Signals and Systems
Spring 2003

Lecture #1

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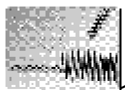
(Slides thanks to A. Willsky, T. Weiss,
Q. Hu, and D. Boning)

1) Administrative details
2) Signals



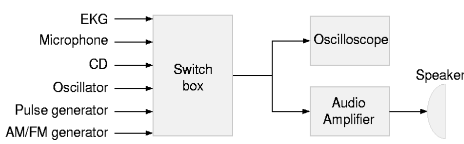
Signals and Systems

6.003 is about using mathematical techniques to help analyze and synthesis systems which process signals.

- Signals are variables that carry information
- Systems process input signals to produce output signals.

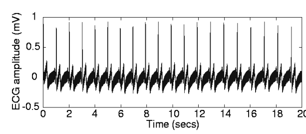
Today: Signals, Next Time: Systems.

Different Types of Signals



Signal Classification
Type of Independent Variable

Time is often the independent variable. Example: the electrical activity of the heart recorded with chest electrodes — the electrocardiogram (ECG or EKG).

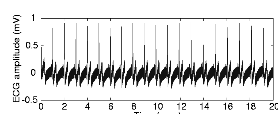


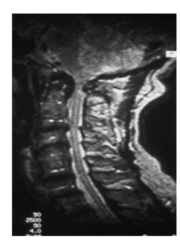
The term *time* is often used generically, to represent the independent variable of a signal. the independent variable may be a spatial variable such as in an image. Here grayscale information is specified as a function of position.

Cervical MRI

Independent Variable Dimensionality

An independent variable can be 1-D (t in the EKG) or 2-D (x,y in the image).

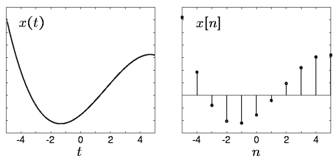




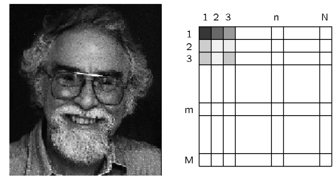
6.003 examples are mostly 1-D, but many applications use multiple dimensions (radar, MRIs, numerical simulation).

Continuous Time (CT) and Discrete-Time (DT) Signals

CT signals take on real or complex values as a function of an independent variable that ranges over the real numbers and are denoted as $x(t)$. DT signals take on real or complex values as a function of an independent variable that ranges over the integers and are denoted as $x[n]$. Note the use of parentheses for CT signals and square brackets for DT signals.



An image example on the left, its DT representation on the right



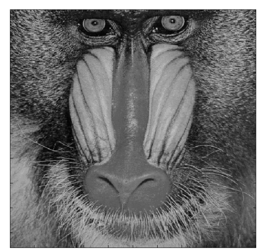
The image on the left consists of 302 × 435 picture elements (pixels) each of which is represented by a triplet of numbers {R,G,B} that encode the color. Thus, the signal is represented by $c[n, m]$ where m and n are the independent variables that specify pixel location and c is a color vector specified by a triplet of hues {R,G,B} (red, green, and blue).

Mandril Example

Blurred Image

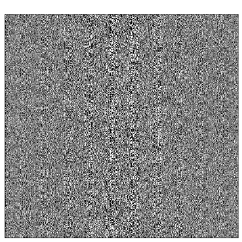
Mandril Example

Unblurred Image – No Noise



Mandril Example

Unblurred Image – 0.1% Noise



Real and Complex Signals

An important class of signals are:

- CT signals of the form $x(t) = e^{st}$
- DT signals of the form $x[n] = z^n$

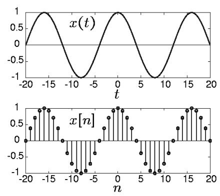
where z and s are complex numbers. For both exponential CT and DT signals, x is a complex quantity and has:

- a real and imaginary part, or
- a magnitude and an angle.

What is most convienient depends on the analysis.

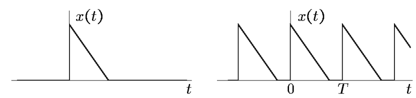
For example, suppose $s = j\pi/8$ and $z = e^{j\pi/8}$, then the real parts are

$$\Re\{x(t)\} = \Re\{e^{j\pi t/8}\} = \cos(\pi t/8),$$
$$\Re\{x[n]\} = \Re\{e^{j\pi n/8}\} = \cos[\pi n/8].$$



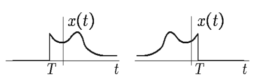
Periodic and A-periodic Signals

Periodic signals are such that $x(t+T) = x(t)$ for all t. The smallest value of T that satisfies the definition is called the *period*. Below on the left below is an aperiodic signal, with a periodic signal shown on the right.

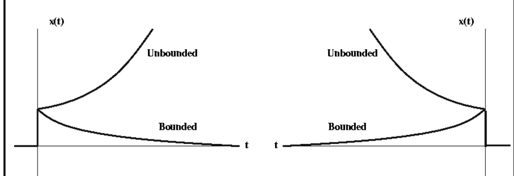


Right- and Left-Sided Signals

A right-sided signal is zero for $t < T$ and a left-sided signal is zero for $t > T$ where T can be positive or negative.



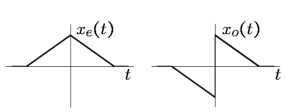
Bounded and Unbounded Signals



Even and Odd Signals

Even signals $x_e(t)$ and odd signals $x_o(t)$ are defined as

$$x_e(t) = x_e(-t) \text{ and } x_o(t) = -x_o(-t).$$

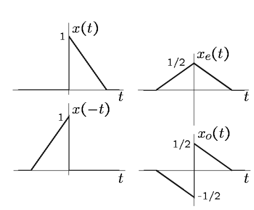


Any signal is a sum of unique odd and even signals. Using

$$x(t) = x_e(t) + x_o(t) \text{ and } x(-t) = x_e(t) - x_o(t),$$

yeilds

$$x_e(t) = \frac{1}{2}(x(t) + x(-t)) \text{ and } x_o(t) = \frac{1}{2}(x(t) - x(-t)).$$



Building Block Signals

Eternal Complex Exponentials

- $x(t) = Xe^{st}$ for all t
- $x[n] = Xz^n$ for all n,

where X, s, and z are complex numbers. We illustrate the richness of this class of functions for CT signals; DT signals are similarly rich. In general s is complex and can be written as

$$s = \sigma + j\omega,$$

where σ and ω are the real and imaginary parts of s.

Eternal, complex exponentials — real s
If $s = \sigma$ is real and X is real then

$$x(t) = Xe^{\sigma t},$$

and we get the family of real exponential functions.
Eternal, complex exponentials — imaginary s
If $s = j\omega$ is imaginary and X is real then

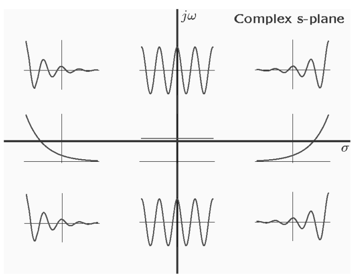
$$x(t) = Xe^{j\omega t} = X(\cos\omega t + j\sin\omega t),$$

and we get the family of sinusoidal functions.
Eternal, complex exponentials — complex s
If $s = \sigma + j\omega$ is complex and X is real then

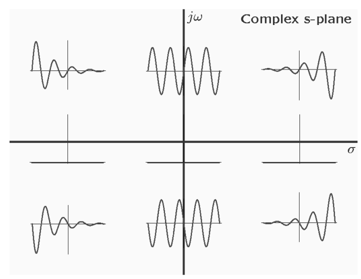
$$x(t) = Xe^{(\sigma + j\omega)t} = Xe^{\sigma t}(\cos\omega t + j\sin\omega t),$$

and we get the family of damped sinusoidal functions.

For $x(t) = Xe^{st}$, $\Re\{x(t)\} = Xe^{\sigma t}\cos\omega t$ is plotted for different values of s superimposed on the complex s-plane.



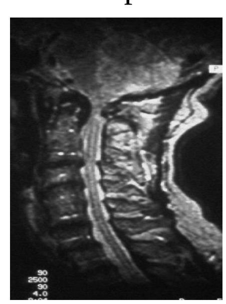
For $x(t) = Xe^{st}$, $\Im\{x(t)\} = Xe^{\sigma t}\sin\omega t$ is plotted for different values of s superimposed on the complex s-plane.



Why are eternal complex exponentials so important

- Almost any signal can be represented as a sums of eternal complex exponentials.
- The output of linear time-invariant (LTI) systems is simple to compute if the inputs are sums of eternal complex exponentials.
- Eternal complex exponentials are the characteristic (unforced, homogeneous) responses of LTI systems (eigenfunctions).

Cervical Spine MRI

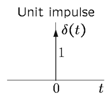


Unit Impulse Function

The unit impulse $\delta(t)$, aka the Dirac delta function, is not a function in the ordinary sense. It is defined by the integral relation

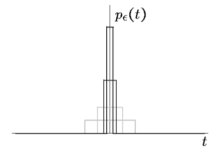
$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0),$$

and is called a *generalized function*. The unit impulse is not defined in terms of its values, but is defined by how it acts inside an integral when multiplied by a smooth function $f(t)$. To see that the area of the unit impulse is 1, choose $f(t) = 1$ in the definition. We represent the unit impulse schematically as shown below; the number next to the impulse is its area.



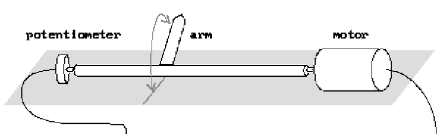
Narrow Pulse Approximation

To obtain an intuitive feeling for the unit impulse, it is often helpful to imagine a set of rectangular pulses where each pulse has width ϵ and height $1/\epsilon$ so that its area is 1.

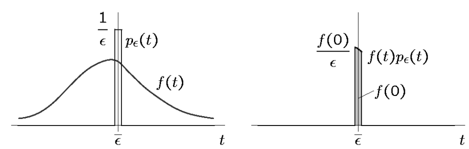


The unit impulse is the quintessential tall and narrow pulse!

Robot Arm System



Intuiting Impulse Definition



As the rectangular pulse gets taller and narrower,

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f(t)p_\epsilon(t)\,dt \to \frac{f(0)}{\epsilon} \cdot \epsilon = f(0).$$

Uses of the Unit Impulse

The unit impulse is a valuable idealization and is used widely in science and engineering. Impulses in time are useful idealizations.

- Impulse of current in time delivers a unit charge instantaneously to a network.
- Impulse of force in time delivers an instantaneous momentum to a mechanical system.

Impulses in space are also useful.

- Impulse of mass density in space represents a point mass.
- Impulse of charge density in space represents a point charge.
- Impulse of light intensity in space represents a point of light.

We can imagine impulses in space and time.

- Impulse of light intensity in space and time represents a brief flash of light at a point in space.

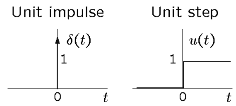
Unit Step Function

Integration of the unit impulse yields the unit step function

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau,$$

which is defined as

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases}$$



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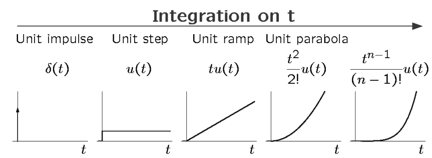
Successive Integrations of the Unit Impulse Function

Successive integration of the unit impulse yields a family of functions.



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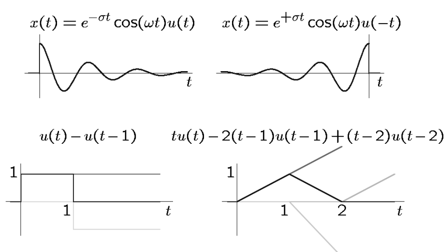
Building Block Signals can be used to create a rich variety of Signals



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Conclusions

- We are awash in a sea of signals.
- Signal categories — identity of independent variable, dimensionality, CT or DT, real or complex, periodic or aperiodic, causality, bounded, even & odd, etc.
- Building block signals — eternal complex exponentials and singularity functions — are a rich class of signals and we will show that they can be summed to represent virtually any signal of physical interest.

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