract it from the above expression to obtain (after simplification)

$$\text{(25)}\quad \operatorname{Var}[f \mid 1] = \frac{\lambda^2}{(\lambda+\mu)^4}\left(1 - e^{-2(\lambda+\mu)T}\right) - \frac{4\lambda\mu}{(\lambda+\mu)^4}\left(1 - e^{-(\lambda+\mu)T}\right) + \frac{2\lambda\mu}{(\lambda+\mu)^3}T\left(1 + e^{-(\lambda+\mu)T}\right) - \frac{2\lambda^2}{(\lambda+\mu)^3}Te^{-(\lambda+\mu)T}.$$

We can find Var[f | 0] in a similar manner. The result is

$$\text{(26)}\quad \operatorname{Var}[f \mid 0] = \frac{\mu^2}{(\lambda+\mu)^4}\left(1 - e^{-2(\lambda+\mu)T}\right) - \frac{4\lambda\mu}{(\lambda+\mu)^4}\left(1 - e^{-(\lambda+\mu)T}\right) + \frac{2\lambda\mu}{(\lambda+\mu)^3}T + \frac{2\mu(\lambda-\mu)}{(\lambda+\mu)^3}Te^{-(\lambda+\mu)T}.$$

Note that as T approaches infinity, both Var[f | 1] / T and Var[f | 0] / T approach $2\lambda\mu/(\lambda+\mu)^3$. This asymptotic result agrees with the general result of Takács (1957a); see also Gnedenko et al. (1969). Further, these authors both show that the asymptotic distribution is Normal.

Lastly, with a substantial amount of algebra, it can be seen that

$$(27)\quad \mathrm{Var}[f\,|\,10] \;=\; \frac{1}{(\lambda+\mu)^2} - \frac{8\lambda\mu}{(\lambda+\mu)^4} + \frac{2\lambda\mu T}{(\lambda+\mu)^3\left(1-e^{-(\lambda+\mu)T}\right)^2} - \frac{2\lambda\mu T}{(\lambda+\mu)^3\left(1-e^{(\lambda+\mu)T}\right)^2} - T^2\left(\frac{\lambda-\mu}{\lambda+\mu}\right)^2 \frac{e^{(\lambda+\mu)T}}{\left(1-e^{(\lambda+\mu)T}\right)^2},$$

$$(28)\quad \mathrm{Var}[f\,|\,11] \;=\; \frac{6\lambda\mu\left(\mu^2 e^{(\lambda+\mu)T}-\lambda^2\right)\left(1-e^{(\lambda+\mu)T}\right)}{(\lambda+\mu)^4\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2} + \frac{2\lambda^2\mu^2\left(1-e^{(\lambda+\mu)T}\right)^2}{(\lambda+\mu)^4\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2} - \frac{2\lambda^3\mu T\left(1+2e^{(\lambda+\mu)T}\right)}{(\lambda+\mu)^3\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2} + \frac{2\lambda\mu^3 T\left(2e^{(\lambda+\mu)T}+e^{2(\lambda+\mu)T}\right)}{(\lambda+\mu)^3\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2} + \frac{\lambda\mu T^2(\lambda-\mu)^2 e^{(\lambda+\mu)T}}{(\lambda+\mu)^2\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2},$$

$$(29)\quad \mathrm{Var}[f\,|\,01] \;=\; \mathrm{Var}[f\,|\,10],$$

$$(30)\quad \mathrm{Var}[f\,|\,00] \;=\; \frac{6\lambda\mu\left(\lambda^2 e^{(\lambda+\mu)T}-\mu^2\right)\left(1-e^{(\lambda+\mu)T}\right)}{(\lambda+\mu)^4\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2} + \frac{2\lambda^2\mu^2\left(1-e^{(\lambda+\mu)T}\right)^2}{(\lambda+\mu)^4\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2} + \frac{2\lambda^3\mu T\left(2e^{(\lambda+\mu)T}+e^{2(\lambda+\mu)T}\right)}{(\lambda+\mu)^3\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2} - \frac{2\lambda\mu^3 T\left(1+2e^{(\lambda+\mu)T}\right)}{(\lambda+\mu)^3\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2} + \frac{\lambda\mu T^2(\lambda-\mu)^2 e^{(\lambda+\mu)T}}{(\lambda+\mu)^2\left(\lambda+\mu e^{(\lambda+\mu)T}\right)^2}.$$

We note that each of these expressions divided by T approaches $2\lambda\mu\ /\ (\lambda+\mu)^3$ as T approaches infinity, and approaches zero as T approaches zero, as we would expect.

2.3 Distribution, transform, mean and variance of time to produce a fixed number of parts

The focus of the following development will be to characterize the time to produce a fixed lot of q parts at some processing speed p, exponentially distributed time between failures with MTBF $1/\lambda$, and exponentially distributed time to repair with MTTR $1/\mu$. Although this problem has been addressed previously by Brouwers (1986) and Kim and Alden (1992), we take a different approach that is simpler and provides insight into the more difficult problem with an arbitrary (general) repair distribution. The key observation is that the random variable of interest can be represented as a power series, from which the transform is easily found. This approach is not new; see, for example, Giffin (1975) for an excellent exposition, or Serfozo (1990) for a rigorous presentation of theoretical results.

Laplace transform and density when machine initially working

We begin by observing that the time to produce a lot can be divided into two mutually-exclusive, collectively-exhaustive components: the processing time to produce parts, plus the time the machine spends in repair. Note that the first component is deterministic and the second is stochastic. Thus the time to produce a lot is given by

$$q/p + \mathbf{R}$$

where **R** is a random variable representing the time spent in repair. We will thus focus our attention on **R**.

Let us first define b as the arrival rate of failures per batch, given by

$$b = \lambda q / p.$$

Using this notation, two parameters characterize the distribution of **R**: b and μ. The key observation of our derivation is that if the machine is working at time 0, we can model the failure process as a Compound Poisson process of the form

$$\mathbf{R}(\bullet; b \mid 1) = \sum_{i=1}^{\mathbf{N}(b)} \mathbf{X}_i$$

where **N**(b) is the number of arrivals in the Poisson failure process with rate b and each $\mathbf{X}_i$ is exponentially distributed with rate μ. Using standard results (Ross, 1983) for Compound Poisson processes,

$$E[\mathbf{R} \mid 1] = b / \mu,$$

$$Var[\mathbf{R} \mid 1] = 2b / \mu^2.$$

In fact, we can easily find all the moments since the Laplace transform is easily found. We note that R is a *mixture* and can be represented by the power series

$$\mathbf{R}(\bullet;\ b \mid 1) = \sum_{i=0}^{\infty} p_i(b)\ \mathbf{X}^{i*}$$

where $\{p_i(b)\}$ is the Poisson distribution with rate b and $\mathbf{X}^{i*}$ represents the i-fold convolution of **X**. With this observation, we may write

$$r^*(s;\ b \mid 1) = \sum_{i=0}^{\infty} p_i \left(x^*(s)\right)^i$$

where r* and x* denote the Laplace transforms of **R** and **X**. Letting φ denote the characteristic function of a random variable,

$$r^*(s;\ b\,|\,1) = \varphi_P\left(x^*(s)\right)$$

(by definition of a characteristic function). Since $\varphi_P(s) = \mathbf{exp}(-b + bs)$, we conclude that

$$(31)\quad r^*(s;\ b\,|\,1) = \mathbf{exp}\left(-b\left(1 - x^*(s)\right)\right),$$

which agrees with the Laplace transform for the Compound Poisson processes (Ross, 1983). In the case of exponentially distributed repairs, $x^*(s) = \mu / (s+\mu)$, and thus,

$$(32)\quad r^*(s;\ b\,|\,1) = \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right).$$

Calculating the first and second derivatives of $r^*(s;\ b\,|\,1)$ at $s = 0$ for the exponential case validates the two moments obtained above; note that from (31) we can easily obtain the moments for any repair distribution.

From this transform we can obtain r, the density of the time spent in repair, for the exponential case. We will need the fact that the Laplace transform of $I_1(2\sqrt{x}) / \sqrt{x}$ is $\mathbf{exp}(1/s) - 1$ (Feller, 1971), which is easily verified by Maclaurin expansion of the transform (Doetsch, 1961). Using this fact and the basic rules of Laplace transforms, we obtain

$$(33)\quad r(t;b\,|1) = u_0(t)\mathbf{exp}(-b) + \mu\ b\ \mathbf{exp}(-\mu t - b)\ I_1\left(2\sqrt{\mu b t}\right)\ (\mu b t)^{-\frac{1}{2}},\quad t \geq 0,$$

where u_0 is the unit impulse function and I_1 is the modified Bessel function of order 1. This density has also been obtained by direct probabilistic argument by Feller (1971), Brouwers (1986), and Kim and Alden (1992).

Laplace transform and density when machine initially failed

We would now like to find r(t; b | 0), the density of time spent in repair given that the machine is currently failed. This is given by the convolution of r(t; b | 1) with the density of time to repair (i.e., the exponential density with parameter μ). The convolution integral is difficult to evaluate, but the Laplace transform is simply the product of the two transforms, and is

$$(34)\quad r^*(s;\ b\,|\,0) = \frac{\mu}{s+\mu}\ \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right).$$

From this transform we can easily find the first two moments of r(t; b | 0). The result is, not surprisingly,

$$E[\mathbf{R}\ |\ 0] = b/\mu + 1/\mu,$$
$$\mathrm{Var}[\mathbf{R}\ |\ 0] = 2b/\mu^2 + 1/\mu^2.$$

To find r(t; b | 0), we need only to invert the above transform. The symbol $\Leftrightarrow$ will be used to represent that the expression on the left is the Laplace transform of the expression on the right. We begin with the following transform identity from standard tables of Laplace transforms (e.g., (29.3.81) in Abramowitz and Stegun),

$$\frac{1}{s^{v+1}}\ \mathbf{exp}(1/s) \quad \Leftrightarrow \quad t^{v/2}\ I_v\left(2\sqrt{t}\right), \quad v > -1,$$

where $I_v(z)$ is the modified Bessel function of order v,

$$I_v(z) = \left(\frac{z}{2}\right)^v \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k!\,(k+v)!}$$

(see (9.6.10) in Abramowitz and Stegun). We now proceed with three simple steps to transform the above identity into r*(s; b | 0). First, taking the identity at v = 0 and scaling s by $1/\mu b$ gives

$$\frac{1}{\mu b}\left(\frac{\mu b}{s}\right) \mathbf{exp}(\mu b/s) \;\Leftrightarrow\; I_0\left(2\sqrt{\mu b t}\right).$$

Next, we replace s by s + μ, which is equivalent to multiplying the inverse transform by **exp**(–μt), and then scale both sides by the constant μ **exp**(–b). We obtain

$$\left(\frac{1}{s+\mu}\right) \mathbf{exp}\left(b\frac{\mu}{s+\mu}\right) \;\Leftrightarrow\; \mathbf{exp}(-\mu t)\; I_0\left(2\sqrt{\mu b t}\right),$$

and

$$\left(\frac{\mu}{s+\mu}\right) \mathbf{exp}\left(-b+b\frac{\mu}{s+\mu}\right) \;\Leftrightarrow\; \mu\; \mathbf{exp}(-\mu t - b)\; I_0\left(2\sqrt{\mu b t}\right).$$

Therefore,

(35) $r(t; b \mid 0) \;=\; \mu\; \mathbf{exp}(-\mu t - b)\; I_0\left(2\sqrt{\mu b t}\right),\;\; t \geq 0.$

We should not be surprised that the impulse term in r(t; b | 1) is not present in r(t; b | 0): since the machine is currently failed, the time spent in repair time is almost surely non-zero.

Since the stochastic process of this section is related to a compound Poisson process, the Laplace transforms (32) and (34) are of a special type, and as a result can be numerically inverted by the rather elegant method of Van Landingham and Shariq (1974). These authors present an efficient method that is specialized for inverting transforms of this type.

We also note that from the above development, we obtain an equivalence between the convolution integral and r(t; b | 0). Writing this equivalence and simplifying, one obtains a surprisingly simple identity,

$$\int_0^z a \; \frac{I_1\left(2\sqrt{at}\right)}{\sqrt{at}} \; dt \;=\; I_0\left(2\sqrt{az}\right) - 1 .$$

2.4 Cumulative distribution of time to produce a fixed number of parts

To find the CDF R(t; b | 1) one could attempt several different approaches. The most obvious is to integrate the density obtained in the previous section. This is possible for the functional form involved; this method is illustrated on a similar function R(t; b | 11) in the next section. Another possible method is direct probabilistic argument, and this has been successfully accomplished by Kim (1994, unpublished). We take a different approach, utilizing the Laplace transform obtained in the previous section.

If f(t) is any non-negative function of t and f*(s) is its Laplace transform, then f*(s) / s is the transform of the integral of f(t) from zero to t, namely, the CDF. We therefore seek the inverse Laplace transform of r*(s; b | 1) / s, i.e.,

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\ \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right)\right\}$$

As in the previous section, we begin with the transform identity

$$\frac{1}{s^{v+1}}\ \mathbf{exp}(1/s) \quad \Leftrightarrow \quad t^{v/2}\ I_v\left(2\sqrt{t}\right), \quad v > -1$$

from standard tables of Laplace transforms (e.g., (29.3.81) in Abramowitz and Stegun). We now proceed with a series of simple steps to transform the above identity into R*(s; b | 1) / s. First, scaling s by $1/\mu b$ gives

$$\frac{1}{\mu b}\left(\frac{\mu b}{s}\right)^{v+1}\mathbf{exp}(\mu b/s) \quad \Leftrightarrow \quad (\mu bt)^{v/2}\ I_v\left(2\sqrt{\mu bt}\right), \quad v > -1.$$

Multiplying both sides by the constant b^{-v} yields

$$\frac{1}{\mu}\left(\frac{\mu}{s}\right)^{v+1}\mathbf{exp}(\mu b/s) \;\Leftrightarrow\; \left(\frac{\mu t}{b}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right), \quad v>-1.$$

Next, we replace s by s + μ, which is equivalent to multiplying the inverse transform by **exp**(–μt), and then scale both sides by the constant **exp**(–b). We obtain

$$\frac{1}{\mu}\left(\frac{\mu}{s+\mu}\right)^{v+1}\mathbf{exp}\left(b\frac{\mu}{s+\mu}\right) \;\Leftrightarrow\; \mathbf{exp}(-\mu t)\left(\frac{\mu t}{b}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right), \quad v>-1,$$

and

$$\frac{1}{\mu}\left(\frac{\mu}{s+\mu}\right)^{v+1}\mathbf{exp}\left(-b+b\frac{\mu}{s+\mu}\right) \;\Leftrightarrow\; \mathbf{exp}(-\mu t-b)\left(\frac{\mu t}{b}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right), \quad v>-1.$$

We are now almost done, as the left-hand side of the above identity is very similar to our desired R*(s; b | 1) / s. Although no simple transformation of the above expression will give us the form that we desire, by the additivity of Laplace transforms we can create the identity we seek, by summing over v from zero to infinity. That is,

$$\frac{1}{\mu}\,\mathbf{exp}\left(-b+b\frac{\mu}{s+\mu}\right)\sum_{v=0}^{\infty}\left(\frac{\mu}{s+\mu}\right)^{v+1} \;\Leftrightarrow\; \sum_{v=0}^{\infty}\mathbf{exp}(-\mu t-b)\left(\frac{\mu t}{b}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right),$$

which, after simplification, gives

$$\frac{1}{\mu}\,\mathbf{exp}\left(-b+b\frac{\mu}{s+\mu}\right)\left(\frac{\mu}{s}\right) \Leftrightarrow \mathbf{exp}(-\mu t-b)\sum_{v=0}^{\infty}\left(\frac{\mu t}{b}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right).$$

Canceling the μ and 1 / μ on the left-hand side, we obtain the desired result, namely, that

$$(36)\quad R(t; b \mid 1) = \mathbf{exp}(-\mu t - b) \sum_{v=0}^{\infty} \left(\frac{\mu t}{b}\right)^{\frac{v}{2}} I_v\left(2\sqrt{\mu b t}\right), \quad t \geq 0.$$

This expression agrees with Kim (1994, unpublished). It is at first surprising that there is not an impulse term at zero, such as the one in our expression for r(t; b | 1). Note, however, that at t = 0 the first term of the infinite series is one and all the others are zero, so that R(0; b | 1) is indeed **exp**(–b). Although this infinite series can not be simplified further, we can evaluate a finite number of terms as an approximation. Press et al. (1989) present an algorithm to compute $I_v(z)$ using downward recurrence in v and the polynomial approximation for $I_0(z)$ given by Abramowitz and Stegun (1964). More sophisticated algorithms exist for computing a sequence of modified Bessel functions, such as the algorithm of Cody (1983), which provides guaranteed error bounds. Several codes are in the public domain and are available via *netlib* (Dongarra and Grosse, 1987); most commercial numerical libraries also provide such routines.

By a simple modification of the above derivation, we can also show that

$$(37)\quad R(t; b \mid 0) = \mathbf{exp}(-\mu t - b) \sum_{v=1}^{\infty} \left(\frac{\mu t}{b}\right)^{\frac{v}{2}} I_v\left(2\sqrt{\mu b t}\right), \quad t \geq 0.$$

2.5 Cumulative distribution of parts produced over a fixed period of time

The task of this section will be to obtain the cumulative distribution of parts produced by a machine at processing speed p over the time period [0, T) when interarrivals of failures and repairs are exponentially distributed with means λ and μ, respectively.

We now show that this distribution follows immediately from the results of the previous section. By simply noting that the Pr { parts produced in [0, T) ≤ q } is equivalent to Pr { time to produce q parts ≥ T }, which is equal to Pr { downtime incurred while producing q parts ≥ T – q / p }, we can write the following *equivalence property*

$$(38)\quad F(q/p;\, T \mid 1) \;=\; 1 \;-\; R(T - q/p;\, \lambda\, q/p \mid 1).$$

Therefore, carefully accounting for impulses and endpoints, the CDF F(t; T | 1) is given by

$$(39)\quad F(t;T\mid 1) = \begin{cases} 0 & t = 0 \\ 1 - e^{-\mu(T-t)-\lambda t} \displaystyle\sum_{v=0}^{\infty} \left(\frac{\mu(T-t)}{\lambda t}\right)^{\frac{v}{2}} I_v\!\left(2\sqrt{\mu\ \lambda\ t\ (T-t)}\right) & 0 < t < T \\ 1 & t \ge T. \end{cases}$$

Barlow and Hunter (1961) give an alternative formula for F(t; T | 1), 0 < t < T,

$$1 - e^{-\lambda t}\left[1 + \sqrt{\mu\ \lambda\ t}\ \int_0^{T-t} e^{-\mu x}\ x^{-1/2}\ I_1\!\left(2\sqrt{\mu\ \lambda\ t\ x}\right)\, dx\right]$$

due to Takács (1957). This integral is simply $1 - f(t; T \mid 11)\, P_{11}(T)$ with (T-t) replaced by x and then integrated from zero to T–t. This has an intuitive physical interpretation; see the end of this subsection. Unfortunately this integral for has no known closed-form solution, and is therefore not much more useful than the integral of the density $f(t; T \mid 1)$.

We can also conclude with analogous logic that

$$(40)\quad F(t;T \mid 0) = \begin{cases} 0 & t = 0 \\ 1 - e^{-\mu(T-t)-\lambda t} \displaystyle\sum_{v=1}^{\infty} \left(\frac{\mu(T-t)}{\lambda t}\right)^{\frac{v}{2}} I_v\left(2\sqrt{\mu\lambda t (T-t)}\right) & 0 < t < T \\ 1 & t \geq T. \end{cases}$$

We have been able to verify this expression and the expression for $F(t; T \mid 1)$ by brute-force integration of their respective densities $f(t; T \mid 0)$ and $f(t; T \mid 1)$. The basic approach is to recognize that

$$1 - F(t; T \mid 1) = \sum_{n=1}^{\infty} \Pr\{\text{ n failures in t time units }\} \Pr\{\, n^{th} \text{ repair occurs at time} \leq T\text{–}t \,\},$$

using an argument similar to the one used to derive $f(t; T \mid 1)$. Pr{ n^{th} repair occurs at time $\leq$ T-t } can be written as an infinite series using (6.5.1) and (6.5.29) of Abramowitz and Stegun (1964). Once this is done, one manipulates the two infinite series to produce a single infinite series of modified Bessel functions, and the above result follows immediately. An example of this technique can be seen at the end of this section.

Distribution function with known starting and terminal machine states

We now turn our attention to finding $F(t; T \mid 11)$, the cumulative distribution of parts produced by a machine at processing speed p over the time period [0, T) given that the machine is working at time 0 and at time T. This distribution will be important to our dynamic programming models in Chapter 3. To find this distribution, we will, as before, first derive an expression in terms of the distribution function R, and then exploit an equivalence between the distribution functions R and F.

In particular, we will now derive the probability that the downtime while producing a batch of size q is at most t, given that the machine starts working and is also working at time $t + q/p$, where p is the production speed of the machine. In our notation, this probability is $R(t; \lambda\, q/p \mid \alpha(0) = 1, \alpha(t+q/p) = 1)$; we will abbreviate this as $R(t; \lambda\, q/p \mid 11)$. Our derivation is a probabilistic argument based on Bayes' theorem. We begin by writing

$$1 - R(t; \lambda\, q/p \mid 11) = 1 - \Pr\{\text{ downtime to produce q parts} \le t \mid 11 \}.$$

The key step is to rewrite this as

$$R(t; \lambda\, q/p \mid 11) = \frac{\Pr\left\{\text{ downtime to produce q parts } \le t \text{ and } \alpha\left(t + q/p\right) = 1 \mid 1\right\}}{\Pr\left\{\alpha\left(t + q/p\right) = 1 \mid 1\right\}}$$

$$= \frac{\int_{y=0}^{t} \text{dens}\left\{\text{ downtime to produce q parts } = y\right\} \Pr\left\{\alpha\left(t + q/p - (q/p + y)\right) = 1 \mid 1\right\}\, dy}{\Pr\left\{\alpha\left(t + q/p\right) = 1 \mid 1\right\}}.$$

The reasoning behind the numerator of the last expression is as follows. First, the event { downtime to produce q parts $\le$ t } has been rewritten as { downtime to produce q parts = y } where y is integrated from 0 to t. If the downtime to produce q parts is y, the

q^{th} part is completed at time $q/p + y$. The machine must be working at the instant $q/p + y$, so in order for the machine to be working at time $t + q/p$, it must be back in the working state after an interval of length $t + q/p - (q/p + y)$. Therefore,

$$R(t; \lambda\, q/p \mid 11) \;=\; \frac{\int_{y=0}^{t} r(y;\; q/p \mid 1)\; P_{11}(t-y)\; dy}{P_{11}(t+q/p)}$$

where $P_{11}(T)$ is the probability that the machine is still working T time units later. $P_{11}(T)$ is simple to derive and appears in many contexts; we first used it in Section 2.1. It is given by

$$P_{11}(T) \;=\; \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)T}.$$

Substituting $P_{11}(T)$ gives

$$R(t; \lambda\, q/p \mid 11) \;=\; \frac{\int_{y=0}^{t} r(y;\; q/p \mid 1) \left[\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)(t-y)}\right] dy}{\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\; e^{-(\lambda+\mu)(t+q/p)}}$$

$$=\; \frac{\mu\;\, R(t;\; q/p \mid 1) \;+\; \lambda\;\; e^{-(\lambda+\mu)t} \int_{y=0}^{t} r(y;\; q/p \mid 1)\, e^{(\lambda+\mu)y}\;\; dy}{\mu \;+\; \lambda e^{-(\lambda+\mu)(t+q/p)}}.$$

The last step is to rewrite the integral in the numerator. Substituting the value for the density r, replacing its Bessel function I_0 with the usual infinite series representation, setting $b = \lambda\, q\, /\, p$, and applying (6.5.2) of Abramowitz and Stegun (1964) to express the integral as an incomplete gamma function, we obtain

$$\int_{y=0}^{t} r(y;\; q/p \mid 1)\; e^{(\lambda+\mu)y}\;\; dy \;=\; e^{-b} \;+\; \sum_{j=0}^{\infty} \frac{(\mu b t)^{j+1} e^{-b}}{j!\;\; (j+1)!}\; (-\lambda t)^{-(j+1)}\;\; \gamma(j+1,\, -\lambda t)$$

where $\gamma(a,z) = \int_0^z e^{-t}\ t^{a-1}\ dt$. Rewriting the incomplete gamma function as an infinite series using (6.5.4) and (6.5.29) gives

$$\int_{y=0}^{t} r(y;\ q/p\ |\ 1)\ e^{(\lambda+\mu)y}\ dy\ =\ e^{-b} + \sum_{j=0}^{\infty} \frac{(\mu bt)^{j+1} e^{-b+\lambda t}}{(j+1)!} \sum_{v=0}^{\infty} \frac{(-\lambda t)^{v}}{(j+v+1)!}.$$

Rearranging terms and applying the infinite series representation for $I_v(z)$, we can write

$$\int_{y=0}^{t} r(y;\ q/p\ |\ 1)\ e^{(\lambda+\mu)y}\ dy\ =\ e^{-b} + \sum_{v=0}^{\infty} (-\lambda t)^{v}\ e^{-b+\lambda t} \left[\frac{I_v\left(2\sqrt{\mu bt}\right)}{\left(\sqrt{\mu bt}\right)^{v}} - \frac{1}{v!} \right].$$

Lastly, recognizing the embedded Taylor series for **exp**(-λt) and simplifying, we obtain

$$\int_{y=0}^{t} r(y;\ q/p\ |\ 1)\ e^{(\lambda+\mu)y}\ dy\ =\ e^{-b+\lambda t} \sum_{v=0}^{\infty} (-1)^{v} \left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right).$$

Thus, our final result is, after simplification,

$$(41)\quad R(t;\, b\ |\ 11)\ =\ \frac{\mu\ R(t;\ b\ |\ 1)\ +\ \lambda\ e^{-b-\mu t} \displaystyle\sum_{v=0}^{\infty} (-1)^{v} \left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right)}{\mu\ +\ \lambda\ e^{-(\lambda+\mu)(t+q/p)}}.$$

To find R(t; b | 10) we simply note using the law of total probability that

$$R(t;\, b\ |\ 1)\ =\ R(t;\, b\ |\ 11)\ P_{11}(t+q/p)\ +\ R(t;\, b\ |\ 10)\ P_{10}(t+q/p),$$

and therefore

$$
\text{(42)}\quad R(t; b \mid 10) = \frac{R\left(t;\ b \mid 1\right) - R\left(t;\ b \mid 11\right)\ P_{11}\left(t + q/p\right)}{P_{10}\left(t + q/p\right)}
$$

$$
= \frac{R\left(t;\ b \mid 1\right) - e^{-b-\mu t} \displaystyle\sum_{v=0}^{\infty} (-1)^{v} \left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_{v}\left(2\sqrt{\mu b t}\right)}{1 - e^{-(\lambda+\mu)(t+q/p)}}.
$$

Through a similar derivation one can also show that

$$
\text{(43)}\quad R(t; b \mid 01) = \frac{R\left(t;\ b \mid 0\right) + e^{-b-\mu t} \displaystyle\sum_{v=0}^{\infty} (-1)^{v} \left(\frac{\lambda t}{\mu q/p}\right)^{\frac{v+1}{2}} I_{v+1}\left(2\sqrt{\mu b t}\right)}{1 - e^{-(\lambda+\mu)(t+q/p)}},
$$

and, of course,

$$
\text{(44)}\quad R(t; b \mid 00) = \frac{R\left(t;\ b \mid 0\right) - R\left(t;\ b \mid 01\right)\ P_{01}\left(t + q/p\right)}{P_{00}\left(t + q/p\right)}
$$

$$
= \frac{\lambda\ R\left(t;\ b \mid 0\right) - \mu\ e^{-b-\mu t} \displaystyle\sum_{v=0}^{\infty} (-1)^{v} \left(\frac{\lambda t}{\mu q/p}\right)^{\frac{v+1}{2}} I_{v+1}\left(2\sqrt{\mu b t}\right)}{\lambda + \mu\ e^{-(\lambda+\mu)(t+q/p)}}.
$$

With a little effort, it can also be seen that $R(t; b \mid 10) = R(t; b \mid 01)$.

We now use the expressions that we have derived for the distribution function R with known starting and terminal machine states to derive new expressions for the distribution function F with known starting and terminal machine states. The modification of the equivalence (38) is immediately obvious,

$$
F(q/p;\, t+q/p \mid 11) = 1 - R(t;\, \lambda\, q/p \mid 11),
$$

and we therefore can write

(45) $$F(t;T \mid 11) = 1 - \frac{\mu\big(1 - F(t;\ T \mid 1)\big) + \lambda\ e^{-\lambda t - \mu(T-t)} \sum_{v=0}^{\infty} (-1)^v \left(\frac{\lambda(T-t)}{\mu t}\right)^{v/2} I_v\left(2\sqrt{\mu\lambda t(T-t)}\right)}{\mu + \lambda\ e^{-(\lambda+\mu)T}},$$

(46) $$F(t;T \mid 10) = 1 - \frac{\big(1 - F(t;\ T \mid 1)\big) - e^{-\lambda t - \mu(T-t)} \sum_{v=0}^{\infty} (-1)^v \left(\frac{\lambda(T-t)}{\mu t}\right)^{v/2} I_v\left(2\sqrt{\mu\lambda t(T-t)}\right)}{1 - e^{-(\lambda+\mu)T}},$$

(47) $$F(t;T \mid 00) = 1 - \frac{\lambda\big(1 - F(t;\ T \mid 1)\big) - \mu\ e^{-\lambda t - \mu(T-t)} \sum_{v=0}^{\infty} (-1)^v \left(\frac{\lambda(T-t)}{\mu t}\right)^{\frac{v+1}{2}} I_{v+1}\left(2\sqrt{\mu\lambda t(T-t)}\right)}{\lambda + \mu\ e^{-(\lambda+\mu)T}},$$

and

(48) $F(t;T \mid 10) = F(t;T \mid 01)$.

Note that in all of the distributions that we have derived in this section, the terms of the infinite series alternate in sign. This is fortunate, since we can exploit numerical methods such as Euler's transformation to accelerate the convergence of these series (Press et al., 1989). As discussed earlier, the modified Bessel functions of order v can be computed using one of several available algorithms, such as Cody (1983) or the one described in Press et al. (1989).

Laplace transform with known starting and terminal machine states

We now derive an expression for the Laplace transform of $R(t; \lambda\ q/p \mid 10)$,

$$L\left\{\frac{R(t;\ b\ |\ 1) - e^{-b-\mu t}\sum_{v=0}^{\infty}(-1)^v\left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_v\left(2\sqrt{\mu b t}\right)}{1 - e^{-(\lambda+\mu)(t+q/p)}}\right\}.$$

We begin by rewriting $1/P_{10}(T)$ as

$$\frac{1}{P_{10}(T)} = \frac{1}{1-e^{-(\lambda+\mu)T}} = \sum_{n=0}^{\infty}\left(e^{-(\lambda+\mu)T}\right)^n \qquad \text{for } \left|e^{-(\lambda+\mu)T}\right| < 1.$$

Note that for all positive T, the convergence condition is satisfied. Our problem can therefore be rewritten as

$$\sum_{n=0}^{\infty} L\left\{\left(e^{-(\lambda+\mu)(t+q/p)}\right)^n R(t;\ b\ |\ 1) - \left(e^{-(\lambda+\mu)(t+q/p)}\right)^n e^{-b-\mu t}\sum_{v=0}^{\infty}(-1)^v\left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_v\left(2\sqrt{\mu b t}\right)\right\}.$$

Noting that multiplying a function of t by **exp**(-at) replaces s by s+a in its transform, we conclude that

$$\begin{aligned}
&\sum_{n=0}^{\infty} L\left\{\left(e^{-(\lambda+\mu)(t+q/p)}\right)^n R(t;\ b\ |\ 1)\right\} \\
&\quad = \sum_{n=0}^{\infty}\left(e^{-(\lambda+\mu)q/p}\right)^n R^*\left(s + (\lambda+\mu)n;\ b\ |\ 1\right) \\
&\quad = \sum_{n=0}^{\infty}\left(e^{-(\lambda+\mu)q/p}\right)^n \frac{1}{s + (\lambda+\mu)\,n}\ \mathbf{exp}\left(-b + b\frac{\mu}{s + \mu + (\lambda+\mu)\,n}\right) \\
&\quad = \sum_{n=0}^{\infty} \frac{1}{s + (\lambda+\mu)\,n}\ \mathbf{exp}\left(-(\lambda+\mu)n\frac{q}{p} - b + b\frac{\mu}{s + \mu + (\lambda+\mu)\,n}\right)
\end{aligned}$$

The more difficult half of the problem is to find

$$\sum_{n=0}^{\infty} L\left\{ \left(e^{-(\lambda+\mu)(t+q/p)}\right)^{n} e^{-b-\mu t} \sum_{v=0}^{\infty} (-1)^{v} \left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_{v}\left(2\sqrt{\mu b t}\right) \right\}.$$

We begin by constructing

$$\sum_{n=0}^{\infty} L\left\{ e^{-b-\mu t} \sum_{v=0}^{\infty} (-1)^{v} \left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_{v}\left(2\sqrt{\mu b t}\right) \right\}$$

from the basic rules of Laplace transforms. As before, the symbol $\Leftrightarrow$ will be used to represent that the expression on the left is the Laplace transform of the expression on the right. We saw in the previous section during our derivation of R(t; b | 1) that

$$\frac{1}{\mu b}\left(\frac{\mu b}{s}\right)^{v+1} \mathbf{exp}(\mu b/s) \Leftrightarrow (\mu b t)^{v/2}\, I_{v}\left(2\sqrt{\mu b t}\right), \quad v > -1.$$

Multiplying both sides by the constant $\mu^{-v}\,(q/p)^{-v}\,(-1)^{v}$ yields

$$(-1)^{v}\, \lambda^{v} \left(\frac{1}{s}\right)^{v+1} \mathbf{exp}(\mu b/s) \Leftrightarrow (-1)^{v} \left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_{v}\left(2\sqrt{\mu b t}\right), \quad v > -1.$$

Next, we replace s by s + μ, which is equivalent to multiplying the inverse transform by **exp**(–μt), and then scale both sides by the constant **exp**(–b). We obtain

$$(-1)^{v}\, \lambda^{v} \left(\frac{1}{s+\mu}\right)^{v+1} \mathbf{exp}\left(b \frac{\mu}{s+\mu}\right) \Leftrightarrow \mathbf{exp}(-\mu t)\, (-1)^{v} \left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_{v}\left(2\sqrt{\mu b t}\right), \quad v > -1.$$

and

$$(-1)^v \frac{1}{\lambda}\left(\frac{\lambda}{s+\mu}\right)^{v+1} e^{-b+b\frac{\mu}{s+\mu}} \Leftrightarrow e^{-\mu t-b}(-1)^v\left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right), \; v>-1.$$

The last step is to sum over v from zero to infinity,

$$\frac{1}{\lambda} e^{-b+b\frac{\mu}{s+\mu}} \sum_{v=0}^{\infty}(-1)^v\left(\frac{\lambda}{s+\mu}\right)^{v+1} \Leftrightarrow e^{-\mu t-b}\sum_{v=0}^{\infty}(-1)^v\left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right),$$

which, after simplification, gives

$$\left(\frac{1}{s+\lambda+\mu}\right) e^{-b+b\frac{\mu}{s+\mu}} \Leftrightarrow e^{-\mu t-b}\sum_{v=0}^{\infty}(-1)^v\left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right).$$

We now see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} L\left\{\left(e^{-(\lambda+\mu)(t+q/p)}\right)^n e^{-b-\mu t}\sum_{v=0}^{\infty}(-1)^v\left(\frac{\lambda t}{\mu q/p}\right)^{v/2} I_v\left(2\sqrt{\mu bt}\right)\right\}\\
&\quad=\sum_{n=0}^{\infty}\left(e^{-(\lambda+\mu)q/p}\right)^n \frac{1}{s+(\lambda+\mu)(n+1)}\,\mathbf{exp}\left(-b+b\frac{\mu}{s+\mu+(\lambda+\mu)n}\right).\\
&\quad=\sum_{n=0}^{\infty}\frac{1}{s+(\lambda+\mu)(n+1)}\,\mathbf{exp}\left(-(\lambda+\mu)n\frac{q}{p}-b+b\frac{\mu}{s+\mu+(\lambda+\mu)n}\right).
\end{aligned}$$

Our Laplace transform of interest is thus, in total,

$$(49)\quad R^*(s;b\mid 10)=\sum_{n=0}^{\infty}\mathbf{exp}\left(-(\lambda+\mu)n\frac{q}{p}-b+b\frac{\mu}{s+\mu+(\lambda+\mu)n}\right)\times \left[\frac{1}{s+(\lambda+\mu)n}-\frac{1}{s+(\lambda+\mu)(n+1)}\right].$$

Further, we can also show by nearly identical arguments that

(50) $$R^*(s; b \mid 11) = \sum_{n=0}^{\infty} \left(-\frac{\lambda}{\mu} e^{-(\lambda+\mu)q/p} \right)^n \mathbf{exp}\left(-b + b \frac{\mu}{s + \mu + (\lambda+\mu)\, n} \right) \times \left[\frac{1}{s + (\lambda+\mu)\, n} + \frac{\lambda}{\mu} \frac{1}{s + (\lambda+\mu)\,(n+1)} \right].$$

(51) $R^*(s; b \mid 01) = R^*(s; b \mid 10)$.

(52) $$R^*(s; b \mid 00) = \sum_{n=0}^{\infty} \left(-\frac{\mu}{\lambda} e^{-(\lambda+\mu)q/p} \right)^n \mathbf{exp}\left(-b + b \frac{\mu}{s + \mu + (\lambda+\mu)\, n} \right) \times \frac{\mu}{s + \mu + (\lambda+\mu)\, n} \left[\frac{1}{s + (\lambda+\mu)\, n} - \frac{1}{s + (\lambda+\mu)\,(n+1)} \right].$$

A simplified Laplace transform with known starting and terminal machine states

It is important to note that the above infinite series for $R^*(s; b \mid 11)$ may not converge if $\lambda > \mu$, and $R^*(s; b \mid 00)$ may not converge if $\mu > \lambda$, since the parenthetical term could be greater than one (Knopp, 1956). However, one possible solution is to work with $R^*(s; b \mid 01)$ and $R^*(s; b \mid 10)$, and then employ an equation such as

$$R(t; b \mid 1) = R(t; b \mid 11)\, P_{11}(t+q/p) + R(t; b \mid 10)\, P_{10}(t+q/p).$$

Furthermore, we can use these results to derive formulae that are simpler and do not suffer from convergence problems. Rearranging the above equation and taking the Laplace transform of each side gives

$$\mathcal{L}\{ R(t; b \mid 11)\, P_{11}(t+q/p) \} = \mathcal{L}\{ R(t; b \mid 1) - R(t; b \mid 10)\, P_{10}(t+q/p) \}$$

$$= \mathcal{L}\{ R(t; b \mid 1) \} - \mathcal{L}\{ R(t; b \mid 10)\, \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)(t+q/p)}\right) \}$$

$$= \frac{1}{s}\, \mathbf{exp}\left(-b + b \frac{\mu}{s+\mu} \right) - \frac{\lambda}{\lambda+\mu} \mathcal{L}\{ R(t; b \mid 10) \} +$$

$$\frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)q/p} \mathcal{L}\{\, R(t;\, b \mid 10)\, e^{-(\lambda+\mu)t} \,\}$$

$$= \frac{1}{s}\, \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right) -$$

$$\frac{\lambda}{\lambda+\mu} \sum_{n=0}^{\infty} \mathbf{exp}\left(-(\lambda+\mu)n\frac{q}{p} - b + b\frac{\mu}{s+\mu+(\lambda+\mu)\, n}\right) \times$$

$$\left[\frac{1}{s+(\lambda+\mu)\, n} - \frac{1}{s+(\lambda+\mu)\,(n+1)}\right] +$$

$$\frac{\lambda}{\lambda+\mu} \sum_{n=0}^{\infty} \mathbf{exp}\left(-(\lambda+\mu)(n+1)\frac{q}{p} - b + b\frac{\mu}{s+\mu+(\lambda+\mu)\,(n+1)}\right) \times$$

$$\left[\frac{1}{s+(\lambda+\mu)\,(n+1)} - \frac{1}{s+(\lambda+\mu)\,(n+2)}\right]$$

Noting that the two infinite series are identical except for a shifted index, we can cancel almost all of the terms and obtain

$$\mathcal{L}\{\, R(t;\, b \mid 11)\, P_{11}(t+q/p) \,\} =$$
$$\frac{1}{s}\, \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right) - \frac{\lambda}{\lambda+\mu}\, \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right)\left[\frac{1}{s} - \frac{1}{s+(\lambda+\mu)}\right],$$

or

$$(53)\quad \mathcal{L}\{\, R(t;\, b \mid 11)\, P_{11}(t+q/p) \,\} = \frac{1}{\lambda+\mu}\, \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right)\left[\frac{\mu}{s} + \frac{\lambda}{s+(\lambda+\mu)}\right].$$

The left-hand side can be interpreted as Pr{downtime $\le t$ & $\alpha(t+q/p) = 1 \mid \alpha(0) = 1$}. We will see later that this probability will be very useful. Of course, if $R(t;\, b \mid 11)$ is desired instead, it is a simple matter to scale by $P_{11}(t+q/p)$.

From the above development we also see that

$$(54)\quad L\{\, R(t;\, b \mid 10)\, P_{10}(t+q/p) \,\} = \frac{\lambda}{\lambda+\mu}\,\mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right)\left[\frac{1}{s} - \frac{1}{s+(\lambda+\mu)}\right].$$

Finally, using the equation

$$R(t;\, b \mid 0) = R(t;\, b \mid 01)\, P_{01}(t+q/p) + R(t;\, b \mid 00)\, P_{00}(t+q/p),$$

we can also obtain, by similar argument,

$$(55)\quad L\{\, R(t;\, b \mid 00)\, P_{00}(t+q/p) \,\} =$$

$$\mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right)\left[\frac{\mu}{s+\mu}\frac{1}{s} - \frac{\mu}{\lambda+\mu}\frac{1}{s} + \frac{\mu}{\lambda+\mu}\frac{1}{s+(\lambda+\mu)}\right].$$

and

$$(56)\quad L\{\, R(t;\, b \mid 01)\, P_{01}(t+q/p) \,\} = \frac{\mu}{\lambda+\mu}\,\mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right)\left[\frac{1}{s} - \frac{1}{s+(\lambda+\mu)}\right].$$

To use these results to compute the distribution F, we begin with the equation

$$F(q/p;\, t+q/p \mid 11) = 1 - R(t;\, \lambda\, q/p \mid 11),$$

and multiplying both sides by $P_{11}(t+q/p)$, obtain

$$P_{11}(t+q/p)\, F(q/p;\, t+q/p \mid 11) = P_{11}(t+q/p) - P_{11}(t+q/p)\, R(t;\, \lambda\, q/p \mid 11).$$

This can be used in several ways, for instance,

$$(57)\quad F(q/p;\, t+q/p \mid 11) =$$

$$1 - \frac{1}{P_{11}\left(t + q/p\right)} \; L^{-1}\left\{ \frac{1}{\lambda + \mu} \,\mathbf{exp}\left(-b + b\frac{\mu}{s + \mu} \right) \left[\frac{\mu}{s} + \frac{\lambda}{s + (\lambda + \mu)} \right] \right\}.$$

Of course, analogous expressions can be written for the other three cases (00, 01, 10).

An important property of the distribution function

Property 1. F(t; T | 1) is a non-increasing function of T.

While this is difficult to prove by calculus, the result follows immediately from the equivalence (38) between the distributions F and R. In particular, for any $\delta > 1$, $T \ge 0$ we wish to show that

$$F(t; \delta T \mid 1) \le F(t; T \mid 1).$$

We can rewrite this using the equivalence (*) as

$$R(\delta T - t;\ \lambda t \mid 1) \ge R(T - t;\ \lambda t \mid 1).$$

Since R(t; b | 1) is a non-decreasing function of t, the result follows.

By the same arguments, the above properties also hold for the CDFs F(t; T | 0), F(t; T | 11), F(t; T | 10), F(t; T | 01), and F(t; T | 00).

2.6 Transient behavior of mean and variance of uptime over a fixed period of time

In this section we explore the transient behavior of the mean and variance of $f(t; T \mid 1)/T$ (derived in Section 2.2) as we vary the parameter T. We are interested in both the behavior of the asymptotes and how quickly these functions approach their asymptotes.

Figures 2.1 – 2.6 depict the results. In each figure we vary the parameter T from 1 to 25. Figure 2.1 will serve as our base case, in which $\lambda = 1$, $\mu = 1$. We will subsequently investigate changes in the failure rate λ and the repair rate μ. We see that in the base case, the mean approaches the asymptote $\mu / (\lambda + \mu) = 50\%$ somewhat slowly, while the variance approaches its asymptote of $2 \lambda \mu / (\lambda + \mu)^3$ even more slowly. The asymptotic mean is sometimes called the *stand-alone availability*.

Next we increase the stand-alone availability (SAA) to 80% in two different ways. In Figure 2.2 we increase μ to 4, and in Figure 2.3 we decrease λ to 0.25. We see vastly different results in each case. Increasing μ leads to a great reduction in the variance asymptote and results in much quicker convergence of both the mean and the variance to their asymptotes. Decreasing λ, however, yields a slight *increase* in the variance asymptote, reduces the rate of convergence of the variance to its asymptote, and does not improve the rate of convergence of the mean as much as increasing μ.

Figures 2.4 and 2.5 tell a similar story. In each case we decrease the SAA to 20%, by increasing λ to 4 in Figure 2.4, and decreasing μ to 0.25 in Figure 2.5. Increasing λ is seen to somewhat improve the rate of convergence of the mean to its asymptote, greatly reduce the variance asymptote, and almost completely eliminate the transient effect

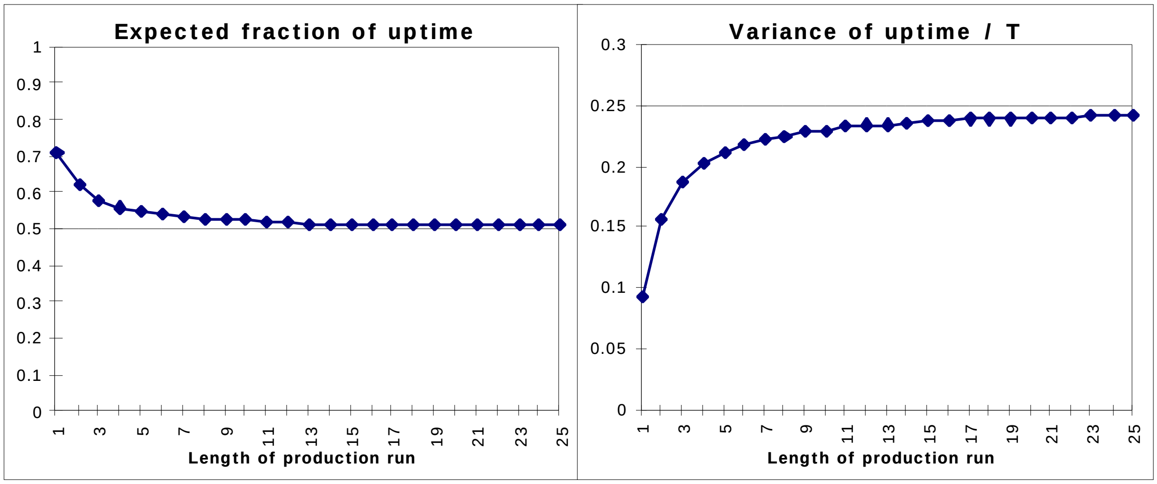

Figure 2.1 Transient behavior at λ = 1, μ = 1, SAA = 50.0%, Var. asympt. = 0.25

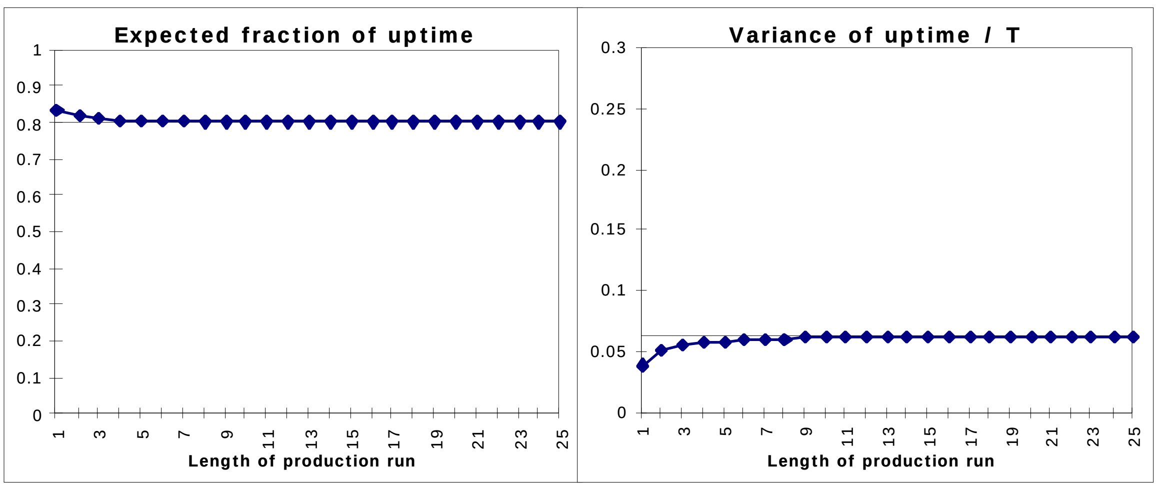

Figure 2.2 Transient behavior at λ = 1, μ = 4, SAA = 80.0%, Var. asympt. = 0.064

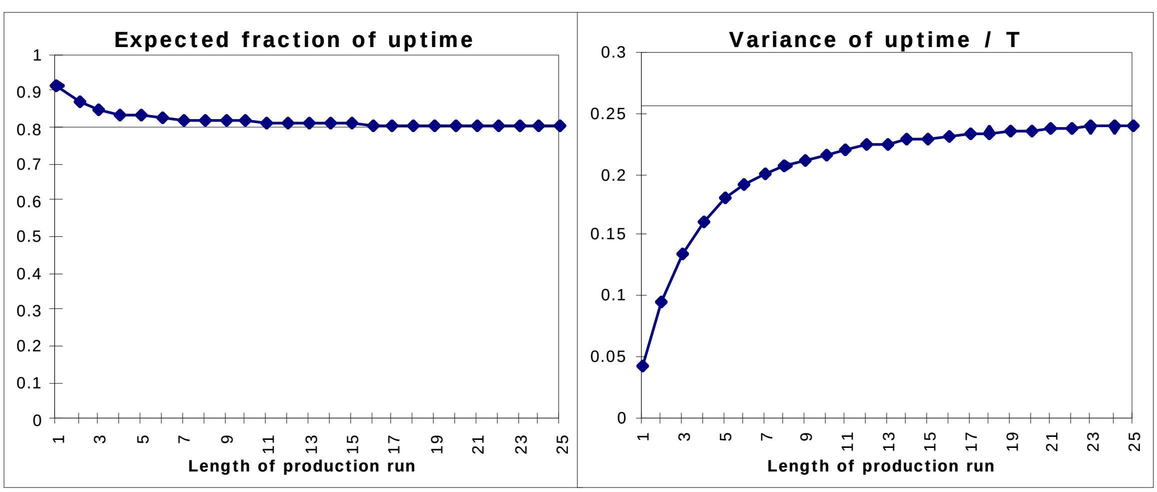

Figure 2.3 Transient behavior at λ = 0.25, μ = 1, SAA = 80.0%, Var. asympt. = 0.256

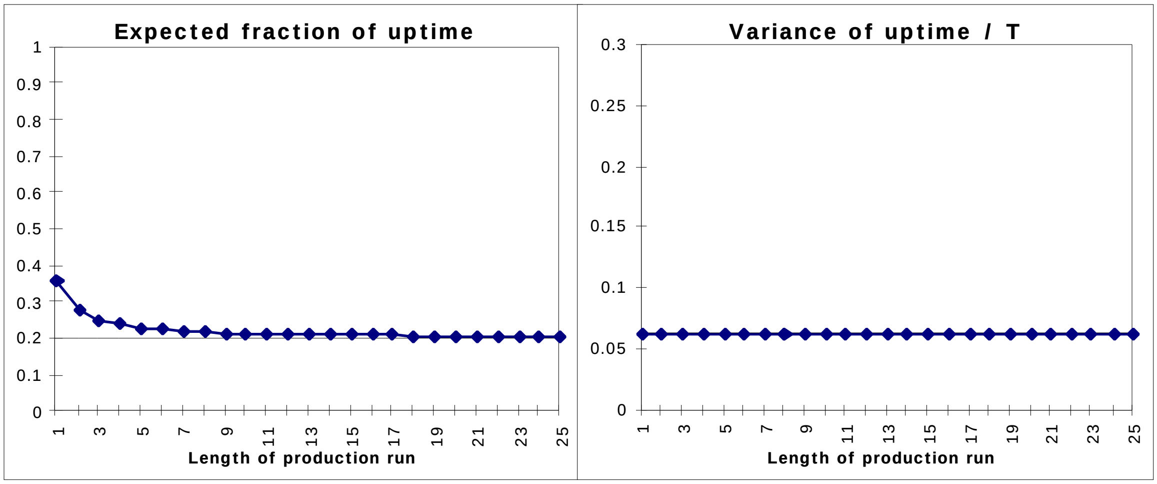

Figure 2.4 Transient behavior at $\lambda = 4$, $\mu = 1$, SAA = 20.0%, Var. asympt. = 0.064

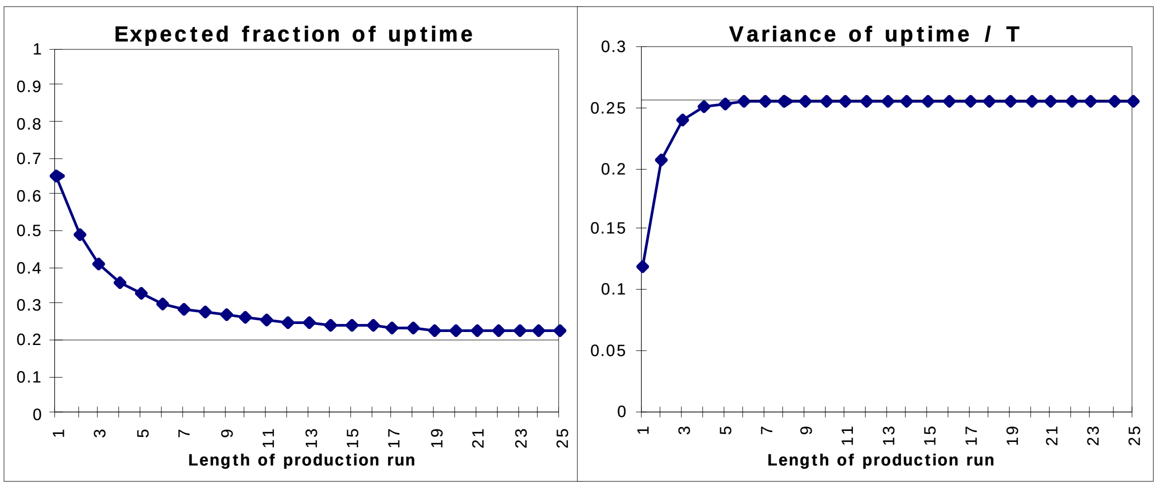

Figure 2.5 Transient behavior at $\lambda = 1$, $\mu = 0.25$, SAA = 20.0%, Var. asympt. = 0.256

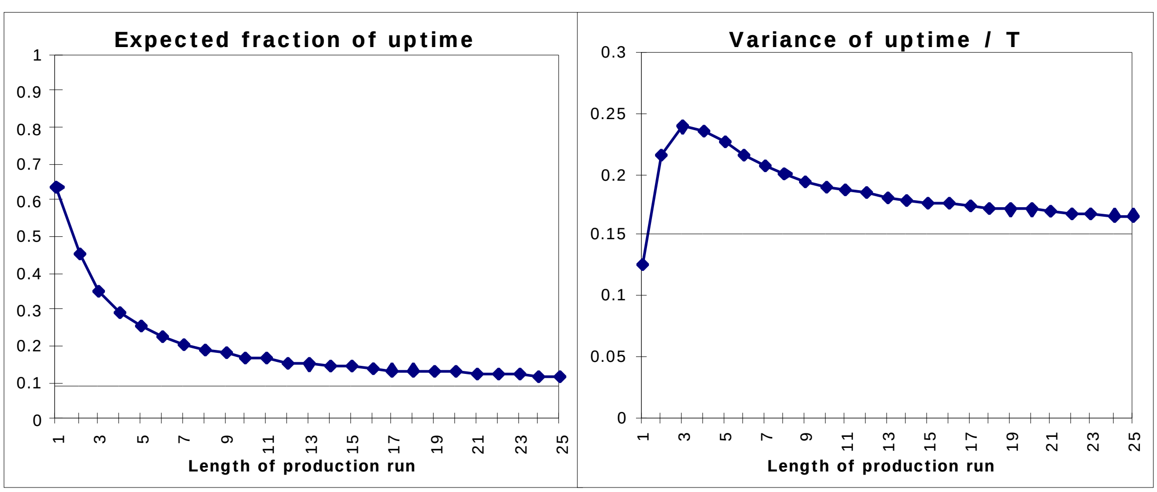

Figure 2.6 Transient behavior at $\lambda = 1$, $\mu = 0.1$, SAA = 9.1%, Var. asympt. = 0.150

associated with the variance. In sharp contrast, decreasing μ yields a slight *increase* in the variance asymptote, does not dramatically improve the rate of convergence of the variance to its asymptote, and worsens the rate of convergence of the mean. We will analytically examine this phenomena below.

Lastly, we observe in Figure 2.6 some of the unusual behavior that can exist at a very low SAA (9%). Here we observe that the variance is initially below the asymptote (at T = 1), increases above the asymptote, then decreases to the asymptote as T increases to infinity. We can understand this behavior intuitively, recognizing that the MTTR is 10 hours, that is, any failure leaves the system failed for a long period of time. This, in combination with the fact that the machine is not failed at time 0 and may still be working two or three hours later (since the MTBF is 1 hour), has the effect of significantly increasing the variability due to the initial startup effect.

There are two ways to improve the reliability of the machine. One is to increase the repair rate μ, and the other is to lower the failure rate λ. The above exploration suggests that for any fixed SAA, we would prefer to have a higher μ instead of a lower λ. We now show this analytically. Recall from Section 2.2 that the asymptotic variance of $f(t; T \mid 1)/T$ is $2\lambda\mu/(\lambda+\mu)^3$. If we increase μ to $\mu+\Delta$, then we must increase λ to $\lambda(\mu+\Delta)/\mu$ in order to maintain a constant SAA. As a result, the asymptotic variance of $f(t; T \mid 1)/T$ becomes

$$\frac{2\lambda\mu}{(\lambda+\mu)^3}\ \frac{\mu}{\mu+\Delta}$$

which is a decreasing function of Δ As a result, increasing μ while holding the SAA constant decreases the asymptotic variance of $f(t; T \mid 1)/T$. It also follows that decreasing λ while holding the SAA constant increases the asymptotic variance. It is also true (but harder to show) that the same is true of the transient variance given by (25).

We now explore relaxation time as one metric for the rate of convergence of the stochastic process, as described by Keilson (1979). First, consider the discrete state Markov Process with two states denoted zero and one. Let the transitions from state zero to state one occur with rate μ, and the transitions from one to zero with rate λ (we do not permit self-transitions), and let the system be in state one at time 0. This discrete state Markov Process is then equivalent to the machine failure process that is the subject of this chapter if we interpret the time that the process spends in state one (zero) as machine uptime (downtime).

Given a function f defined on the Markov chain state space N, Keilson defines the covariance function $r_f(\tau)$ as

$$r_f(\tau) = \sum_{m \in N} \sum_{n \in N} f(m)\ e_m\ \left(p_{mn}(\tau) - e_n\right)\ f(n)$$

where e_m are the ergodic probabilities and $p_{mn}(\tau)$ represents the probability that the chain is in state n at time t+τ if the chain is in state m at time t. Let us define the function f such that $f(0) = 0$ and $f(1) = 1$. Thus the function f serves as an indicator function for machine uptime. In this case $r_f(\tau)$ reduces to

$$r_f(\tau) = e_1\ \left(p_{11}(\tau) - e_1\right)$$

Furthermore, for our chain,

$$e_1 = \frac{\mu}{\lambda + \mu}$$

and it is easily shown (Barlow and Proschan, 1965 or Gross and Harris, 1985) that

$$p_{11}(\tau) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)\tau}.$$

Thus, given our definition of *f*, the covariance function for this process after simplification is

$$r_f(\tau) = \frac{\lambda\mu}{(\lambda + \mu)^2} \; e^{-(\lambda + \mu)\tau}.$$

The relaxation time for the process is then defined as

$$T_{REL} = \int_0^\infty \frac{r_f(\tau)\ d\tau}{r_f(0)}$$

which is easily seen to equal $1/(\lambda + \mu)$ in this case. This is consistent with our empirical observations above: the rate of convergence of expected fraction of machine uptime appears affected equally by λ and μ. This is also consistent with results of Baxter (1985), who shows that for a machine starting in steady state, the autocorrelation of the indicator function *f* is given by $\mathbf{exp}(-(\lambda + \mu)\ |\tau|)$. Furthermore, it is easy to see from the expressions for $P_{11}(T)$ and $P_{01}(T)$ that the rate of convergence of the so-called *availability coefficient* $\Pr\{\alpha(t) = 1\}$ is exponential with rate determined by $\lambda + \mu$. See Gnedenko et al. (1969) for a further discussion.

Keilson points out the familiarity of the relaxation time expression with a survival function, and notes that the relaxation time is essentially a survival function for the dependence of the process on its initial condition. Keilson also shows that the survival function can be rewritten in terms of the fundamental matrix of the process and then it is easily seen that the relaxation time derived above is in fact the largest eigenvalue of the fundamental matrix. For another discussion of relaxation time, see Morse (1958).

2.7 Normal approximation to the distribution of parts produced over a fixed period of time

In this section we briefly explore the accuracy of approximating $f(t; T \mid 1)$ by a Normal distribution. Takács (1957a, 1957b) has proven that this distribution is asymptotically Normal as a function of T. We will therefore approximate $f(t; T \mid 1)$ by its first two moments, as derived in Section 2.2. Those results will serve as the mean and variance in our Normal approximation.

To facilitate numerical evaluation of the Normal approximation, we propose two metrics. The first is the so-called Kolmogorov distance (which we denote by $\overline{K}$), the largest absolute difference between the two cumulative distributions. The second is the average absolute difference between the two cumulative distributions over [0, T) (which we will denote by $\overline{E}$). For a rigorous discussion of these metrics, see Kalashnikov (1994).

In Figure 2.7 we plot $f(t; T \mid 1)$ and the resulting Normal approximation for $\lambda = 2$, $\mu = 4$, and $T = 5$. We see that the approximation is quite good over the full range of the distribution ($\overline{E} = 0.5\%$, $\overline{K} = 2.0\%$). However, as T is decreased (with λ and μ fixed), the approximation worsens. Figures 2.8 and 2.9 show the results for $T = 2$ ($\overline{E} = 1.3\%$, $\overline{K} = 3.7\%$) and $T = 1$ ($\overline{E} = 2.8\%$, $\overline{K} = 7.1\%$). A maximum absolute error of 7% in the cumulative distribution suggests that the approximation should not be used for numerical work other than first order approximation.

Intuitively, we would expect that it is not the magnitude of T that dictates the accuracy of the approximation, but rather, the number of failure/repair cycles that occur within the interval [0, T). Figure 2.10 confirms this, where T is held at 1 while λ is increased to

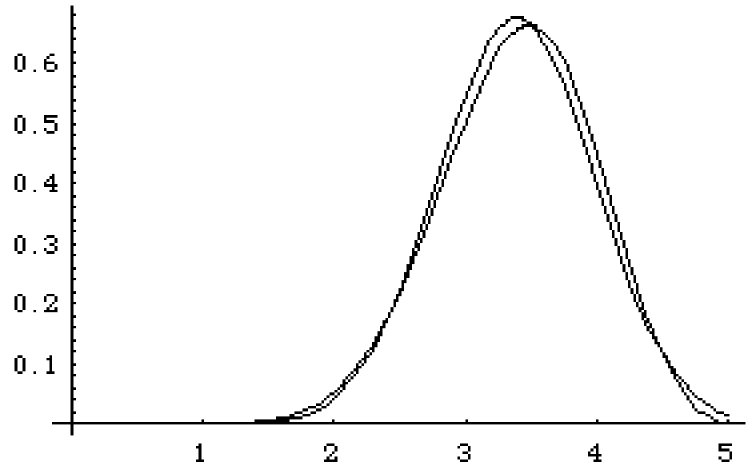

Figure 2.7 f(t;T | 1) and normal approximation at λ = 2, μ = 4, T = 5

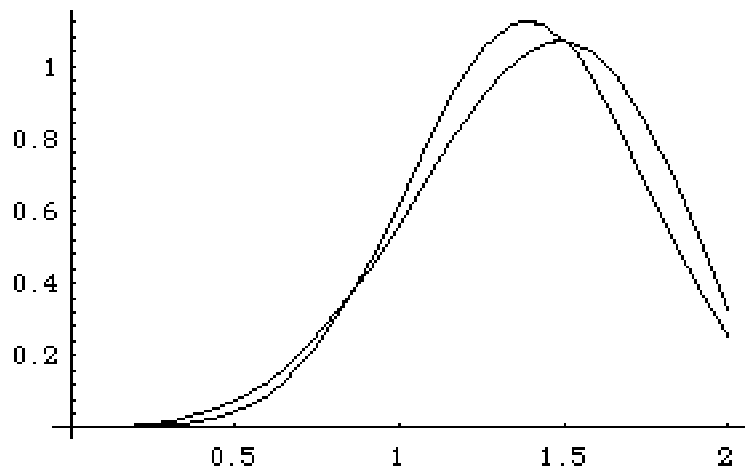

Figure 2.8 f(t;T | 1) and normal approximation at λ = 2, μ = 4, T = 2

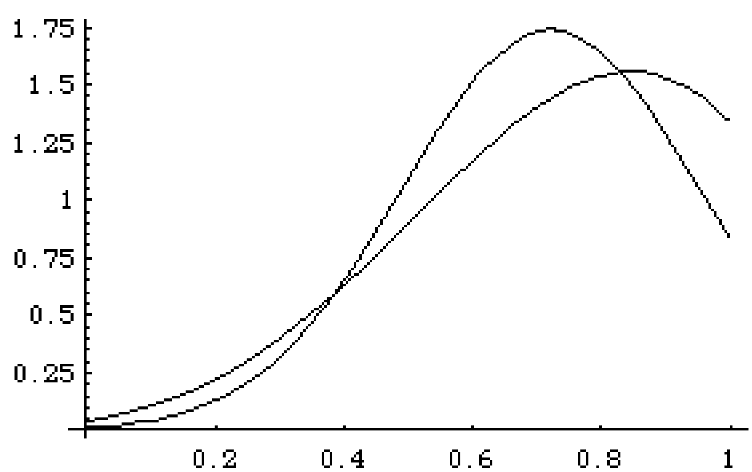

Figure 2.9 f(t;T | 1) and normal approximation at λ = 2, μ = 4, T = 1

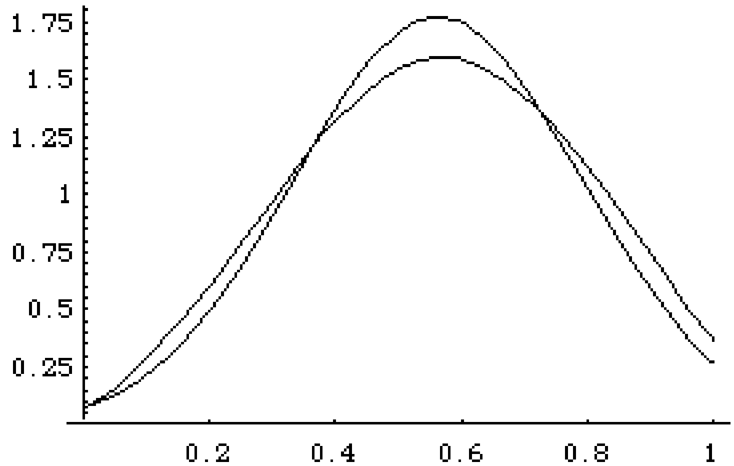

Figure 2.10 f(t;T | 1) and normal approximation at λ = 4, μ = 4, T = 1

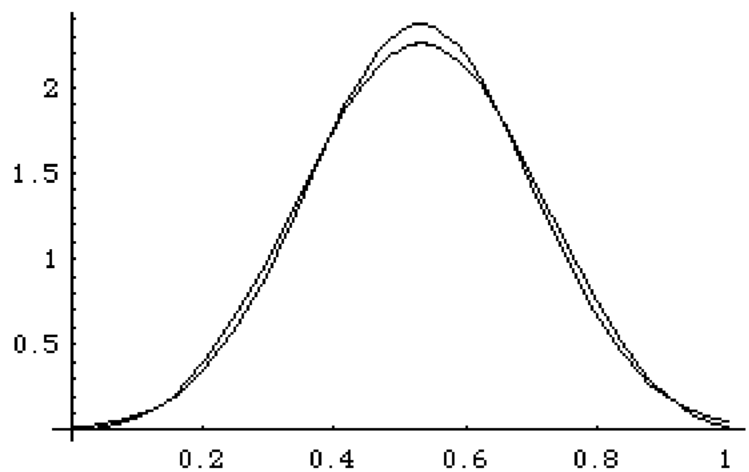

Figure 2.11 f(t;T | 1) and normal approximation at λ = 8, μ = 8, T = 1

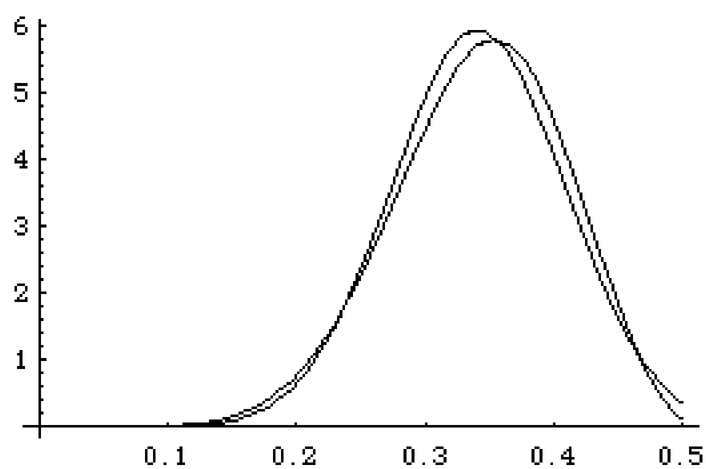

Figure 2.12 f(t;T | 1) and normal approximation at λ = 15, μ = 30, T = 0.5

4, and a dramatic improvement results ($\bar{E}$ = 1.1%, $\bar{K}$ = 2.3%). In Figure 2.11, λ and μ are increased to 8 (with T held at 1) and the approximation improves further ($\bar{E}$ = 0.4%, $\bar{K}$ = 1.0%).

Figure 2.12 further demonstrates this principle. Decreasing T to 0.5 but increasing λ to 15 and μ to 30 results in an approximation which is once again reasonable over the full range of the distribution ($\bar{E}$ = 0.7%, $\bar{K}$ = 2.4%).

To see this analytically, recall from equation (3) that

$$f(\,t; T \mid 1\,) = \left[\lambda\mu t \,\frac{I_1\left(2\sqrt{x}\right)}{\sqrt{x}} + \lambda \;\; I_0\left(2\sqrt{x}\right) \right] e^{-\lambda t-\mu(T-t)} + u_0(T-t)\, e^{-\lambda T}, \qquad 0 \le t \le T.$$

It can be seen that if t and T are scaled by k, and λ and μ by $1/k$, then $f(\,t; T \mid 1\,) \,/\, T$ is unchanged. Therefore, the shape of the density $f(\,t; T \mid 1\,)$ is determined not by T alone, but rather, by the relative magnitude of T in relation to λ and μ.

We have shown that in certain circumstances, the Normal approximation can be quite good even in short time intervals. Conversely, the approximation can be quite poor even over long time intervals. The conclusion we reach is that care must be taken before the approximation is used.

2.8 Distribution of time to produce multiple batches of parts

We now turn our attention to the problem of finding the distribution of time to produce multiple batches on a single machine. The probability density function of the time to produce n different batches is the convolution of n probability density functions of type $r(t; b)$. Since the Laplace transform $r^*(s; b)$ of $r(t; b)$ is known, we can find the Laplace transform of the density of time to produce multiple batches by simply multiplying the transforms of the probability density function of time to produce each batch. Given the transform of the density of time to produce multiple batches, it is then easy to obtain the moments of the time to produce multiple batches. The remainder of this section deals with the more difficult problem of finding the probability density function of time to produce multiple batches.

First consider the simplest possible problem: two batches with equal failure and repair rates; that is, $b_1 = b_2 = b$ and $\mu_1 = \mu_2 = \mu$. This problem is equivalent to finding the two-fold convolution of $r(t; b)$ which, from its Laplace transform, is easily seen to be equivalent to a one-batch problem with failure rate 2b. The intuition behind this result is the following. Since each process is a Compound Poisson process on the interval [0, 1] with rate b, the superposition of the two processes is a Compound Poisson process on the interval [0, 1] with rate 2b. It is also easily seen from the Laplace transform that this easily generalizes to the case $b_1 \neq b_2$, in which the problem is equivalent to a one-batch problem with failure rate $b_1 + b_2$. These results are a consequence of the fact that the superposition of two Poisson processes with rates λ_1 and λ_2 is itself a Poisson process of rate $\lambda_1 + \lambda_2$ (Ross, 1989). Furthermore, this result extends directly to $n > 2$ batches.

The case with different repair rates ($\mu_1 \neq \mu_2$) is much more difficult. The Laplace transforms easily multiply but the product is not easily inverted. One reasonable guess is to assume that this problem is equivalent to the one-batch problem with failure rate $b_1 + b_2$ and repair rate given by

$$\mu_1 \frac{b_1}{b_1 + b_2} + \mu_2 \frac{b_2}{b_1 + b_2}.$$

This approximation is exact for $\mu_1 = \mu_2$ and worsens as $|\mu_1 - \mu_2|$ grows. In fact, for reasonable values of μ_1, μ_2, b_1 and b_2, the approximation is not very good.

A two moment approximation

We instead propose the following: let us assume that the two-batch distribution can be represented as an equivalent one-batch distribution; this is a reasonable guess since the case $\mu_1 = \mu_2$ reduced to a one-batch distribution. Further, since we can find the moments of the two-batch distribution, we can find the μ_0 and b_0 for the one-batch distribution such that the first two moments of the one-batch distribution are the same as the first two moments of the two-batch distribution. This equates to solving two equations (one for each moment) in two unknowns (μ_o and b_o):

$$\begin{aligned} \frac{b_0}{\mu_0} &= \frac{b_1}{\mu_1} + \frac{b_2}{\mu_2}, \\ \frac{2b_0}{\mu_0^2} &= \frac{2b_1}{\mu_1^2} + \frac{2b_2}{\mu_2^2}, \end{aligned}$$

whose solution is

$$\begin{aligned} b_0 &= \frac{\left(b_1\mu_2 + b_2\mu_1\right)^2}{b_1\mu_2^2 + b_2\mu_1^2}, \\ \mu_0 &= \frac{\left(b_1\mu_2 + b_2\mu_1\right)\left(\mu_1\mu_2\right)}{b_1\mu_2^2 + b_2\mu_1^2}. \end{aligned}$$

This approximation has the desirable property that in the case $\mu_1 = \mu_2 = \mu$ for which the exact result is known, the approximation produces the correct exact result $b_0 = b_1 + b_2$ and $\mu_0 = \mu$.

It can be shown that the third moment of the two-batch distribution does not equal the third moment of the one-batch distribution with μ_o and b_o given above. From this we can conclude that *in general, there does not exist an equivalent one-batch distribution for the multiple batch distribution*. In other words, the resulting stochastic process is no longer Compound Poisson.

This two-batch procedure can be applied iteratively to approximate the distribution of $n \geq 3$ batches as follows: compute the one-batch approximation to the two-batch distribution yielding a one-batch distribution. Use this result along with the parameters for the third batch to compute the one-batch approximation to the three-batch distribution, and so forth.

Accuracy of two moment approximation

We would now like to evaluate the accuracy of the two moment approximation described above. Let us first take a moment to consider the limiting behavior of the multiple batch distribution. First note that as b_1 increases, the distribution of time to produce batch 1 approaches the Normal distribution. The same is true for batch 2 as b_2 increases. Therefore as b_1 and b_2 increase, the convolution of the two distributions approaches the Normal distribution, since the convolution of two Normal distributions is itself a Normal distribution. This result is discussed with great rigor and depth by Feller (1971). Since the Normal distribution is completely described by its first two moments, we can conclude that our two moment approximation is asymptotically exact.

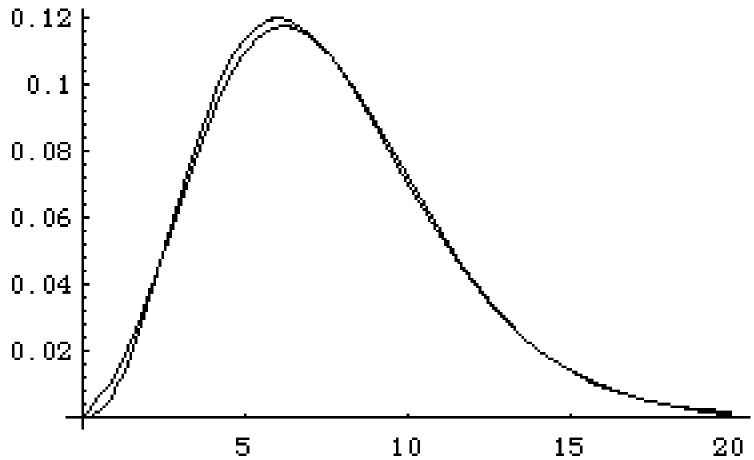

Figure 2.13 Exact and approximate convolution of two densities of type r(t; b | 1) with parameters $b_1 = 6$, $b_2 = 6$, $\mu_1 = 4$, $\mu_2 = 1$

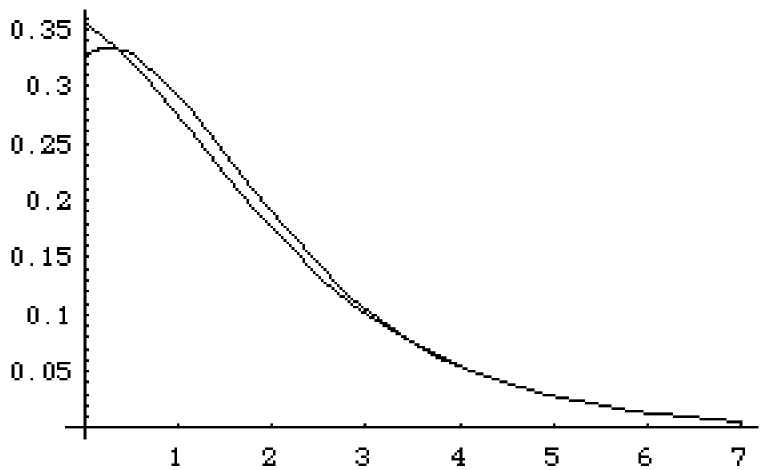

Figure 2.14 Exact and approximate convolution of two densities of type r(t; b | 1) with parameters $b_1 = 1$, $b_2 = 1$, $\mu_1 = 1$, $\mu_2 = 2$

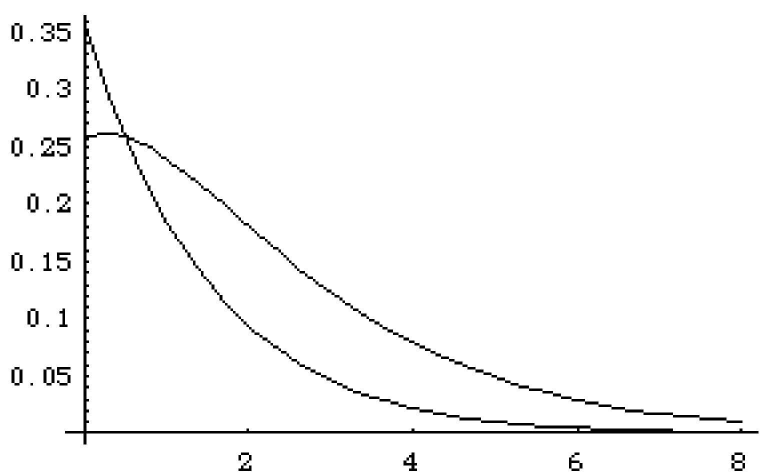

Figure 2.15 Exact and approximate convolution of two densities of type r(t; b | 1) with parameters $b_1 = 2$, $b_2 = 0.1$, $\mu_1 = 4$, $\mu_2 = 0.4$

Accordingly, we limit our attention to modest values of b. To evaluate the accuracy of the approximation we compute the Kolmogorov distance $\overline{K}$, the largest absolute difference between the exact cumulative distribution and the approximation*. Three exact convolved densities with varied parameters and their approximations are plotted on the previous page. In all examples we assume that the machine is initially working. Figure 2.13 shows that the approximation is very good for $b_1 = b_2 = 6$, $\mu_1 = 4$, $\mu_2 = 1$ ($\overline{K}$ = 0.8%). This would correspond to running two batches for three hours with MTBF = 30 minutes, and MTTR = 15 minutes for one part and 60 minutes for the other. Since the approximation is asymptotically correct and is also exact when $\mu_1 = \mu_2$, it is encouraging that the approximation is quite good even with a relatively small b and with μ_1 and μ_2 differing by a factor of 4. Figure 2.14 shows the result in the case $b_1 = b_2 = 1$, $\mu_1 = 1$, $\mu_2 = 2$. We see that in this case, even with a small b, the quality of the approximation appears reasonable ($\overline{K}$ = 3.5%). The quality of the approximation does degrade if we continue to increase the difference between μ_1 and μ_2 while keeping b and μ small. Figure 2.15 shows that the approximation breaks down at $b_1 = 2$, $b_2 = 0.1$, $\mu_1 = 4$, $\mu_2 = 0.4$ ($\overline{K}$ = 37.2%).

An equivalent convolution

We have shown above how to approximate the distribution of time to produce multiple batches. This subsection examines the probability that we are able produce two batches with differing parameters in an interval of fixed length. We will show how to write this probability as a convolution, which is important for computational purposes. Without such a result, the evaluation of this probability would require a n-fold integral. If numerical integration were used, the computational effort would grow exponentially in n. Because we are able to express the probability of interest as a convolution of n

* Note that the average absolute difference between the two cumulative distributions $\overline{E}$ will be zero since the distribution and its approximation both have infinitely long right tails.

distributions, we can write the Laplace transform of this probability as a product of n terms. As a result, if numerical Laplace transform inversion is used, the computational effort grows only linearly in n.

We now wish to evaluate the probability that we are able to produce two batches of size q_1 and q_2 in an interval of length T, where the processing speeds are p_1 and p_2, the failure rates are λ_1 and λ_2, and the repair rates are μ_1 and μ_2. Assuming that we produce batch one first, denote the probability that we produce at most q_2 parts by the distribution function $G(q_2\,;T, q_1, p_1, p_2, \lambda_1, \lambda_2, \mu_1, \mu_2)$, which we will abbreviate as $G(q_2\,;T, q_1)$. This distribution function can be written as

$$G(q_2\,;T, q_1) = \Pr\{\text{ time to produce } 1^{st} \text{ batch} > T - q_2/p_2\ \} + \int_{q_1/p_1}^{T-q_2/p_2} \Pr\{\ \#\text{ of parts produced in } 2^{nd} \text{ batch} \le q_2 \mid \text{time to produce } 1^{st} \text{ batch} = y\ \} \times \text{dens}\{\text{ time to produce } 1^{st} \text{ batch} = y\ \}\ dy$$

$$= \left[1 - R_1\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2};\ \lambda_1 \frac{q_1}{p_1}\right)\right] + \int_{q_1/p_1}^{T-q_2/p_2} F_2\left(\frac{q_2}{p_2};\ T - y\right) r_1\left(y - \frac{q_1}{p_1};\ \lambda_1 \frac{q_1}{p_1}\right) dy\,.$$

By the equivalence property,

$$= \left[1 - R_1\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2};\ \lambda_1 \frac{q_1}{p_1}\right)\right] + \int_{q_1/p_1}^{T-q_2/p_2} \left[1 - R_2\left(T - \frac{q_2}{p_2} - y;\ \lambda_2 \frac{q_2}{p_2}\right)\right] r_1\left(y - \frac{q_1}{p_1};\ \lambda_1 \frac{q_1}{p_1}\right) dy\,,$$

$$= \left[1 - R_1\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2};\ \lambda_1 \frac{q_1}{p_1}\right)\right] + R_1\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2};\ \lambda_1 \frac{q_1}{p_1}\right) - r_1\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2};\ \lambda_1 \frac{q_1}{p_1}\right) \star R_2\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2};\ \lambda_2 \frac{q_2}{p_2}\right),$$

where $\star$ denotes the convolution operator, and the argument of the convolution is $T - q_1/p_1 - q_2/p_2$. Not surprisingly, it is now seen that

$$G(q_2\,;\,T,\,q_1) = 1 - r_1\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2}\,;\ \lambda_1\frac{q_1}{p_1}\right) \star R_2\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2}\,;\ \lambda_2\frac{q_2}{p_2}\right),$$

that is, a convolution of the type that is the subject of this section. Through a similar derivation, we can also show that

$$G(q_2\,;\,T,\,q_1) = 1 - r_2\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2}\,;\ \lambda_2\frac{q_2}{p_2}\right) \star R_1\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2}\,;\ \lambda_1\frac{q_1}{p_1}\right).$$

Although we have derived this result for the case of two batches, the result extends by induction to any number of batches.

In the discussion above we have not described how to handle known initial (and possibly ending) machine states. Suppose for example that we know that the machine is initially working but is failed T time units later. Then

$$G(q_2\,;\,T,\,q_1\ |\ 10) = 1 - r_1\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2}\,;\ \lambda_1\frac{q_1}{p_1}\ \middle|\ 1\right) \star R_2\left(T - \frac{q_1}{p_1} - \frac{q_2}{p_2}\,;\ \lambda_2\frac{q_2}{p_2}\ \middle|\ 10\right).$$

In general, the density for the first part should be conditioned on the initial state of the machine. The distribution for the last part should be conditioned on 11 or 10 depending if the ending machine state is working or failed (respectively). The distribution for each intermediate part should be conditioned on the machine initially working, since the machine must have been working at the end of the previous batch (a failed machine can not complete a batch).

To prevent misunderstanding, we wish to highlight the fact that *G is not a convolution of two F distributions*. Such a convolution would correspond to the probability that we produce at most q parts given two production opportunities, one of length T_1 and the other of length T_2. Although such a convolution may have important practical uses, we do not need it for our work and do not consider it here. We mention only that the convolution of two F distributions is itself a distribution of type F if and only if the failure *and* repair rates for the two distributions are identical.

Appendix: Algorithms for numerical Laplace transform inversion

Since the Laplace transforms derived earlier in this chapter must be inverted numerically, identifying effective inversion algorithms is important for implementation of our results. The classic paper by Davies and Martin (1979) compares a variety of different Laplace transform inversion methods and measures their applicability to a variety of different types of inversion problems. Their broad conclusion is that Laguerre polynomial methods are the most effective, although no one method is optimal in all circumstances. We are not aware of a more recent survey that has followed the improvements in algorithms for numerical Laplace transform inversion over the last 15 years. In this section we present a small study of our own, briefly describing our experience using two relatively new algorithms for Laplace transform inversion that have appeared in the literature: an implementation of Talbot's Method (a contour integration method) and an implementation of Weeks' Method (a Laguerre polynomial method).

Talbot's Method

Murli and Rizzardi (1990) have developed an implementation of Talbot's (1979) method for numerically inverting Laplace transforms via contour integration. The method requires that the Laplace transform f*(s) satisfy the following criteria:

(1) The locations of the singularities $s_1, s_2, \ldots$ of f*(s) must be known; let $\gamma_0 \equiv \max_j \mathbf{Re}\left(s_j\right)$

(2) $| f^*(s) | \to 0$ uniformly as $| s | \to \infty$ in $\mathbf{Re}(s) < \gamma_0$,

(3) $| \mathbf{Im}(s_j) | < K$ j, and K is known.

Further, Murli and Rizzardi's algorithm performs best when the following additional conditions are satisfied:

(4) $|\,\mathbf{Re}(s_j)\,| \le 0 \quad j$.

(5) No singularities exist at zero.

Our tests were performed with the function

$$R^*(s;\ b \mid 1) = \frac{1}{s}\ \mathbf{exp}\left(-b + b\frac{\mu}{s+\mu}\right).$$

Based on the behavior of this function in the complex plane (Copson, 1935), we can conclude that this function is ideally suited for use with Murli and Rizzardi's algorithm. The two singularities are a simple pole at zero and an essential singularity at $-\mu$, and thus $\gamma_0 = 0$. Condition (2) is satisfied since the exponential part of R*(s) approaches **exp**(-b) as $|\,s\,| \to \infty$, and therefore R*(s) $\to$ 0. Condition (3) is satisfied for any small positive ε, and fortunately (4) is satisfied as well, although (5) is not. The authors report that their method is influenced near singularities, so that the singularity in R*(s) at zero can affect the results for large t.

The results of three experiments are shown in Tables 2.1 - 2.3. Table 2.1 represents the base case with b = 2, μ = 4, and the argument t varying from 0.001 to 100. This case is intended to be representative of the typical inputs one might expect, e.g., mean time between failures = 30 minutes, q = 100 parts, p = 100 parts/hour, mean time to repair = 15 minutes. The other two cases are intended to stress the inversion code. In Table 2.2 we set b = 20, μ = 0.4 and in Table 2.3 we set b = 0.2, μ = 40. For each case, the range of the argument t was selected so that the extremes of the tails of R(t; T | 1) were reached.

In these experiments, the exact values reported were obtained via numerical integration of the density r(t; T | 1) using *Mathematica* (Wolfram, 1988) to 14 decimal digits of precision, as reported in tables. In all of the runs, 13 decimal digits of precision were requested of the inversion code. The code used was identical to that used by Murli and Rizzardi in their experiments, except for adjustment of machine dependent parameters, and modifications that we made to utilize double-precision real and complex arithmetic.

We see from Table 2.1 that the algorithm performs extremely well over the entire range of t. Observe that at t=10, the CDF has reached unity to within 12 decimal digits of precision, and at t=1E-14, the inverse transform is **exp**(-b) to precision within the last reported decimal digit. Further, for each value of t, only 25 evaluations of the function R*(s) were required; this metric is sometimes used as an indication of the efficiency of an inversion code.

Table 2.2 shows the results of increasing b to 20 and decreasing μ to 0.4. We see that for this case, the algorithm performs extremely well for small t, but begins to produce significant errors as t grows large (>50). This is likely due to the singularity in R*(s) at zero. Talbot suggests a simple solution: increase the parameter λ in the algorithm. λ is a geometric parameter that in part determines the shape of the contour; increasing λ will shift the contour of integration away from the singularity. Increasing λ carelessly can, according to Murli and Rizzardi, result in a significant increase in computation time.

As an example of this method, we increased λ to 2λ for t = 100. The resulting approximation was then 0.99638150355243, for a relative error of -3.58E-6. Increasing λ to 3λ resulted in an approximation of 0.99638508385793, for a relative error of 1.90E-13.

t	Approximation	Exact	Rel. error	Pct. error
1E-14	0.13533528323662	0.13533528323662	0	0
1E-13	0.13533528323672	0.13533528323672	0	0
1E-12	0.13533528323769	0.13533528323770	-1.00E-14	-7.38E-12
1E-11	0.13533528324744	0.13533528324744	0	0
1E-10	0.13533528334488	0.13533528334488	0	0
1E-09	0.13533528431929	0.13533528431929	0	0
1E-08	0.13533529406343	0.13533529406344	-1.00E-14	-7.38E-12
1E-07	0.13533539150484	0.13533539150484	0	0
1E-06	0.13533636591888	0.13533636591888	0	0
1E-05	0.13534611005927	0.13534611005927	0	0
1E-04	0.13544355146224	0.13544355146224	0	0
1E-03	0.13641796454108	0.13641796454108	0	0
0.01	0.14616115308902	0.14616115308902	0	0
0.02	0.15698138198830	0.15698138198830	0	0
0.03	0.16779053519637	0.16779053519637	0	0
0.04	0.17858339797306	0.17858339797306	0	0
0.05	0.18935497016687	0.18935497016688	-1.00E-14	-5.28E-12
0.06	0.20010046086437	0.20010046086437	0	0
0.07	0.21081528310081	0.21081528310081	0	0
0.08	0.22149504863448	0.22149504863448	0	0
0.09	0.23213556278690	0.23213556278691	-1.00E-14	-4.30E-12
0.1	0.24273281935103	0.24273281935103	0	0
0.2	0.34556977358660	0.34556977358660	0	0
0.3	0.44088974695003	0.44088974695003	0	0
0.4	0.52708093194976	0.52708093194976	0	0
0.5	0.60350096061199	0.60350096061199	0	0
0.6	0.67017687350024	0.67017687350024	0	0
0.7	0.72757277323626	0.72757277323626	0	0
0.8	0.77641534042106	0.77641534042106	0	0
0.9	0.81756690473868	0.81756690473868	0	0
1	0.85193635694241	0.85193635694241	0	0
2	0.98527653589128	0.98527653589128	0	0
3	0.99888038022251	0.99888038022242	9.00E-14	9.01E-12
4	0.99992806460700	0.99992806460307	3.93E-12	3.93E-10
5	0.99999589578403	0.99999589578587	-1.84E-12	-1.84E-10
6	0.99999978591957	0.99999978591942	1.50E-13	1.50E-11
7	0.99999998959658	0.99999998959718	-6.00E-13	-6.00E-11
8	0.99999999952322	0.99999999952291	3.10E-13	3.10E-11
9	0.99999999997913	0.99999999997915	-2.00E-14	-2.00E-12
10	0.99999999999912	0.99999999999913	-1.00E-14	-9.99E-13

Table 2.1 Results of Murli and Rizzardi's algorithm for b = 2, μ = 4

t	Approximation	Exact	Rel. error	Pct. error
1E-15	2.0611536224386E-9	2.0611536224386E-9	0	0
1E-14	2.0611536224387E-9	2.0611536224387E-9	0	0
1E-13	2.0611536224402E-9	2.0611536224402E-9	0	0
1E-12	2.0611536224550E-9	2.0611536224550E-9	0	0
1E-11	2.0611536226034E-9	2.0611536226035E-9	-1.00E-22	-4.86E-12
1E-10	2.0611536240875E-9	2.0611536240875E-9	0	0
1E-09	2.0611536389278E-9	2.0611536389278E-9	0	0
1E-08	2.0611537873308E-9	2.0611537873309E-9	-1.00E-22	-4.86E-12
1E-07	2.0611552713617E-9	2.0611552713618E-9	-1.00E-22	-4.86E-12
1E-06	2.0611701116972E-9	2.0611701116972E-9	0	0
1E-05	2.0613185176964E-9	2.0613185176964E-9	0	0
1E-04	2.0628028421636E-9	2.0628028421636E-9	0	0
1E-03	2.0776725529970E-9	2.0776725529970E-9	0	0
0.01	2.2290350045993E-9	2.2290350045993E-9	0	0
0.1	4.0285688996247E-9	4.0285688996247E-9	0	0
1	7.7718249473702E-8	7.7718249473702E-8	0	0
2	4.8418621380097E-7	4.8418621380097E-7	0	0
3	1.8638678233342E-6	1.8638678233342E-6	0	0
4	5.5448015001308E-6	5.5448015001308E-6	0	0
5	1.3951529953470E-5	1.3951529953470E-5	0	0
6	3.1128461416655E-5	3.1128461416655E-5	0	0
7	6.3366983528633E-5	6.3366983528633E-5	0	0
8	1.1991655923654E-4	1.1991655923654E-4	0	0
9	2.1374957527165E-4	2.1374957527165E-4	0	0
10	3.6234082052527E-4	3.6234082052527E-4	0	0
20	1.3027549946722E-2	1.3027549946739E-2	-1.70E-14	-1.30E-10
30	9.0853144998461E-2	9.0853144737580E-2	2.61E-10	2.87E-07
40	0.27969010773392	0.27969010486059	2.87E-09	1.03E-06
50	0.53163935704165	0.53163913993762	2.17E-07	4.08E-05
60	0.75157619547114	0.75157367485922	2.52E-06	3.35E-04
70	0.89095222433758	0.89095380605233	-1.58E-06	-1.78E-04
80	0.95957587129784	0.95957851871227	-2.65E-06	-2.76E-04
90	0.98709474700440	0.98709153853762	3.21E-06	3.25E-04
100	0.99638377411774	0.99638508385774	-1.31E-06	-1.31E-04
110	0.99909886512638	0.99909869355890	1.72E-07	1.72E-05
120	0.99979760432365	0.99979733975725	2.65E-07	2.65E-05
130	0.99995828713949	0.99995845751716	-1.70E-07	-1.70E-05
140	0.99999217241571	0.99999216508717	7.33E-09	7.33E-07
150	0.99999864556966	0.99999862976019	1.58E-08	1.58E-06
160	0.99999977440395	0.99999977628191	-1.88E-09	-1.88E-07
170	0.99999996487716	0.99999996570170	-8.25E-10	-8.25E-08
180	0.99999999504371	0.99999999503739	6.32E-12	6.32E-10
190	0.99999999934825	0.99999999931933	2.89E-11	2.89E-09
200	0.99999999991661	0.99999999991796	-1.35E-12	-1.35E-10
210	0.99999999998919	0.99999999998893	2.60E-13	2.60E-11
220	0.99999999999867	0.99999999999821	4.60E-13	4.60E-11
230	0.99999999999984	0.99999999999986	-2.00E-14	-2.00E-12
240	0.99999999999999	1.00000000000000	-9.99E-15	-9.99E-13
250	1.00000000000000	1.00000000000000	0	0

Table 2.2 Results of Murli and Rizzardi's algorithm for b = 20, μ = 0.4

Furthermore, recomputing the results of all the entries in Table 2.2 resulted in absolute errors smaller than 1.24E-12 for all t reported.

Although we are encouraged by the success of increasing λ to 3λ, we should heed the warning of Murli and Rizzardi and first investigate the impact of such a change on the speed of the algorithm. Timing tests were conducted on a Power Macintosh 7100/80 in emulation mode with SANE-based math instructions. The parameters used were the same as for the results of Table 2.2. For the base case (λ unadjusted), 1000 inversions required approximately 26 seconds, or 0.26 seconds per inversion. Increasing λ to 3λ resulted in no measurable increase in computation time. Lastly, an unrelated test was conducted to determine the impact of decreasing the requested accuracy from 13 significant decimal digits to 6. The result was a two-fold performance improvement, decreasing the time per inversion to approximately 0.13 seconds.

The results of the third experiment are reported in Table 2.3. Here we see that for $b = 0.2$, $\mu = 40$, λ unadjusted, the algorithm once again performs extremely well. At t = 1E-15, the inverse transform is equal to **exp**(-b) and at t = 0.9 the inverse transform equals unity, to accuracy within the last decimal digit reported.

Further experiments were conducted and showed that an increase in λ was helpful to improve accuracy whenever b was several (e.g., 3) orders of magnitude larger than μ.

Experiments were also performed at $b = 20$, $\mu = 40$, and the results were identical to those obtained at $b = 20$, $\mu = 0.4$, except with t 100 times smaller. Similarly, the results for $b = 0.2$, $\mu = 0.4$ were identical to those obtained for $b = 0.2$, $\mu = 40$, except with t 100 times larger.

Weeks' Method

Garbow et al. (1988) have developed an implementation of Weeks' method, which approximates a function from its Laplace transform by expansion in a Laguerre polynomial. The method requires that the function has continuous derivatives of all orders.

The algorithm consists of two distinct stages. The first stage computes the coefficients of the Laguerre polynomial to achieve a desired accuracy level. Once the coefficients are determined, the inversion for any particular value of the function is accomplished simply by evaluating the Laguerre polynomial. This two stage approach means that the algorithm of Garbow et al. will be particularly efficient when the same function needs to be evaluated for many different values of t.

Timing experiments were performed on the same hardware as before, on the problem of Table 2.1. Like many numerical inversion algorithms, geometric parameters can be specified which can influence the accuracy of the result. In our experiments we set σ_0 to one and allowed the algorithm to set σ and b; see Lyness and Giunta (1986) for theoretical details. The first stage of the algorithm computed a Laguerre polynomial with 128 coefficients in 0.14 seconds. The algorithm also determined that accuracy could not be improved further (i.e., by computing more coefficients). Using this polynomial, 1000 inversions were performed in 8 seconds, which is about 0.008 seconds per inversion.

For this particular function and choice of λ and μ, we see that the algorithm of Garbow et al. is faster than that of Murli and Rizzardi *even if only one function value is required.* This need not always be true. We have found that some Laguerre polynomial expansions require a much larger number of coefficients to achieve a high degree of

t	Approximation	Exact	Rel. error	Pct. error
1E-15	0.81873075307798	0.81873075307799	-1E-14	-1E-12
1E-14	0.81873075307804	0.81873075307805	-1E-14	-1E-12
1E-13	0.81873075307863	0.81873075307864	-1E-14	-1E-12
1E-12	0.81873075308453	0.81873075308453	0	0
1E-11	0.81873075314348	0.81873075314348	0	0
1E-10	0.81873075373296	0.81873075373297	-1E-14	-1E-12
1E-09	0.81873075962782	0.81873075962783	-1E-14	-1E-12
1E-08	0.81873081857643	0.81873081857643	0	0
1E-07	0.81873140806140	0.81873140806141	-1E-14	-1E-12
1E-06	0.81873730280611	0.81873730280611	0	0
1E-05	0.81879623974991	0.81879623974991	0	0
1E-04	0.81938456011584	0.81938456011585	-1E-14	-1E-12
1E-03	0.82516409833765	0.82516409833766	-1E-14	-1E-12
0.01	0.87373116093797	0.87373116093798	-1E-14	-1E-12
0.02	0.91208423778107	0.91208423778107	0	0
0.03	0.93881527657833	0.93881527657834	-1E-14	-1E-12
0.04	0.95743704762819	0.95743704762819	0	0
0.05	0.97040356552837	0.97040356552838	-1E-14	-1E-12
0.06	0.97942821305528	0.97942821305529	-1E-14	-1E-12
0.07	0.98570660604161	0.98570660604161	0	0
0.08	0.99007261462709	0.99007261462710	-1E-14	-1E-12
0.09	0.99310751643234	0.99310751643235	-1E-14	-1E-12
0.1	0.99521631196769	0.99521631196769	0	0
0.2	0.99987807193596	0.99987807193596	0	0
0.3	0.99999697260719	0.99999697260720	-1E-14	-1E-12
0.4	0.99999992636591	0.99999992636588	3E-14	3E-12
0.5	0.99999999823878	0.99999999823881	-3E-14	-3E-12
0.6	0.99999999995846	0.99999999995846	0	0
0.7	0.99999999999903	0.99999999999903	0	0
0.8	0.99999999999997	0.99999999999998	-1E-14	-1E-12
0.9	1.00000000000000	1.00000000000000	0	0

Table 2.3 Results of Murli and Rizzardi's algorithm for b = 0.2, μ = 40

accuracy, and in these cases the first stage of the algorithm can take many times longer than in the present case. Of course, if a sufficiently large number of function values is desired, then the algorithm of Garbow et al. will always be faster.

A naive attempt to implement the algorithm of Garbow et al. for the problem of Table 2.2 (b = 20, μ = 0.4) produces unacceptable results. The algorithm performed extremely well for small to moderate values of t, but for t ≥ 40 the algorithm did not output meaningful answers. The problem is easily corrected by scaling the problem. In particular, we multiplied μ by b and the argument t by $1/b$. This resulted in less accuracy for small t but uniformly good results over the entire range of t. The results are summarized in Table 2.4. To achieve even better results, one could employ a combination of a scaled and unscaled usage of the algorithm depending on parameter and argument values.

Timing experiments were also conducted for the problem of Table 2.4. The first stage of the algorithm computed a Laguerre polynomial with 256 coefficients in 0.24 seconds, and determined that accuracy could not be improved further by computing more coefficients. Using this polynomial, 1000 inversions were performed in 15 seconds, which is about 0.015 seconds per inversion.

t	Approximation	Exact	Rel. error	Pct. error
1E-15	2.0611536183944E-9	2.0611536224386E-9	-4E-18	-2E-07
1E-14	2.0611536197266E-9	2.0611536224387E-9	-3E-18	-1E-07
1E-13	2.0611536188208E-9	2.0611536224402E-9	-4E-18	-2E-07
1E-12	2.0611536186600E-9	2.0611536224550E-9	-4E-18	-2E-07
1E-11	2.0611536184981E-9	2.0611536226035E-9	-4E-18	-2E-07
1E-10	2.0611536208407E-9	2.0611536240875E-9	-3E-18	-2E-07
1E-09	2.0611536347846E-9	2.0611536389278E-9	-4E-18	-2E-07
1E-08	2.0611537845659E-9	2.0611537873309E-9	-3E-18	-1E-07
1E-07	2.0611552677738E-9	2.0611552713618E-9	-4E-18	-2E-07
1E-06	2.0611701080297E-9	2.0611701116972E-9	-4E-18	-2E-07
1E-05	2.0613185140643E-9	2.0613185176964E-9	-4E-18	-2E-07
1E-04	2.0628028388098E-9	2.0628028421636E-9	-3E-18	-2E-07
1E-03	2.0776725493649E-9	2.0776725529970E-9	-4E-18	-2E-07
0.01	2.2290350018216E-9	2.2290350045993E-9	-3E-18	-1E-07
0.1	4.0285688983664E-9	4.0285688996247E-9	-1E-18	-3E-08
1	7.7718249472495E-8	7.7718249473702E-8	-1E-18	-2E-09
2	4.8418621379980E-7	4.8418621380097E-7	-1E-18	-2E-10
3	1.8638678233330E-6	1.8638678233342E-6	-1E-18	-6E-11
4	5.5448015001291E-6	5.5448015001308E-6	-2E-18	-3E-11
5	1.3951529953469E-5	1.3951529953470E-5	-1E-18	-7E-12
6	3.1128461416655E-5	3.1128461416655E-5	0	0
7	6.3366983528632E-5	6.3366983528633E-5	-1E-18	-2E-12
8	1.1991655923653E-4	1.1991655923654E-4	-1E-17	-8E-12
9	2.1374957527165E-4	2.1374957527165E-4	0	0
10	3.6234082052526E-4	3.6234082052527E-4	-1E-17	-3E-12
20	1.3027549946739E-2	1.3027549946739E-2	0	0
30	9.0853144737558E-2	9.0853144737580E-2	-2E-14	-2E-11
40	0.27969010486059	0.27969010486059	0	0
50	0.53163913993762	0.53163913993762	0	0
60	0.75157367485920	0.75157367485922	-2E-14	-3E-12
70	0.89095380605233	0.89095380605233	0	0
80	0.95957851871226	0.95957851871227	-1E-14	-1E-12
90	0.98709153853762	0.98709153853762	0	0
100	0.99638508385774	0.99638508385774	0	0
110	0.99909869355937	0.99909869355890	5E-13	5E-11
120	0.99979733975724	0.99979733975725	-1E-14	-1E-12
130	0.99995845751714	0.99995845751716	-2E-14	-2E-12
140	0.99999216508717	0.99999216508717	0	0
150	0.99999862976019	0.99999862976019	0	0
160	0.99999977628198	0.99999977628191	7E-14	7E-12
170	0.99999996570231	0.99999996570170	6E-13	6E-11
180	0.99999999503693	0.99999999503739	-5E-13	-5E-11
190	0.99999999931367	0.99999999931933	-6E-12	-6E-10
200	0.99999999990377	0.99999999991796	-1E-11	-1E-09
210	1.00000000000000	0.99999999998893	1E-11	1E-09
220	0.99999999999743	0.99999999999821	-8E-13	-8E-11
230	0.99999999998623	0.99999999999986	-1E-11	-1E-09
240	0.99999999955673	1.00000000000000	-4E-10	-4E-08
250	0.99999999932218	1.00000000000000	-7E-10	-7E-08

Table 2.4 Results of the algorithm of Garbow et al. for b = 20, μ = 0.4

References for Chapter 2

Abate, Joseph and Ward Whitt. "Numerical Inversion of Probability Generating Functions". Operations Research Letters, 12, pp. 245-251, 1992.

Abramowitz, Milton and Irene A. Stegun. Handbook of Mathematical Functions, Applied Mathematics Series, vol. 55. Washington: National Bureau of Standards, 1964 (reprinted by Dover Publications, New York, 1965).

Barlow, Richard E. and Larry C. Hunter. "Reliability Analysis of a One-Unit System". Operations Research, 9, pp. 200-208, 1961.

Barlow, Richard E. and Frank Proschan. Mathematical Theory of Reliability. New York: John Wiley & Sons, Inc., 1965.

Baxter, Laurence A. "Availability Measures for a Two-State System". Journal of Applied Probability, 18, pp. 227-235, 1981a.

Baxter, Laurence A. "Some Remarks on Numerical Convolution". Communications in Statistics: Simulation and Computation, B10(3), pp. 281-288, 1981b.

Baxter, Laurence A. "Some Notes on Availability Theory". Microelectronics and Reliability, 25(5), pp. 921-926, 1985.

Bharucha-Reid, A. T. Elements of the Theory of Markov Processes and Their Applications. New York: McGraw-Hill, 1960.

Boisvert, Ronald F. and Bonita V. Saunders. "Algorithm 713: Portable Vectorized Software for Bessel Function Evaluation". ACM Transactions on Mathematical Software, 18(4), pp. 456-469, December 1992.

Brouwers, J. J. H. "Probabilistic Descriptions of Irregular System Downtime". Reliability Engineering, 15, pp. 263-281, 1986.

Cléroux, R. and D. J. McConalogue. "A Numerical Algorithm for Recursively-Defined Convolution Integrals Involving Distribution Functions." Management Science, 22, pp. 1138-1146, 1976.

Cody, W. J. "Algorithm 597: Sequence of Modified Bessel Functions of the First Kind." ACM Transactions on Mathematical Software, 9(2), pp. 242-245, June 1983.

Copson, E. T. An Introduction to the Theory of Functions of a Complex Variable. Oxford: Oxford University Press, 1935.

Cox, D. R. Renewal Theory. London: Methuen & Co., Ltd., 1962.

Crump, Kenny S. "Numerical Inversion of Laplace Transforms Using a Fourier Series Approximation". Journal of the Association for Computing Machinery, 23(1), pp. 89-96, 1976.

Davies, Brian and Brian Martin. "Numerical Inversion of the Laplace Transform: a Survey and Comparison of Methods". Journal of Computational Physics, 33(1), pp. 1-32, 1979.

De Hoog, F. R., J. H. Knight, and A. N. Stokes. "An Improved Method for Numerical Inversion of Laplace Transforms". SIAM Journal of Scientific and Statistical Computing, 3(3), pp. 357-366, 1982.

Dickey, J. M. "The Renewal Function for an Alternating Renewal Process, Which Has a Weibull Failure Distribution and a Constant Repair Time". Reliability Engineering and System Safety, 31, pp. 321-343, 1991.

Doetsch, G. Guide to the Application of Laplace Transforms. New York: Van Nostrand, 1961.

Dongarra, J. J. and E. Grosse. "Distribution of Mathematical Software via Electronic Mail". Communications of the ACM, 30(5), pp. 403-407, May 1987.

Feller, William. An Introduction to Probability Theory and Its Applications, Volume 2, Second Edition. New York: John Wiley & Sons, Inc., 1971.

FitzHugh, Richard. "Statistical Properties of the Asymmetric Random Telegraph Signal, with Applications to Single-Channel Analysis". Mathematical Biosciences, 64, pp. 75-89, 1983.

Garbow, B. S., G. Giunta, J. N. Lyness and A. Murli. "Software for an Implementation of Weeks' Method for the Inverse Laplace Transform Problem". ACM Transactions on Mathematical Software, 14(2), pp. 163-170, June 1988.

Gershwin, Stanley B. Manufacturing Systems Engineering. Englewood Cliffs, NJ: Prentice-Hall, 1994.

Gnedenko, B. V., Yu K. Belyayev and A. D. Solovyev. Mathematical Methods of Reliability Theory. New York: Academic Press, 1969.

Gross, Donald and Carl M. Harris. Fundamentals of Queueing Theory. New York: John Wiley & Sons, Inc., 1985.

Grundy, R. E. "Laplace Transform Inversion Using Two-point Rational Approximants". Journal of the Institute of Mathematics and Its Applications, 20, pp. 299-306, 1977.

Kabak, Irwin W. "System Availability and Some Design Implications". Operations Research, 17, pp. 827-837, 1969.

Kalashnikov, Vladimir V. Mathematical Methods in Queueing Theory. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1994.

Keilson, Julian. Markov Chain Models – Rarity and Exponentiality. New York: Springer-Verlag, 1979.

Kim, David S. Personal communication, 1994.

Kim, David S. and Jeffrey M. Alden. "Estimating the Distribution of Time Required to Produce a Fixed Lot Size on a Serial Production Line". Research Report GMR-7811, General Motors Research Laboratories, 1992.

Knopp, Konrad. Infinite Sequences and Series. New York: Dover, 1956.

Krylov, Vladimir Ivanovich and Nadezhda Sergeevna Skoblya. Handbook of Numerical Inversion of Laplace Transforms. Jerusalem: Israel Program for Scientific Translations, 1969.

Lie, C. H., C. L. Hwang, and F. A. Tillman. "Availability of Maintained Systems: A State-of-the-Art Survey". AIIE Transactions, 9(3), pp. 247-259, 1977.

Lyness, J. N. and G. Giunta. "A Modification of the Weeks Method for Numerical Inversion of the Laplace Transform". Mathematics of Computation, 47(175), pp. 313-322, July 1986.

Martz, H. F. Jr. "On Single-Cycle Availability". IEEE Transactions on Reliability, R-20(1), pp. 21-23, 1971.

McConalogue, D. J. "Convolution Integrals Involving Probability Distribution Functions (Algorithm 102)". Computer Journal, 21(3), pp. 270-272, 1978.

McConalogue, D. J. "Numerical Treatment of Convolution Integrals Involving Distributions With Densities Having Singularities at the Origin". Communications in Statistics: Simulation and Computation, B10(3), pp. 265-280, 1981.

Morse, Philip M. Queues, Inventories and Maintenance. New York: John Wiley & Sons, Inc., 1958.

Mortensen, R. E. "Alternating Renewal Process Models for Electric Power System Loads". IEEE Transactions on Automatic Control, 35(11), pp. 1245-1249, 1990.

Munford, A. G. "Moments of a Filtered Binary Process". IEEE Transactions on Information Theory, 32(6), pp. 824-826, 1986.

Murli, A. and M. Rizzardi. “Algorithm 682: Talbot’s Method for the Laplace Inversion Problem”. ACM Transactions on Mathematical Software, 16(2), pp. 158-168, June 1990.

Nahmias, Stephen. Production and Operations Analysis. Homewood, Illinois: Irwin, 1989.

Piessens, R. and R. Huysmans. “Algorithm 619: Automatic Numerical Inversion of the Laplace Transform”. ACM Transactions on Mathematical Software, 10(3), pp. 348-353, September 1984.

Press, William H., Brian P. Flannery, Saul A. Teukolsky and William T. Vetterling. Numerical Recipes in Pascal. Cambridge: Cambridge University Press, 1989.

Ross, Sheldon M. Stochastic Processes. New York: John Wiley & Sons, Inc., 1983.

Ross, Sheldon M. Introduction to Probability Models, Fourth Edition. San Diego, CA: Academic Press, Inc., 1989.

Serfozo, R. F. “Point Processes”, Chapter 1 in Heyman, D. P. and M. J. Sobel, eds., Stochastic Models, Handbooks in Operations Research & Management Science, Vol. 2, Amsterdam: Elsevier (North-Holland), 1990.

Sericola, Bruce. “Closed-form Solution for the Distribution of the Total Time Spent in a Subset of States of a Homogeneous Markov Process During a Finite Observation Period”. Journal of Applied Probability, 27, pp. 713-719, 1990.

Shaked, Moshe and J. George Shanthikumar. “Reliability and Maintainability”, Chapter 13 in Heyman, D. P. and M. J. Sobel, eds., Stochastic Models, Handbooks in Operations Research & Management Science, Vol. 2, Amsterdam: Elsevier (North-Holland), 1990.

Sookne, D. J. “Bessel Functions of Real Argument and Integer Order”. National Bureau of Standards Journal of Research B, 77A, pp. 125-132, 1973.

Takács, Lajos. “Occurrence and Coincidence Phenomena in Case of Happenings with Arbitrary Distribution Law of Duration”. *Acta Mathematica* (Academiae Scientiarum Hungaricae), 2, pp. 275-298, 1951.

Takács, Lajos. “On Certain Sojurn Time Problems in the Theory of Stochastic Processes”. *Acta Mathematica* (Academiae Scientiarum Hungaricae), 8, pp. 169-191, 1957a.

Takács, Lajos. “On Limiting Distributions Concerning a Sojourn Time Problem”. *Acta Mathematica* (Academiae Scientiarum Hungaricae), 8, pp. 279-294, 1957b.

Takács, Lajos. "On a Sojurn Time Problem in the Theory of Stochastic Processes". Transactions of the American Mathematical Society, 93, pp. 531-540, 1959.

Talbot, A. "The Accurate Numerical Inversion of Laplace Transforms". Journal of the Institute of Mathematics and Its Applications, 23, pp. 97-120, 1979.

Wolfram, Stephen. *Mathematica*: A System for Doing Mathematics by Computer. Redwood City, CA: Addison-Wesley, 1988.