

Electronic Companion—“A Single-Product Inventory Model for Multiple Demand Classes” by Hasan Arslan, Stephen C. Graves, and Thomas A. Roemer, *Management Science*, doi 10.1287/mnsc.1070.0701.

Online Appendices

Appendix I. Properties of the Single-Pass Algorithm (SPA)

The purpose of this appendix is to provide the proofs for four propositions that characterize the solution from the Single-Pass Algorithm (SPA) to the Service Level Problem (SLP). We restate the propositions as they appear in the paper; the notation is defined in the paper and the equation numbers refer to equations in the paper.

PROPOSITION 1. Consider two stocking policies \mathbf{s}^1 and \mathbf{s}^2 where for some j , $s_j^2 = s_j^1 - 1$, $s_{j-1}^2 = s_{j-1}^1 + 1$, and $s_i^2 = s_i^1, \forall i \neq j, j - 1$. We have that:

- (i) $IL_i(s_j^2, \dots, s_N^2) \geq IL_i(s_j^1, \dots, s_N^1)$ for $i \in \{|1, j - 1|\}$, and
- (ii) $\sum_{i=1}^j B_{i,i}(s_j^2, \dots, s_N^2) \geq \sum_{i=1}^j B_{i,i}(s_j^1, \dots, s_N^1)$.

PROOF. From (6) and (7) we find that $IL_j(s_j^2, \dots, s_N^2) = IL_j(s_j^1 - 1, s_{j+1}^1, \dots, s_N^1) = IL_j(s_j^1, \dots, s_N^1) - 1$. Thus from (8) and (9), we obtain $B_j(s_j^2, \dots, s_N^2) \leq B_j(s_j^1, \dots, s_N^1) + 1$ and $B_{j,j-1}(s_j^2, \dots, s_N^2) \leq B_{j,j-1}(s_j^1, \dots, s_N^1) + 1$. We can now use this result in (7) to show that:

$$IL_{j-1}(s_{j-1}^2, \dots, s_N^2) = s_{j-1}^2 - B_{j,j-1}(s_j^2, \dots, s_N^2) \geq s_{j-1}^1 + 1 - B_{j,j-1}(s_j^1, \dots, s_N^1) - 1 = IL_{j-1}(s_{j-1}^1, \dots, s_N^1)$$

From $IL_{j-1}(s_{j-1}^2, \dots, s_N^2) \geq IL_{j-1}(s_{j-1}^1, \dots, s_N^1)$ and $s_i^2 = s_i^1, i \in \{|1, j - 2|\}$, we find that $IL_i(s_i^2, \dots, s_N^2) \geq IL_i(s_i^1, \dots, s_N^1)$ for $i \in \{|1, j - 2|\}$, which proves the first result.

For the second result, we can make a sample path comparison. Suppose at time t , we have $B_{j+1,j}(t) < s_j^1$; then there are no backorders for either case: $\sum_{i=1}^j B_{i,i}(t | s_i^2, \dots, s_N^2) = \sum_{i=1}^j B_{i,i}(t | s_i^1, \dots, s_N^1) = 0$. If $B_{j+1,j}(t) \geq s_j^1$, then $B_j(t | s_j^2, \dots, s_N^2) = B_j(t | s_j^1, \dots, s_N^1) + 1$ and either $B_{j,j}(t | s_j^2, \dots, s_N^2) = B_{j,j}(t | s_j^1, \dots, s_N^1) + 1$ or $B_{j,j-1}(t | s_j^2, \dots, s_N^2) = B_{j,j-1}(t | s_j^1, \dots, s_N^1) + 1$. In the former case we have $\sum_{i=1}^j B_{i,i}(t | s_i^2, \dots, s_N^2) = \sum_{i=1}^j B_{i,i}(t | s_i^1, \dots, s_N^1) + 1$; in the latter case we have $\sum_{i=1}^j B_{i,i}(t | s_i^2, \dots, s_N^2) = \sum_{i=1}^j B_{i,i}(t | s_i^1, \dots, s_N^1)$. This proves the result. □

PROPOSITION 2. For all feasible solutions \mathbf{s} for the SLP, we have $\sum_{i=j}^N s_i \geq \sum_{i=j}^N \hat{s}_i$ for all j where $\hat{\mathbf{s}}$ is the solution found by the SPA.

PROOF. Suppose we have a feasible solution \mathbf{s}^1 such that $\sum_{i=j}^N s_i^1 < \sum_{i=j}^N \hat{s}_i$. We will iteratively construct a series of feasible solutions, which leads to a contradiction of the supposition.

In order for \mathbf{s}^1 to be a feasible solution, we must have $s_N^1 \geq \hat{s}_N$; otherwise the fill-rate constraint for class N is violated. If $s_N^1 = \hat{s}_N$ then we must have $s_{N-1}^1 \geq \hat{s}_{N-1}$ by the same logic. If both $s_N^1 = \hat{s}_N, s_{N-1}^1 = \hat{s}_{N-1}$, then we must have $s_{N-2}^1 \geq \hat{s}_{N-2}$ and so on.

Iterative Step: Let k be the largest index such that $s_k^1 > \hat{s}_k$; that is, $s_N^1 = \hat{s}_N, \dots, s_{k+1}^1 = \hat{s}_{k+1}$, and $s_k^1 > \hat{s}_k$. If $k \leq j$, we have a contradiction of the original supposition that $\sum_{i=j}^N s_i^1 < \sum_{i=j}^N \hat{s}_i$. If there does not exist an index k , then we must have $\mathbf{s}^1 = \hat{\mathbf{s}}$, which is also a contradiction of the original supposition. Given $k > j$, then we construct a new solution: $s_k^2 = s_k^1 - 1, s_{k-1}^2 = s_{k-1}^1 + 1$, and $s_i^2 = s_i^1, \forall i \neq k, k - 1$. By application of Proposition 1(i), we can show that this new solution is feasible. Since $k > j$, we have that $\sum_{i=j}^N s_i^1 = \sum_{i=j}^N s_i^2 < \sum_{i=j}^N \hat{s}_i$.

We now use the new solution to repeat the Iterative Step. At each step we move one unit of reserve stock from a lower-priority class to a higher-priority class to create a new feasible solution. The number of possible iterative steps is finite as each unit of reserve stock can be moved at most $N - 1$

times. Therefore, at some step n we have either $s_i^n = \hat{s}_i$ for $i = j, \dots, N$ or $s_j^n > \hat{s}_j$ and $(s_{j+1}^n, \dots, s_N^n) = (\hat{s}_{j+1}, \dots, \hat{s}_N)$, both of which are contradictions of the original supposition. \square

PROPOSITION 3. Consider two stocking policies \mathbf{s}^1 and \mathbf{s}^2 with $\sum_{i=1}^j s_i^1 \leq \sum_{i=1}^j s_i^2 \forall j \in \{1, N-1\}$ and $\sum_{i=1}^N s_i^1 = \sum_{i=1}^N s_i^2$. Then we have $z(\mathbf{s}^1) \leq z(\mathbf{s}^2)$.

PROOF. From (13) and the assumption that $\sum_{i=1}^N s_i^1 = \sum_{i=1}^N s_i^2$, we need to show that

$$\sum_{i=1}^N E[B_{i,i}(s_i^1, \dots, s_N^1)] \leq \sum_{i=1}^N E[B_{i,i}(s_i^2, \dots, s_N^2)].$$

This result follows directly from application of Proposition 1(ii). Starting with the stocking policy \mathbf{s}^2 , we can construct a series of new policies in which we move one unit of reserve stock from class $j-1$ to class j , and eventually reach the stocking policy \mathbf{s}^1 . From Proposition 1(ii), each such move reduces the external backorders in classes $1, 2, \dots, j$, and has no impact on backorders at class $j+1, \dots, N$. Thus we show that $\sum_{i=1}^N B_{i,i}(s_i^1, \dots, s_N^1) \leq \sum_{i=1}^N B_{i,i}(s_i^2, \dots, s_N^2)$, and we get the desired result by taking expectations. \square

Appendix II. Equivalence to Threshold Clearing Mechanism Developed by Deshpande et al. (2003)

In this appendix we show that our model yields the same equations for expected backorders and expected on-hand inventory as presented by Deshpande et al. (2003) for a two-class system.

1. *Distribution of Backorders for Class 2.* Deshpande et al. (2003) derive the following expression for the steady-state distribution of backorders for class 2, where $IP(\infty)$ is uniformly distributed on the range $[r+1, r+Q]$:

$$\Pr[BO_2(\infty) = j \mid IP(\infty) = y] = \sum_{x=(j+y-K)^+}^{\infty} \binom{x-y+K}{j} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^j \cdot \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^{x-y+K-j} \cdot p(x; \lambda\tau).$$

K is the reserve stock level for class 1, i.e., $s_1 = K$ and $p(x, \lambda\tau)$ is the probability a Poisson random variable with mean $\lambda\tau$ equals x , where τ is the replenishment lead-time ($\tau = L$). We will show that our model results in an equivalent expression. From (6)–(9), when $N = 2$, we have that

$$\Pr[B_{2,2} = j \mid IP_2 = k] = \sum_{i=j}^{\infty} \Pr[B_2 = i \mid IP_2 = k] \times \binom{i}{j} p_2^j (1-p_2)^{i-j}$$

where IP_2 is a discrete random variable uniformly distributed on $[s_2 + 1, s_2 + Q]$. We can now express the backorders in terms of demand over the lead-time:

$$\Pr[B_{2,2} = j \mid IP_2 = k] = \sum_{i=j}^{\infty} \Pr[D_1^L + D_2^L = i + k] \times \binom{i}{j} p_2^j (1-p_2)^{i-j}.$$

In our model, $IP = IP_1 + IP_2 = s_1 + IP_2$, and IP is uniformly distributed on the range $[s_1 + s_2 + 1, s_1 + s_2 + Q] = [r+1, r+Q]$. Thus, we can re-express the above as

$$\Pr[B_{2,2} = j \mid IP = k + s_1] = \sum_{i=j}^{\infty} \Pr[D_1^L + D_2^L = i + k] \times \binom{i}{j} p_2^j (1-p_2)^{i-j}.$$

Finally, we substitute $y = k + s_1$ and change indices in the summation to get:

$$\begin{aligned} \Pr[B_{2,2} = j \mid IP = y] &= \sum_{i=j}^{\infty} \Pr[D_1^L + D_2^L = i + y - s_1] \times \binom{i}{j} p_2^j (1-p_2)^{i-j} \\ &= \sum_{x=[j+y-s_1]^+}^{\infty} \Pr[D_1^L + D_2^L = x] \times \binom{x-y+s_1}{j} p_2^j (1-p_2)^{x-y+s_1-j}. \end{aligned}$$

As $s_1 = K$, $L = \tau$ and $\Pr[D_1^L + D_2^L = x] = p(x, \lambda L)$, this shows that $\Pr[B_{2,2} = j \mid IP = y] = \Pr[BO_2(\infty) = j \mid IP(\infty) = y]$ for all $j \in \mathbb{Z}^+ \cup \{0\}$. Hence, the steady-state distribution of class 2 backorders is the same for the FCFS clearing mechanism and the threshold clearing mechanism.

2. *Distribution of Backorders for Class 1.* Deshpande et al. (2003) derive the following expression for the steady-state distribution of backorders for class 1,

$$\Pr(BO_1(\infty) = j \mid IP(\infty) = y) = \sum_{x=y+j}^{\infty} \binom{x-(y-K)}{K+j} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{K+j} \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{x-y-j} \cdot p(x; \lambda\tau), \quad \text{for } j > 0.$$

We will show that our model results in an equivalent expression. From (6)–(9) we have that

$$\Pr[B_{1,1} = j \mid IP_2 = k] = \Pr[B_1 = j \mid IP_2 = k] = \Pr[B_{2,1} = j + s_1 \mid IP_2 = k]$$

where IP_2 is uniformly distributed on the range $[s_2 + 1, s_2 + Q]$. We can now express the backorders in terms of demand over the lead time:

$$\Pr[B_{2,1} = j + s_1 \mid IP_2 = k] = \sum_{i=j+s_1}^{\infty} \Pr[D_1^L + D_2^L = i + k] \times \binom{i}{j+s_1} p_1^{j+s_1} (1-p_1)^{i-j-s_1}.$$

In our model, $IP = IP_1 + IP_2 = s_1 + IP_2$, and IP is uniformly distributed on the range $[s_1 + s_2 + 1, s_1 + s_2 + Q] = [r + 1, r + Q]$. Thus, we can re-express the above as

$$\Pr[B_{2,1} = j + s_1 \mid IP = k + s_1] = \sum_{i=j+s_1}^{\infty} \Pr[D_1^L + D_2^L = i + k] \times \binom{i}{j+s_1} p_1^{j+s_1} (1-p_1)^{i-j-s_1}.$$

Finally, we substitute $y = k + s_1$ and change indices in the summation to get:

$$\begin{aligned} \Pr[B_{2,1} = j + s_1 \mid IP = y] &= \sum_{i=j+s_1}^{\infty} \Pr[D_1^L + D_2^L = i + y - s_1] \times \binom{i}{j+s_1} p_1^{j+s_1} (1-p_1)^{i-j-s_1} \\ &= \sum_{x=j+y}^{\infty} \Pr[D_1^L + D_2^L = x] \times \binom{x-y+s_1}{j+s_1} p_1^{j+s_1} (1-p_1)^{x-y-j}. \end{aligned}$$

As $s_1 = K$, $L = \tau$ and $\Pr[D_1^L + D_2^L = x] = p(x, \lambda L)$, this shows that $\Pr(B_{1,1} = j \mid IP = y) = \Pr(BO_1(\infty) = j \mid IP(\infty) = y)$ for all $j \in \mathbb{Z}^+ \cup \{0\}$. Hence, the steady-state distribution of class 1 backorders is the same for the FCFS clearing mechanism and the threshold clearing mechanism.

3. *Distribution of On-hand Inventory.* Deshpande et al. (2003) derive the following expressions for the steady-state distribution of on-hand inventory,

$$P(OH = j \mid IP(\infty) = y) = \begin{cases} p(y - j; \lambda\tau), & \text{if } j \in \{K, y\}, \\ \sum_{x=y-j}^{\infty} \binom{x-(y-K)}{K-j} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{K-j} \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{x-y+j} \cdot p(x; \lambda\tau), & \text{if } j \in \{0, K-1\}. \end{cases}$$

We will show that our system results in an equivalent expression. From (6)–(9) we consider two cases:

For $j \geq s_1$

$$\Pr[IL_1^+ + IL_2^+ = j \mid IP_2 = k] = \Pr[IL_1^+ = s_1, IL_2^+ = j - s_1 \mid IP_2 = k] = \Pr[D_1^L + D_2^L = k + s_1 - j].$$

For $0 \leq j < s_1$

$$\Pr[IL_1^+ + IL_2^+ = j \mid IP_2 = k] = \Pr[IL_1^+ = j, IL_2^+ = 0 \mid IP_2 = k] = \Pr[B_{2,1} = s_1 - j \mid IP_2 = k]$$

where IP_2 is uniformly distributed on the range $[s_2 + 1, s_2 + Q]$. We can use the results from the prior developments for the backorders to find:

For $j \geq s_1$

$$\Pr[IL_1^+ + IL_2^+ = j \mid IP = y] = \Pr[IL_1^+ = s_1, IL_2^+ = j - s_1 \mid IP = y] = \Pr[D_1^L + D_2^L = y - j].$$

For $0 \leq j < s_1$

$$\begin{aligned} \Pr[IL_1^+ + IL_2^+ = j \mid IP = y] &= \Pr[B_{2,1} = s_1 - j \mid IP = y] \\ &= \sum_{i=s_1-j}^{\infty} \Pr[D_1^L + D_2^L = i + y - s_1] \times \binom{i}{s_1-j} p_1^{s_1-j} (1-p_1)^{i+j-s_1} \\ &= \sum_{x=y-j}^{\infty} \Pr[D_1^L + D_2^L = x] \times \binom{x-y+s_1}{s_1-j} p_1^{s_1-j} (1-p_1)^{x-y+j}. \end{aligned}$$

As $s_1 = K$, $L = \tau$ and $\Pr[D_1^L + D_2^L = x] = p(x, \lambda L)$, this shows that $\Pr(IL_1^+ + IL_2^+ = j \mid IP = y) = \Pr(OH = j \mid IP(\infty) = y)$ for all $j \in [0, y]$. Hence, the steady-state distribution of on-hand inventory level is the same for FCFS clearing mechanism and threshold clearing mechanism developed by Deshpande et al. (2003).

Appendix III. Probability of a Nonpriority Allocation

The purpose of this appendix is to explore the quality of the FCFS allocation rule. The primary concern with the allocation rule is that it might occasionally fill a backorder from a lower-priority demand class before a backorder from a higher-priority demand class. In this section we explore the likelihood of this event. In particular, we examine a system with two demand classes and with order quantity $Q = 1$.

Derivation of Equation to Compute Probability of a Nonpriority Allocation

Suppose a demand occurs at time t . Since $Q = 1$, this triggers a replenishment that arrives at time $t + L$. We ask the following question: what is the likelihood that at the time of replenishment there are backorders for both demand classes and this unit replenishment is used to fill a backorder for demand class 2? We term this event a nonpriority allocation and will develop an equation to compute the probability of this event.

We denote the time just before the arrival of the replenishment as $t + L^-$. We need to consider the possible outcomes for the demand over the replenishment lead time, namely over the interval $(t, t + L)$ where t is chosen to be a demand epoch. We let D_i^τ denote the demand for class i over an interval of length τ ; thus, D_i^τ is a Poisson random variable with mean $\lambda_i \tau$.

For ease of presentation we assume a nonnegative reserve stock for class 2; that is we assume $s_2 = R - s_1 \geq 0$. We do so without great loss of practical value, as $s_2 < 0$ results in a zero fill rate for class 2 demand. Nevertheless, the following development applies for negative values for s_2 , where we decide the allocation of the replenishment triggered at time t by the $|s_2|$ th demand occurrence prior to time t .

We first characterize the probability that there is one or more external backorders for class 2 at time $t + L^-$ and that the replenishment at time $t + L$ is allocated to a class 2 backorder. Suppose $D_1^L + D_2^L < s_2 = R - s_1$. That is, the demand over the lead time is less than the reserve stock for class 2. Then there are no backorders, internal or external, at time $t + L^-$. We use the replenishment to restore the reserve stock for class 2.

Suppose $D_1^L + D_2^L \geq s_2 = R - s_1$. In this case, since there was a demand at time t and since the subsequent demand over the lead time equals or exceeds the reserve stock, there are one or more backorders for class 2 at time $t + L^-$; that is, $B_2(t + L^-) = D_1^L + D_2^L - (s_2 - 1) \geq 1$. These backorders could consist of external backorders for class 2 ($B_{2,2}(t + L^-) > 0$) or internal backorders for replenishing class 1 ($B_{2,1}(t + L^-) > 0$) or a mix of both. The allocation rule assigns the replenishment at time $t + L$ to the oldest external or internal backorder at class 2. The oldest backorder corresponds to the s_2 th demand during the replenishment lead time. [For $s_2 = 0$, the s_2 th demand during the replenishment lead time is the demand at time t .] If the s_2 th demand were from class 1, then we apply the replenishment at time $t + L$ to an internal backorder for class 2 and reduce $B_{2,1}(t + L^-)$ by one. If the s_2 th demand were from class 2, then we apply the replenishment at time $t + L$ to an external backorder for class 2 and reduce $B_{2,2}(t + L^-)$ by one. Thus, the probability that the replenishment at time $t + L$ is used to clear a class 2 backorder equals the probability that there are at least s_2 demands during the lead time and that the s_2 th demand is from class 2:

$$\Pr[D_1^L + D_2^L \geq s_2] \times \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

We now characterize the probability for a nonpriority allocation. That is, we will find the probability that there exists one or more class 1 backorders at time $t + L^-$, there are at least s_2 demands during the lead time and the s_2 th demand is from class 2.

Suppose that there are at least s_2 demands during the lead time with the s_2 th demand being from class 2; suppose that the s_2 th demand occurs at time epoch $t + \tau$ for $0 < \tau < L$.¹ Thus, we have $D_1(t, t + \tau) + D_2(t, t + \tau) = s_2$, where $D_i(s, t)$ denotes external demand for class i over interval $(s, t]$. Furthermore since we assume that the s_2 th demand is from class 2, we can express the backorders at time $t + L^-$ as:

$$\begin{aligned} B_2(t + L^-) &= D_1^L + D_2^L - (s_2 - 1) \\ B_{2,2}(t + L^-) &= 1 + D_2(t + \tau, t + L) \\ B_{2,1}(t + L^-) &= D_1(t + \tau, t + L) \\ B_1(t + L^-) &= [-IL_1(t + L^-)]^+ = [D_1(t + \tau, t + L) - s_1]^+ \end{aligned}$$

In order for there to be a backorder for class 1 at time $t + L^-$, for a given τ , we need $D_1(t + \tau, t + L) > s_1$.

We can now write down the probability for a nonpriority allocation:

$$\int_{\tau=0}^L \Pr[s_2 \text{th demand at time } t + \tau] \times \Pr[s_2 \text{th demand from class 2}] \times \Pr[D_1(t + \tau, t + L) > s_1] d\tau$$

where

$$\begin{aligned} \Pr[s_2 \text{th demand at time } t + \tau] &= \frac{\lambda^{s_2} \tau^{s_2-1}}{(s_2 - 1)!} e^{-\lambda \tau} \quad \text{for } \lambda = \lambda_1 + \lambda_2 \\ \Pr[s_2 \text{th demand from class 2}] &= \frac{\lambda_2}{\lambda_1 + \lambda_2} \\ \Pr[D_1(t + \tau, t + L) > s_1] &= \sum_{i=s_1+1}^{\infty} \frac{e^{-\chi} \chi^i}{i!} \quad \text{for } \chi = \lambda_1 \times (L - \tau). \end{aligned}$$

To compute this probability, we develop a recursive procedure. We define $G(n, m)$ as follows:

$$G(n, m) = \int_{\tau=0}^L \frac{\lambda^n \tau^{n-1}}{(n-1)!} e^{-\lambda \tau} \times \sum_{i=m}^{\infty} \frac{e^{-\lambda_1(L-\tau)} (\lambda_1(L-\tau))^i}{i!} d\tau.$$

Thus, we interpret $G(n, m)$ to be the probability that during the replenishment lead time there are n demands from both classes, followed by at least m additional demands from class 1. We can write the probability of a nonpriority allocation in terms of $G(n, m)$:

$$\frac{\lambda_2}{\lambda_1 + \lambda_2} \times G(s_2, s_1 + 1).$$

By integration by parts, we can show that

$$G(n, m) = \frac{\lambda_1 + \lambda_2}{\lambda_2} \times G(n - 1, m) - \frac{\lambda_1}{\lambda_2} \times G(n, m - 1)$$

with boundary conditions:

$$G(n, 0) = \Pr[D_1^L + D_2^L \geq n] \quad \text{and} \quad G(0, m) = \Pr[D_1^L \geq m].$$

Thus it is quite easy to compute the probability of a nonpriority allocation.

¹ When $s_2 = 0$, the equivalent conditions are that the demand at time t is from class 2, and we set $\tau = 0$.

Numerical Experiment

To test the quality of the allocation rule, we compute the probability of a nonpriority allocation event as we vary the fill-rate targets and the demand rates for a two-demand class system.

For each test problem in the experiment, there are two demand classes, the reorder quantity is $Q = 1$, and replenishment lead-time from the outside supplier is $L = 1/4$ year. There are five possible values for the fill-rate target for each of the demand classes: $\beta_1 = 0.7, 0.8, 0.9, 0.95, 0.99$ and $\beta_2 = 0.6, 0.7, 0.8, 0.9, 0.95$. We only consider combinations with $\beta_1 \geq \beta_2$; thus, we have 19 combinations of fill-rates. Finally, we have eleven possible settings for the demand rates: $\{\lambda_1 = 12, \lambda_2 = 24\}$, $\{\lambda_1 = 24, \lambda_2 = 12\}$, $\{\lambda_1 = 18, \lambda_2 = 18\}$, $\{\lambda_1 = 4, \lambda_2 = 8\}$, $\{\lambda_1 = 8, \lambda_2 = 4\}$, $\{\lambda_1 = 6, \lambda_2 = 6\}$, $\{\lambda_1 = 1, \lambda_2 = 11\}$, $\{\lambda_1 = 11, \lambda_2 = 1\}$, $\{\lambda_1 = 1, \lambda_2 = 3\}$, $\{\lambda_1 = 3, \lambda_2 = 1\}$, and $\{\lambda_1 = 2, \lambda_2 = 2\}$ units/year.

We specify a test problem by setting the number of demand classes (1 candidate), the replenishment lead-time (1 candidate), the reorder quantity (1 candidate), the set of desired fill-rates (19 candidates), and the set of demand rates (11 candidates). This provides a total of 209 test problems.

For each test problem we compute the optimal solution for reserve stocks. Next, we compute the probability of a nonpriority allocation event. From the results of this numerical experiment we conclude that nonpriority allocations can happen, but do not occur very frequently. The maximum value of the probability of a nonpriority allocation in our test problems is 0.073. The probability of a nonpriority allocation is higher than 0.05 in only 2.87% of the cases and less than 0.02 in 74.64% of the cases. Average value of the probability of the nonpriority allocation is 0.0093. Average values of the probability of the nonpriority allocation are shown below:

Pr(nonpriority allocation)	$\{\lambda_1 = 12, \lambda_2 = 24\}$	$\{\lambda_1 = 24, \lambda_2 = 12\}$	$\{\lambda_1 = 18, \lambda_2 = 18\}$
Average	0.0197	0.0158	0.0200

Pr(nonpriority allocation)	$\{\lambda_1 = 4, \lambda_2 = 8\}$	$\{\lambda_1 = 8, \lambda_2 = 4\}$	$\{\lambda_1 = 6, \lambda_2 = 6\}$	$\{\lambda_1 = 1, \lambda_2 = 11\}$	$\{\lambda_1 = 11, \lambda_2 = 1\}$
Average	0.0098	0.0084	0.0101	0.0035	0.0026

Pr(nonpriority allocation)	$\{\lambda_1 = 1, \lambda_2 = 3\}$	$\{\lambda_1 = 3, \lambda_2 = 1\}$	$\{\lambda_1 = 2, \lambda_2 = 2\}$
Average	0.0037	0.0032	0.0048

Pr(nonpriority allocation)	$\beta_1 - \beta_2 = 0$	$0.04 \leq \beta_1 - \beta_2 < 0.15$	$0.15 \leq \beta_1 - \beta_2 < 0.25$	$0.25 \leq \beta_1 - \beta_2 < 0.35$	$0.35 \leq \beta_1 - \beta_2$
Average	0.0157	0.0107	0.0081	0.0040	0.0021
>0.05 (%)	6.81	4.54	0	0	0
<0.02 (%)	70.45	80.30	86.36	100	100
<0.01 (%)	56.81	71.21	77.27	88.63	100

We note that the highest probabilities occur for the case when $\beta_1 = \beta_2$, that is, when we have the same fill rate target for both classes. In this case, there is no reserve stock for class 1, $s_1 = 0$, and we effectively operate as a one-class system. As such, this is not a very realistic case, but we include it as a limiting case to two-class scenarios where there is a slight difference in the fill rate targets for the two classes.

If we were to exclude these test problems, we would have 165 problems with $\beta_1 > \beta_2$. In the following table we report the summary statistics for this reduced set of test problems in comparison with the full set.

Pr(nonpriority allocation)	$\beta_1 > \beta_2$ 165 test problems	$\beta_1 \geq \beta_2$ 209 test problems
Maximum	0.0730	0.0730
Average	0.0075	0.0093
>0.05 (%)	1.8100	2.8700
<0.02 (%)	88.4800	84.6800
<0.01 (%)	79.3900	74.6400

Discussion

In this appendix we show that the likelihood of a nonpriority allocation is quite small for the case we can analyze, a two class system with order quantity of one. The question arises as to how these findings might extend to more complex settings with more demand classes and/or nonunitary lot sizes.

More than two demand classes. We expect that developing a general approach for finding the probability of a nonpriority allocation is quite complex. However, the analysis of the two-class system can be helpful in developing bounds. For instance, one might want to determine the probability of an allocation to a backorder from a specified set of lower priority classes, say classes $\{m+1, \dots, N\}$, when backorders exist in higher priority classes, say classes $\{1, \dots, m\}$. Then one might obtain an upper bound on this probability from the analysis of a two-class system. The lower priority class would have a reserve stock $\sum_{i=m+1}^N s_i$ and demand rate $\sum_{i=m+1}^N \lambda_i$, while the higher priority class has a reserve stock s_m and demand rate $\sum_{i=1}^m \lambda_i$.

General order quantities. In order for there to be a nonpriority allocation, there need to be more than Q external backorders at the time of replenishment. For instance, for a two-class system, one condition for a nonpriority allocation is that $D_1^L + D_2^L > s_2 + Q - 1$. For a given set of fill rate targets and the rationing policy that we find by solving the SLP, we expect that the $\Pr[D_1^L + D_2^L > s_2 + Q - 1]$ decreases with Q . As a consequence, our intuition is that the case with $Q = 1$ provides the largest values for the probability of a nonpriority allocation, all else being equal.

Appendix IV. Relative Fill Rate Performance of Priority Allocation

The purpose of this appendix is to examine the fill rate performance of a priority allocation rule relative to the FCFS allocation rule. We consider the priority allocation rule that first fills all backorders and restores the reserve stock for a higher priority class before allocating any inventory to the backorders for a lower priority class. We provide two results. First we show that the class- N fill rate is the same for both allocation rules. Second we show that the priority allocation rule has a higher fill rate than FCFS allocation for all other classes.

We will argue these results using the mapping of the demand-class system (DCS) to a serial stage system (SSS).

The fill rate for class N is the same for both the FCFS and the priority allocation rule.

We contend that the inventory position at stage N does not depend on the allocation rule. Whenever there is a demand, the inventory position drops by one; a reorder for quantity Q is placed at the demand epoch that would decrease IP_N to the stage- N reserve stock level s_N . As a consequence, the inventory position is always in the range $[s_N + 1, s_N + Q]$; with the assumption of Poisson demand, we know that IP_N is uniformly distributed over the range $[s_N + 1, s_N + Q]$, independent of the allocation rule.

The inventory level for stage N is given by Equation (6):

$$IL_N = IP_N - \sum_{i=1}^N D_i^L$$

where D_i^L is the random variable for the external demand at stage i over an interval of length L ; thus, it represents a Poisson random variable with mean $\lambda_i L$. Again, this expression does not depend on

the allocation rule; by the standard argument, the inventory level is just the inventory position, net of the demand over the lead time. Since the fill rate for class N is given by

$$\text{Fillrate}_N = \Pr(IL_N > 0),$$

we conclude that this is independent of the allocation rule.

The priority allocation rule has a higher fill rate than FCFS allocation for classes 1, 2, ..., N - 1.

To show this result, it is sufficient to show that the on-hand inventory under priority allocation is greater than or equal to the on-hand inventory under the FCFS rule.

Let IOH^P , IOH^F denote the on-hand inventory under priority and FCFS rules. Then the positive part of the inventory level for each stage is given by:

$$IL_j^+(t) = \min[s_j, IOH(t) - c_{j-1}].$$

Thus, if we can show that $IOH^P(t) \geq IOH^F(t)$ then we have shown that for each stage $IL_j^{+P}(t) \geq IL_j^{+F}(t)$. Since the fill rate for each stage is given by

$$\text{Fillrate}_i = \begin{cases} \Pr(IL_i > 0) & \text{if } s_i > 0 \\ \text{Fillrate}_{i+1} & \text{if } s_i = 0 \end{cases} \quad \text{for } i \in \{|1, N-1|\},$$

this assures that the fill rate under priority allocation is greater than or equal to the fill rate under FCFS allocation.

To show that $IOH^P(t) \geq IOH^F(t)$ at any time t , we will establish that the shortfalls for each stage are also ordered, that is, that $SF_i^P(t) \leq SF_i^F(t)$ for each stage i .

Suppose we start at time zero with $SF_i^P(0) \leq SF_i^F(0)$. Then we will show that this ordering is preserved.

Between any two replenishments, the inventory is depleted according to the critical-level control policy, which operates the same for both allocation rules. It is easy to see that $SF_i^P(t) \leq SF_i^F(t)$ is always preserved by the critical-level control policy. Whenever there is demand from class j , the demand increases by one the shortfall at stages $j, j+1, \dots, N-1$. This is regardless of the allocation rule.

Now let us consider what happens when the system receives a replenishment. For the priority rule, the allocation starts with the highest priority class and allocates as much as possible to reduce its shortfall; if we denote the amount allocated to the shortfall at stage k by Q_k , we have for the priority rule:

$$Q_1 = \min[SF_1(t), Q].$$

This allocation reduces the shortfall for class 1 and all lower priority classes:

$$SF_j(t) := SF_j(t) - Q_1 \quad \text{for } j \in \{|1, N-1|\}.$$

The priority allocation process then repeats this allocation with class 2 with the remaining inventory, namely with $Q - Q_1$. The allocation is given by

$$Q_2 = \min[SF_2(t), Q - Q_1].$$

This allocation reduces the shortfall for class 2 and all lower priority classes:

$$SF_j(t) := SF_j(t) - Q_2 \quad \text{for } j \in \{|2, N-1|\}.$$

This process continues until the original replenishment quantity is depleted.

This allocation process, by design, results in the greatest possible reduction to the shortfall for each stage. Any other feasible allocation, including FCFS, cannot result in a greater reduction to the shortfall of any class. As a consequence, if $SF_i^P(t) \leq SF_i^F(t)$ holds just prior to the allocation, then it remains true after the allocation.

Now we will argue that $SF_i^P(t) \leq SF_i^F(t)$ for all stages is sufficient to show that $IOH^P(t) \geq IOH^F(t)$. We do this by contradiction and will consider two cases.

Case 1. Suppose that $IOH^P(t) < IOH^F(t) \leq c_{N-1}$. We define the critical indices kP, kF as

$$c_{kP-1} < IOH^P(t) \leq c_{kP} \quad \text{and} \quad c_{kF-1} < IOH^F(t) \leq c_{kF}$$

where we set the index to zero when $IOH = 0$. Since we suppose $IOH^P(t) < IOH^F(t)$, we have that $kP \leq kF$. First we consider the case when $kP = kF = k$; then from (1) and the critical-level policy, we have

$$SF_k^P(t) = c_k - IOH^P(t) \quad \text{and} \quad SF_k^F(t) = c_k - IOH^F(t),$$

as there are no backorders at stage k . Given the supposition that $IOH^P(t) < IOH^F(t)$, this implies that $SF_k^P(t) > SF_k^F(t)$. Thus we have a contradiction when $kP = kF = k$.

Next we consider the case when $kP < kF$; again from (1) and the critical-level policy, we have

$$SF_{kF}^P(t) = c_{kF} - IOH^P(t) + \sum_{j=kP+1}^{kF} B_{j,j}(t) \quad \text{and} \quad SF_{kF}^F(t) = c_{kF} - IOH^F(t).$$

Given the supposition that $IOH^P(t) < IOH^F(t)$, this implies that $SF_{kF}^P(t) > SF_{kF}^F(t)$. Thus we have a contradiction when $kP < kF$.

Case 2. Suppose that $IOH^P(t) < IOH^F(t)$ and $IOH^F(t) > c_{N-1}$. Since $IOH^F(t) > c_{N-1}$, the system with FCFS allocation has no backorders. In the terms of the SSS, we have

$$\begin{aligned} IL_i^F(t) &= s_i \quad \text{for } i \in \{1, N-1\} \\ IL_N^F(t) &= IOH^F(t) - c_{N-1} > 0 \\ IL^F(t) &= \sum_{i=1}^N IL_i^F(t) = IOH^F(t). \end{aligned}$$

But the inventory level for the system does not depend on the allocation rule; that is, we have $IL^F(t) = IL^P(t)$ for all t . Hence we must have

$$IL^P(t) = IOH^F(t).$$

But for any allocation rule, the inventory level equals the on-hand inventory minus the backorders; that is, for the priority allocation rule we have

$$IL^P(t) = IOH^P(t) - \sum_{j=1}^N B_{j,j}(t).$$

By equating the two above expressions, we see that $IOH^P(t) \geq IOH^F(t)$, which is a contradiction of the supposition that $IOH^P(t) < IOH^F(t)$.

This completes the argument that $IOH^P(t) \geq IOH^F(t)$ which assures that the priority allocation rule has a higher fill rate than FCFS allocation for classes $1, 2, \dots, N-1$.

Reference

See references list in the main paper.