Sam Kendig
6.946j/8.351j/12.620j - Structure and Interpretation of Classical Mechanics
Problem Set 6

- Exercise 3.1: Deriving Hamilton's Equations

  a.) $L(t, \theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

  $p_\theta = \partial_2 L(t, q, v) = ml^2\dot{\theta}$
  $\dot{\theta} = \frac{p_\theta}{ml^2}$

  $H(t, \theta, p_\theta) = pv - L(t, q, v) = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$

  b.) $L(t; x, y; \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x, y)$
  $V(x, y) = \frac{x^2+y^2}{2} + x^2y - y^3/3$

  $p_x = m\dot{x}, p_y = m\dot{y}$
  $\dot{x} = \frac{p_x}{m}, \dot{y} = \frac{p_y}{m}$

  $H(t; x, y; p_x, p_y) = \frac{p_x^2+p_y^2}{2m} + V(x, y)$
  $H(t; x, y; p_x, p_y) = \frac{p_x^2+p_y^2}{2m} + \frac{x^2+y^2}{2} + x^2y - y^3/3$

  c.) $L(t; \theta, \varphi; \dot{\theta}, \dot{\varphi}) = \frac{1}{2}mR^2(\dot{\theta}^2 + (\dot{\varphi}\sin\theta)^2)$

  $p = \partial_2 L(t; \theta, \varphi; \dot{\theta}, \dot{\varphi}) = \begin{pmatrix} mR^2\dot{\theta} \\ mR^2\dot{\varphi}\sin^2\theta \end{pmatrix}$
  $p_\theta = mR^2\dot{\theta}$
  $p_\varphi = mR^2\dot{\varphi}\sin^2\theta$
  $\dot{\theta} = \frac{p_\theta}{mR^2}$
  $\dot{\varphi} = \frac{p_\varphi}{mR^2\sin^2\theta}$

  $H(t; \theta, \varphi; p_\theta, p_\varphi) = \frac{1}{2mR^2}(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta})$

- Computing Hamilton's Equations

```
(define ((L-pend m l g) local)
  (let ((theta (coordinate local))
(thetadot (velocity local)))
    (+ (* 1/2 m (square (* l thetadot)))
       (* m g l (cos theta)))))

(se ((Lagrangian->Hamiltonian
      (L-pend 'm 'l 'g))
     (up 't 'theta 'p_theta)))
```

$$-glm\cos(\theta) + \frac{\frac{1}{2}p_\theta^2}{l^2m}$$

```
(define ((L-V m V) local)
  (let ((x (ref local 1 0))
(y (ref local 1 1))
(vel (velocity local)))
    (- (* 1/2 m (square vel)) (V x y))))

(define (V x y)
  (+ (/ (+ (square x) (square y)) 2)
     (* (square x) y)
     (/ (cube y) 3)))

(se ((Lagrangian->Hamiltonian
      (L-V 'm V))
     (up 't (up 'x 'y)
 (down 'p_x 'p_y))))
```

$$x^2y + \frac{1}{3}y^3 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{\frac{1}{2}p_x^2}{m} + \frac{\frac{1}{2}p_y^2}{m}$$

```
(define ((L-sphere m R) local)
  (let ((theta (ref local 1 0))
 (phi (ref local 1 1))
 (thetadot (ref local 2 0))
 (phidot (ref local 2 1)))
    (* 1/2 m (square R)
       (+ (square thetadot)
  (square (* phidot (sin theta)))))))

(se ((Lagrangian->Hamiltonian
      (L-sphere 'm 'R))
     (up 't (up 'theta 'varphi)
 (down 'p_theta 'p_varphi))))
```

$$\frac{\frac{1}{2}p_\theta^2}{R^2m} + \frac{\frac{1}{2}p_\varphi^2}{R^2m\left(\sin(\theta)\right)^2}$$

- Exercise 3.4:

  a.) $F(x) = ax + bx^2$

  $G(y) = xy - F(x)$

  $y = DF(x) = a + 2bx \to x = \frac{y-a}{2b}$

  $G(y) = \frac{y^2-ay}{2b} - \left(a\left(\frac{y-a}{2b}\right) + b\left(\frac{y-a}{2b}\right)^2\right) = \frac{(y-a)^2}{4b}$

  c.) $F(x, y; \dot{x}, \dot{y}) = x\dot{x}^2 + 3\dot{x}\dot{y} + y\dot{y}^2$

  $[p_x, p_y] = \partial_1 F(x, y; \dot{x}, \dot{y}) = (2x\dot{x} + 3\dot{y}, 2y\dot{y} + 3\dot{x})$

  $(\dot{x}, \dot{y}) = \left(\frac{p_x-p_y}{2(x-y)}, \frac{p_y x - p_x y}{3(x-y)}\right)$

  $G(x, y; p_x, p_y) = \frac{p_x(p_x-p_y)}{2(x-y)} + \frac{p_y(p_y x - p_x y)}{3(x-y)} - \left(x\left(\frac{p_x-p_y}{2(x-y)}\right)^2 + 3\left(\frac{p_x-p_y}{2(x-y)}\right)\left(\frac{p_y x p_x y}{3(x-y)}\right) + y\left(\frac{p_y x - p_x y}{3(x-y)}\right)\right)$

- Exercise 3.5:

  Using Hamilton's equations, show directly that the Hamiltonian is a conserved quantity if it has no explicit time dependence.

  $E = H \circ \sigma$
  $DE = D(H \circ \sigma) = \partial_0 H \circ \sigma + \{H, H\} \circ \sigma = \partial_0 H \circ \sigma$

  If $\partial_0 H \circ \sigma = 0$, there is no explicit time dependence, and energy and the hamiltonian are conserved.