Sam Kendig
6.946j/8.351j/12.620j - Structure and Interpretation of Classical Mechanics
Problem Set 8

- 4.1

$H = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$

$D\theta(t) = \frac{p_\theta}{ml^2}$
$Dp_\theta(t) = -mgl\sin\theta$

$$M = \begin{bmatrix} 0 & \frac{1}{ml^2} \\ -mgl\cos\theta & 0 \end{bmatrix}$$

$\lambda = \pm\sqrt{-\frac{g}{l}\cos\theta}$

For the stable fixed point at $\theta = 0$, the eigenvalues are purely imaginary, $\pm\sqrt{\frac{g}{l}}i$.
The magnitude of the eigenvalues is the frequency of the small-amplitude oscillations around this stable fixed point.

For the unstable fixed point at $\theta = \pi$, the eigenvalues are purely real, $\pm\sqrt{\frac{g}{l}}$.
The corresponding eigenvectors are $\left(-\frac{1}{m\sqrt{gl^3}}, 1\right)$ and $\left(\frac{1}{m\sqrt{gl^3}}, 1\right)$. These are the trajectories with which approach and diverge from the fixed point exponentially.

- 4.2: Elliptical oscillation

$\xi_a(n) = e^{An}(u\cos Bn - v\sin Bn)$
$\xi_b(n) = e^{An}(u\sin Bn + v\cos Bn)$

For $A = 0$:

$\xi(n) = \alpha\xi_a(n) + \beta\xi_b(n) = (\alpha u + \beta v)\cos Bn + (\beta u - \alpha v)\sin Bn$

Since u and v are linearly independent vectors, the combinations $\alpha u + \beta v$ and $\beta u - \alpha v$ form a new basis of linearly independent vectors. In this vector space, if B is a rational multiple of π, then this will trace distinct points on an ellipse, otherwise it will trace the full ellipse.

- 4.3: Standard map

$I' = (I + K\sin\theta)\mathrm{mod}2\pi$
$\theta' = (\theta + I')\mathrm{mod}2\pi = (I + \theta + K\sin\theta)\mathrm{mod}2\pi$

$$DT(I,\theta) = \begin{bmatrix} 1 & K\cos\theta \\ 1 & 1 + K\cos\theta \end{bmatrix}$$

$$M = DT(0,0) = \begin{bmatrix} 1 & K \\ 1 & 1 + K \end{bmatrix}$$

$\det(M - \rho I) = 0 = (1-\rho)(1+K-\rho) - K$

$\rho = \frac{1}{2}(K + 2 \pm \sqrt{K^2 + 4K})$

The point $I = 0, \theta = 0$ is unstable for $-4 < K < 0$.

$$M = DT(0,\pi) = \begin{bmatrix} 1 & -K \\ 1 & 1-K \end{bmatrix}$$

$\det(M - \rho I) = 0 = (1-\rho)(1-K-\rho) + K$

$\rho = \frac{1}{2}(K - 2 \pm \sqrt{K^2 - 4K})$

The point $I = 0, \theta = \pi$ is unstable for $0 < K < 4$.

- 4.4: Quartet

  For a given 4-volume around a fixed point, the volume evolves such that it spirals outwards in 2 of the dimensions, and spirals inwards in the other 2 dimensions, conserving the total 4-volume.