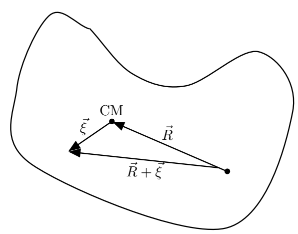
Sam Kendig
6.946j/8.351j/12.620j - Structure and Interpretation of Classical Mechanics
Problem Set 4

- Exercise 2.2: Steiner's Theorem

  Show that $I' = I + MR^2$ where I is the moment of inertia through the center of mass, I' is a moment parallel to the axis of I, M is the total mass, and R is the distance between the two parallel axes:

  Consider the mass shown below:



  $I' = \sum_\alpha m_\alpha (R + \xi_\alpha)^2$
  $I' = \sum_\alpha m_\alpha \xi_\alpha^2 + \sum_\alpha m_\alpha R^2 + \sum_\alpha 2 m_\alpha \xi_\alpha R$
  $I' = I + MR^2 + 2R \sum_\alpha \alpha \xi_\alpha$
  The last term is identically 0, as it is the formula for center of mass, which is 0 at the center of mass.

  Thus, $I' = I + MR^2$.

- 2.3: Moments of Inertia

  a) Solid sphere of uniform density

  $\rho = \frac{M}{\frac{4}{3}\pi R^3}$
  $I = \int_0^R \int_0^{2\pi} \int_0^\pi \rho (r \sin\phi)^2 r^2 \sin\phi \; d\phi d\theta dr$
  $I = \frac{8}{15}\pi \rho R^5 = \frac{2}{5} MR^2$

  b) Spherical shell of uniform density

  $\rho = \frac{M}{4\pi R^2}$
  $I = \int_0^{2\pi} \int_0^\pi \rho (r \sin\phi)^2 r^2 \sin\phi \; d\phi d\theta|_{r=R}$
  $I = \frac{8}{3}\pi \rho R^4 = \frac{2}{3} MR^2$

- 2.4: Jupiter

  a) With the mass concentrated at the center, the average perpendicular distance each mass element is shorter than if the mass were uniformly distributed.

  b) $\rho(r) = \frac{M}{R^3}\frac{\sin(\pi r/R)}{4r/R}$
  $I = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho(r)(r\sin\phi)^2 r^2 \sin\phi \; d\phi d\theta dr$
  $I = (1 - \frac{6}{\pi^2})\frac{2}{3}MR^2$

- 2.5: A constraint on the moments of inertia

$$\mathbf{I} = \begin{bmatrix} \sum_\alpha m_\alpha(\eta_\alpha^2 + \zeta_\alpha^2) & -\sum_\alpha m_\alpha \xi_\alpha \eta_\alpha & -\sum_\alpha m_\alpha \xi_\alpha \zeta_\alpha \\ -\sum_\alpha m_\alpha \eta_\alpha \xi_\alpha & \sum_\alpha m_\alpha(\xi_\alpha^2 + \zeta_\alpha^2) & -\sum_\alpha m_\alpha \eta_\alpha \zeta_\alpha \\ -\sum_\alpha m_\alpha \zeta_\alpha \eta_\alpha & -\sum_\alpha m_\alpha \zeta_\alpha \eta_\alpha & \sum_\alpha m_\alpha(\xi_\alpha^2 + \eta_\alpha^2) \end{bmatrix}$$

  Thus the moments of inertia are $\sum_\alpha m_\alpha(\eta_\alpha^2 + \zeta_\alpha^2), \sum_\alpha m_\alpha(\xi_\alpha^2 + \zeta_\alpha^2), \sum_\alpha m_\alpha(\xi_\alpha^2 + \eta_\alpha^2)$

  Selecting an arbitrary moment, we find that:
  $\sum_\alpha m_\alpha(\eta_\alpha^2 + \zeta_\alpha^2) \leq \sum_\alpha m_\alpha(\xi_\alpha^2 + \zeta_\alpha^2) + \sum_\alpha m_\alpha(\xi_\alpha^2 + \eta_\alpha^2) = \sum_\alpha m_\alpha(\eta_\alpha^2 + \zeta_\alpha^2 + 2\xi_\alpha^2)$

  Since (ξ, η, ζ) are in an arbitrary orthonormal coordinate system, we can extend this to all moments of inertia.

- 2.6: Principle moments of inertia

  a) Consider the tetrahedron centered at the origin with vertices at $(1,1,1), (-1,-1,1), (1,-1,-1)$,and $(-1,1,-1)$ each of mass m:

$$\mathbf{I} = \begin{bmatrix} \sum_\alpha m_\alpha(\eta_\alpha^2 + \zeta_\alpha^2) & -\sum_\alpha m_\alpha \xi_\alpha \eta_\alpha & -\sum_\alpha m_\alpha \xi_\alpha \zeta_\alpha \\ -\sum_\alpha m_\alpha \eta_\alpha \xi_\alpha & \sum_\alpha m_\alpha(\xi_\alpha^2 + \zeta_\alpha^2) & -\sum_\alpha m_\alpha \eta_\alpha \zeta_\alpha \\ -\sum_\alpha m_\alpha \zeta_\alpha \eta_\alpha & -\sum_\alpha m_\alpha \zeta_\alpha \eta_\alpha & \sum_\alpha m_\alpha(\xi_\alpha^2 + \eta_\alpha^2) \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 8m & 0 & 0 \\ 0 & 8m & 0 \\ 0 & 0 & 8m \end{bmatrix}$$

  Since the matrix is diagonal, the principle moments of inertia are: $A = B = C = 8m$

  The principle axes are: $\hat{x}, \hat{y}, \hat{z}$

  b) For a solid cube of side length l and mass M centered about the origin:
  $\rho = \frac{M}{l^3}$

$A = B = C = \int_{-l/2}^{l/2} \int_{-l/2}^{l/2} \int_{-l/2}^{l/2} \rho(x^2 + y^2) dx dy dz$

$A = B = C = \frac{\rho l^5}{6} = \frac{M l^2}{6}$

The principle axes are: $\hat{x}, \hat{y}, \hat{z}$ (from symmetry of the system)

c) The rigid system of 5 point masses of mass m at coordinates:
$(-1, 0, 0), (1, 0, 0), (1, 1, 0), (0, 0, 0), (0, 0, 1)$

The center of mass is at $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

$$\mathbf{I} = \begin{bmatrix} \frac{32}{25}m & \frac{16}{25}m & 0 \\ \frac{16}{25}m & \frac{84}{25}m & 0 \\ 0 & 0 & \frac{84}{25}m \end{bmatrix}$$

Diagonalizing the matrix, we find that the principle moments of inertia are:
$A = 1.0989, B = 3.36, C = 3.5411$

The principle axes are:
$\hat{a} = -0.9622\hat{x} + 0.2723\hat{y}$
$\hat{b} = \hat{z}$
$\hat{c} = 2.723\hat{x} + 0.9622\hat{y}$