

Private Risk

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Abstract

We extend the traditional decision analytic approach to calculation of the buying (selling) price of a lottery by allowing a risk averse (risk prone) decision maker to rebalance his financial portfolio in the course of determination of these prices. Building on the classical portfolio allocation problem in complete markets, we generalize the standard treatment to include both traded and non-traded unique risks. Our principal focus is on *private* risks—risks that are not tradable or traded in financial markets. We show that allowing portfolio rebalancing in a distributive bargaining setting with risk averse negotiators expands the zone of possible agreement [ZOPA] relative to the ZOPA yielded when rebalancing is not allowed.

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1 Unique, Private and Market Risks

Among approaches to valuing investment under uncertainty, contingent claim analysis plays a central role. It is the bridge that ties financial markets to valuation of investment projects available to managers: the Law of One Price says that if it is possible to construct a portfolio of financial market securities whose probability distribution over time perfectly mimics the probability distribution of cash flows of a project over time, then the value of the project is the value of the portfolio. In a simpler vernacular, we shall call a portfolio of traded market securities whose returns perfectly mimic the outcome of a lottery, a *market lottery*. If an investment project can be represented as a market lottery and financial markets are in equilibrium, the price of the lottery—hence the value of the project-- is uniquely determined by the Law of One Price. Said differently, if a spanning portfolio exists, *no arbitrage* dictates that the price of the lottery equals the current value of the spanning portfolio.

A *unique or unsystematic risk* is a risk that is uncorrelated with market risk. A consequence is that the market prices a unique risk lottery as bearing no risk premium. Unique risks are, from the market's perspective, *diversifiable* in the sense that such a risk can, in principle, be “atomized” by partitioning it into smaller and smaller components that can be sold individually in the market at closer and closer to a generic individual investor's zero risk level as the magnitude of each atom decreases.

We define a *private risk* to be a risk that may either be correlated with the market or be unique, but has the following additional characteristics:

- It represents a substantial portion of the investor's current wealth,

- It is either not tradable in securities markets or is inhibited from trading by large agency costs.

If a private risk is not tradable, it cannot be diversified away. Then subjective expected utility is an indispensable tool for rational pricing of that risk.

Our aim here is to explain how the owner of a private risk lottery should go about determination of his selling price to a (single) investor who has expressed interest in buying it. There is one buyer and one seller negotiating a selling (buying price) for this lottery and neither buyer nor seller can observe or deduce the price of the lottery from market prices by building a spanning portfolio from market securities. However, both seller and buyer are allowed to rebalance their market security portfolios in the course of determination of their respective buying and selling prices for this private lottery. We shall explain how the Zone of Possible Agreement (ZOPA) is affected by enlarging the choice sets of buyer and seller in this fashion.

2 Literature Review

Lessard and Miller (2001) classify types of risks faced in large engineering projects. They define *residual risks* to be those risks that remain after strategizing to reduce, shift, transform and diversify away identifiable risks.¹ Sponsors of a project who possess a *comparative advantage* in bearing residual risks often embrace them. Their comparative advantage may

“...arise for any one of three reasons: some parties may have more information about particular risks and their impacts than others; some parties or stakeholders may have different degrees of influence over outcomes; or some investors differ in their ability to diversify risks”.²

Residual risks are, in the terminology adopted here, *private risks*. Lessard and Miller give several examples where local partners load-up on these private or residual risks in recognition of their ability to influence outcomes. While possessing a competitive

¹ Lessard and Miller (2000) Ch. 3 Mapping and Facing the Landscape of Risks pp. 87-88.

² Lessard and Miller (2000) Ch. 3 Mapping and Facing the Landscape of Risks pp. 89

advantage relative to private risks, local partners may "...have little ability to diversify risks and little knowledge about commercial prospects worldwide". For example, a Chilean firm Endesa is planning on buying a power-generating plant in Argentina. Endesa has prior experience in privatization and knows more about the future of the Argentine power sector than the local government. "Based on its experience as an operator, Endesa has a clear information and influence advantage when it comes to operating risk".³

Private risks are often private because the investor *chooses* to hold them to exploit a comparative advantage, despite the fact that they may be diversifiable in a market context. To sell successfully such a private risk in the market, he would have to find a mechanism that compensates market participants that do not possess his comparative advantage.

If the consequences of a lottery are uncorrelated with the market (a *unique risk lottery*) then, from the market's perspective this lottery is *diversifiable* and is priced in the market with no risk premium--just take the expected value of lottery consequences at each point in time and discount at the risk-free rate. This is the standard nostrum adopted by financial engineers to price diversifiable risk. Luenberger (1998) defines this valuation procedure as *zero-level pricing*:

"One way to assign a value to such a project is to make believe that the project value is a price, and then set the price so that you would be indifferent between either purchasing a small portion of the project or not. This is called zero-level pricing since you will purchase the project at zero level...If there is only private uncertainty the zero-level price is just the discounted expected value of the project (using actual probabilities)." ⁴

He generalizes zero-level pricing to lotteries (or projects) that have both market and unique risk components. Suppose that the consequences of a lottery *Y* depend on both the

³ Example From Lessard and Miller (2000) pp90.

⁴ Luenberger (1998) p 458

state s of the market and a state e of a unique lottery. Define y_{se} to be the consequence of the joint event $s \cap e$. Define the marginal probability that a market event $s \in S$ occurs to be p_s and the marginal probability that a unique lottery event $e \in E$ occurs to be q_e . Assuming that market events and unique lottery events are probabilistically independent, the lottery Y can be represented symbolically as $L_Y = \{(y_{se}; p_s q_e) \mid s \in S, e \in E\}$. By definition, if there exists a set of market securities that spans all market states then there exists a unique risk-neutral probability π_s for each market state $s \in S$. Zero-level pricing requires that the risk neutral probability π_{se} of state $s \cap e$ satisfy $\pi_{se} = \pi_s q_e$. As we show in later in the paper, for an individual investor-decision maker, π_{se} depends on the investor's utility function and is in general, cannot be represented as a product $\pi_{se} = \pi_s q_e$ of probabilities, so zero-level pricing is a special case.

In a similar vein, Neely (1998)⁵ argues that:

“Simply put, endogenous project uncertainties are not correlated with the external market events. Therefore, the beta of cash-flows that are functions of endogenous uncertainties is zero, and the proper discount rate for evaluating these cash-flows is the risk-free rate”.

As does Luenberger, he applies zero-level pricing to value contingent claims contracts on real assets and real option problems. According to Trigeorgis (1998)⁶,

“...no premium would be required for the part of an asset's risk (i.e., the unique or firm specific risk) that can be diversified away”.

However Trigeorgis does not go further in pricing unique risks and limits his treatment of real options to market risks.

This short discussion of pricing unique risk is, in our view, a reasonable representation of financial economists' approach to valuing unique risk. The key assumption driving these valuation procedures is that project specific risks are uncorrelated with the market portfolio and can be diversified away, so investors do not require a risk premium in pricing these risks. When an investor owns a private risk with consequences that

⁵ Neely (1998) pp79

⁶ Trigeorgis (1998) pp 41

represent a substantial proportion his wealth, we expect his subjective beliefs *and* preferences for risk bearing to come into play. If he is risk averse, we intuitively expect that he would assign a positive risk premium to such a private risk.

Once risks are outside the realm of financial markets, subjective expected utility is the sensible alternative for measuring the value of risk to an individual investor. Any other defined (statistical, mathematical) measure of risk can be justified as an approximation to subjective expected utility evaluation. For the owner of a private risk, subjective expected utility is the analytical glue that binds financial market valuation to private risk valuation.

Luenberger expands his treatment of unique risk to the case in which unique risk cannot be diversified away, in particular when "...the cash outlay required may represent a significant portion of one's investment capital". This is similar to our definition of private risk where the risk is not traded and the investor cannot diversify it away. He proposes a *buying price analysis* for valuing a cash flow lottery of this type. The buying price b of a private risk lottery is the price at which the investor is indifferent between owning the lottery or not. Then the investor's expected utility without the lottery equals his expected utility with the lottery purchased at price b . For an investor who is not risk neutral, this cash flow buying price clearly depends on the investor's risk preferences and probability beliefs about the unique risk component of such a private cash flow lottery. Methods for calculating buying and selling prices in the absence of portfolio rebalancing go back to Raiffa (1968).

Luenberger does a buying price analysis of cash flow lotteries with both market and private risks that differs from a zero-level pricing approach: he assumes that the investor is constantly risk averse; i.e. the investor's utility function for wealth is exponential at each of a discrete set of future time points. He first calculates the certainty equivalent for uncertain cash flow at each discrete point in time and then computes the discounted value of cash flow certainty equivalents. Exponential utility for terminal wealth has the advantage of mathematical tractability: the certainty equivalent for a single stage lottery is functionally independent of initial wealth prior to observation of the outcome of the

lottery. A consequence is that one does not need to address the problem of portfolio rebalancing when the private lottery is added. The price paid is lack of flexibility in capturing the shape of preferences for investors who may be decreasingly or increasingly risk averse as their level of wealth changes.

Similarly, Smith and Nau (1995) propose an integrated valuation procedure for pricing projects under uncertainty with private risks. The investor's subjective probabilities and utility function are used to compute the certainty equivalent for the private risk component of a cash flow at each of a discrete set of points in time. Market risks are priced using complete market risk-neutral probabilities. These authors prove that (a) if the market is complete, then the investor's [firm's] buying and selling prices are the same and (b) if the market is incomplete, then the buying (selling) price for a private risk lottery lies between bounds given by an option pricing analysis.. As they employ exponential utility, they do not need to address the issue of portfolio rebalancing in the course of calculating the buying and the selling price for an uncertain cash flow.

When the investor is faced with a private lottery that is *perfectly correlated* with the market the decision variable is not the price of the lottery – since it is observed in the market – but it is how much of the risky asset to hold. The market prices this lottery. If the market is in equilibrium, there are no arbitrage opportunities and the equilibrium investor has to adjust his holdings of other risky and non-risky assets so as to align his own risk neutral probabilities with the market risk neutral probabilities. The investor can use his personal risk neutral probabilities to price the private lottery and arrive at a higher or lower price than the market quoted price. If so, he will engage in buying or selling of other risky assets in order to create a hedge strategy for his private risk. The investor will continue to buy or sell risky assets until his subjective expected utility prices are in line with market prices.

David Mayers (1973, 1976) derives a pricing model for investors who hold “two kinds of assets, perfectly liquid (marketable) or perfectly non-liquid (non-marketable)”. Labor income is an example of a perfectly non-liquid asset. Mayers constructs and solves this

special portfolio problem. He builds a single period “extended model” of capital asset pricing. He also shows that the composition of the market portfolio varies widely across investors:

“Each investor holds a portfolio of marketable assets that solves his personal and possibly unique portfolio problem”.⁷

A principal difference between Mayers’ analysis and ours is that non-marketable assets are correlated with market securities in Mayers’ analysis and so are not unique risks. Truly private risks, unlike human capital, may be uncorrelated with all market securities. When there is only one market security, Mayers’ analysis implies that adding a private risk to the investor’s portfolio will not affect the composition of his market portfolio (it will only affect the proportion of his total wealth invested in the market portfolio as we show later). As is the case with “modern portfolio theory”, all investors will hold the same market portfolio. When a private risk asset is uncorrelated with market securities, capital asset pricing models cannot be applied to price it. Here we show how to use subjective expected utility to price such private risk assets.

Hoff (1997) develops a valuation approach for uncertain payoffs when markets are incomplete. The basic idea of his research is derived from the field of financial economics especially the application of portfolio optimization and valuation using state contingent securities. Hoff assumes the investor’s utility function is of the form: $U(c_0, w_1) = U(c_0) + U(w_1)$ where c_0 is the consumption in year 0, and w_1 is wealth in year one. $U(w_1)$ is a utility vector over states se . In order to render the relation between our analysis and Hoff’s precise, we re-derive Hoff’s result in our notation in Appendix 2. The main distinction between our analysis and Hoff’s is that we consider only year one wealth without consumption in the previous year. This difference does not change the character of results in the context of valuation of uncertain investments.

From now on, assume the investor’s utility is of the form $U(w_1)$. An investor, endowed with wealth w_0 at $t = 0$, wishes to determine the buying price b of a private risk lottery $Y \equiv \{(y_e; q_e) \mid e \in E\}$. He must decide how to allocate w_0 among N market securities

⁷ Mayers (1973) pp 259-260

given that Y is purchased at b. Hoff argues that the buying price b of Y is determined as follows,

$$b = \boldsymbol{\psi} \cdot \mathbf{y} \quad (1)$$

Here $\boldsymbol{\psi}$ is a vector of risk neutral probabilities defined in the following way:

$$\boldsymbol{\psi} = \frac{1}{r} \frac{\nabla \mathbf{u}}{\nabla \mathbf{u} \cdot \mathbf{1}} \quad (2)$$

with $r \equiv 1 + r_f$, r_f the risk free rate and

$$\nabla \mathbf{u} = [\nabla u_1, \nabla u_2, \dots, \nabla u_e]$$

$$\nabla u_e = \left\{ \begin{array}{ll} \frac{E_{X,Y} [U(\hat{w}_{se})] - E_{X,Y} [U(w_{se})]}{\Delta w_{se} + y_e - b}, & \text{if } (\Delta w_{se} + y_e - b) \neq 0 \\ E[U'(w_{se})], & \text{if } (\Delta w_{se} + y_e - b) = 0 \end{array} \right\} \quad (3)$$

Here w_{se} ($= w_s$) is the period 1 wealth in state se before purchase the private risk, and \hat{w}_{se} is period 1 wealth after the buying (selling) of the lottery and re-optimizing the market portfolio. Δw_{se} is the change in market wealth (proportion of the investor wealth allocation to market securities) in period one due to re-optimization. y_e is the payoff of the private lottery in state e .

Hoff proves that (1)- (3) hold for both complete and incomplete markets. However his definition of an incomplete market is based on two assumptions: (1) The private risk Y is not spanned by market securities and (2) the investor can only trade in the risk free asset – he can not buy the private risk lottery and trade in market securities at the same time. This second assumption is a severe practical limitation as investors can always trade a current market portfolio.

The main disadvantage of this approach is that in order to calculate risk neutral probabilities one must first calculate $\nabla \mathbf{u}$. However one needs to know b in order to calculate $\nabla \mathbf{u}$. If we approximate $\nabla \mathbf{u}_i$ by $E[U'(w_i)]$ then we are *ignoring* the portfolio rebalancing effect. Hoff seems to ignore the portfolio rebalancing in his thesis. By

limiting rebalancing to the risk free security alone, he argues that the portfolio rebalancing effect on buying and selling price is minimal and can be ignored. Had he allowed trading in general when evaluating buying and selling price of private risk, the above analysis may fail as we will show by example in sections 3 and 4.

Our definition of “incomplete markets” differs from that of Hoff’s: a private risk lottery is not spanned by existing market securities and the investor will rebalance his existing market portfolio (not just the risk free asset) as he or she adjusts for the addition or sale of a private lottery. This freedom to rebalance expands the domain of choice and yields buying and selling prices that are decisively different from buying and selling prices in the absence of rebalancing.

3 Private risk and the Investor’s Portfolio Problem

The focus of our development of an expanded theory of buying and selling price for a private lottery is this: if an investor can simultaneously rebalance his market portfolio and buy or sell a private lottery, then both buying and selling prices must account for rebalancing opportunities. This broader perspective leads to new insights. If rebalancing is ignored or not allowed the calculating buying and selling price of a private lottery is well understood (Raiffa 1968). However, when rebalancing is allowed:

- How does the buying (or selling) price of a private lottery change?
- What happens of the ZOPA for buyer and seller?

3.1 The Generic Investor’s Problem

We adopt a formulation of the investor portfolio problem similar to that of Huang and Litzenberger (1988) and Leroy and Werner (2001). These authors provide an exhaustive review of the literature which we do not reproduce here. An investor endowed with wealth w_0 at $t=0$ must decide to allocate w_0 among N market securities. With the exception of one risk free security, values (market price per share) of the remaining $N-1$ securities are

uncertain. If the investor wishes to behave rationally – maximize his subjective expected utility for the wealth w_1 at $t = 1$, how should he behave?

The following standard formulation of his problem will serve as a benchmark for our treatment of private risk. Let $P_i^{(0)}$ be the price of security i at $t = 0$, and set x_{is} equal to the change in value $P_i^{(1)} - P_i^{(0)}$ plus any cash flow or dividend from security i if market state $s \in S = \{s_1, \dots, s_N\}$ obtains at time $t = 1$. If he buys (sells short) α_i shares of security i at $t = 0$, his wealth at $t = 1$ given that states s obtains is

$$w_{1,s} = w_0 + \sum_{i=1}^N \alpha_i x_{is} \quad (4)$$

Define $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$ and

$$\mathbf{X}_{N \times N} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NN} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{bmatrix} = [\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)}] \quad (5)$$

Namely, \mathbf{x}_i is the vector composed of the elements of the i^{th} row of the $(N \times N)$ matrix \mathbf{X} with x_{ij} the payoff or change in value of security i from $t = 0$ to $t = 1$ if the market state $s = j$ obtains and $\mathbf{x}^{(s)}$ is the s^{th} column vector of \mathbf{X} . This terminology allows us to write $w_1(s, \boldsymbol{\alpha})$ at $t = 1$ given $(s, \boldsymbol{\alpha})$ as

$$w_1(s; \boldsymbol{\alpha}) = w_0 + \boldsymbol{\alpha} \mathbf{x}^{(s)} \quad (6)$$

If our investor assigns probability p_s to the state s for each $s \in S$ and possesses a monotone increasing concave utility function U for terminal wealth at $t = 1$, then there exists a unique solution to his investment problem:

$$\max_{\boldsymbol{\alpha}} E_{\mathbf{X}} [U(w_0 + \alpha_1 \mathbf{x}_1 + \cdots + \alpha_N \mathbf{x}_N)] \quad (7)$$

subject to $\boldsymbol{\alpha} \mathbf{P}^{(0)} = w_0$; if no short selling is allowed, $\boldsymbol{\alpha} \geq 0$. Here $\mathbf{P}^{(0)} = (P_1^{(0)}, \dots, P_N^{(0)})^t$.

Because we have chosen \mathbf{X} to be nonsingular, a unique solution $\boldsymbol{\alpha} = \boldsymbol{\alpha}^*$ exists and satisfies the LaGrangian conditions:

$$\frac{\partial L_i}{\partial \alpha_i} = \sum_{s \in S} p_s U'(w_0 + \alpha \mathbf{x}^{(s)}) x_{is} = \lambda P_i^{(0)}, \quad i = 1, 2, \dots, N \quad (8)$$

and $w_0 = \alpha \mathbf{P}^{(0)}$.

These conditions can be recast in terms of risk neutral probabilities as follows:

$$\pi_s \equiv \frac{1}{r} \frac{p_s U'(w_0 + \alpha \mathbf{x}^{(s)})}{\sum_{s \in S} p_s U'(w_0 + \alpha \mathbf{x}^{(s)})}, \quad s = 1, \dots, N. \quad (9)$$

Summing over states,

$$\sum_{s=1}^N \pi_s = \frac{1}{1+r_f} \equiv \frac{1}{r} \quad (10)$$

with $\boldsymbol{\pi}^{(s)} \equiv (\pi_1, \dots, \pi_s, \dots, \pi_N)^t$ and at $\alpha = \alpha^*$:

$$\mathbf{X} \boldsymbol{\pi}^{(s)} = \mathbf{P}^{(o)} \quad \text{or} \quad \boldsymbol{\pi}^{(s)} = \mathbf{X}^{-1} \mathbf{P}^{(0)}, \quad (11)$$

provided that \mathbf{X} is nonsingular (complete markets).

3.2 Private Risk

Suppose that this investor possess the opportunity to buy a *private risk* that will unfold on (a) a domain of mutually exclusive and collectively exhaustive events $E \equiv \{e_1, \dots, e_M\}$ and possibly on both E and S at once. Which particular event will obtain in E is uncertain, so our investor assigns marginal probabilities q_j to the event “ e_j obtains at the time $t = 1$ ”, $j = 1, 2, \dots, M$., and a joint probability q_{sj} to each $(s, e) \in S \times E$. Only our investor observes $e \in E$, at time $t = 1$. If we label a random variable or uncertain quantity (rv or uq) “private” we mean that this rv or uq is *probabilistically independent* of the uncertain event “state of the market at $t + 1$ ”. We suppose that the investor receives a payoff from ownership of the private risk lottery Y . The rv Y has domain $\{s \cap e \mid s \in S, e \in E\} \equiv S \times E$ and range $\mathfrak{R} = (-\infty, \infty)$ or some measurable subset of \mathfrak{R} . When market and private risk events are independent, Y is a lottery $\{(y_{se}; p_s q_e) \mid s \in S, e \in E\}$.

In traditional discussions of portfolio optimization, Ingersoll (1987) Huang and Litzenberger (1993) LeRoy and Werner (2000) assume that all risks are market risks or

that alternatively that markets are totally incomplete (all risks are private risks). We construct a portfolio problem in which the investor can buy or sell private risk and simultaneously rebalance his market portfolio.

3.3 Buying and Selling Price

The buying price of a private lottery is the maximum amount the investor is willing to pay given his current wealth allocation. In other word, it is the value that makes him indifferent between buying the lottery and the status quo Let b_{max} be the buying price for the lottery Y :

At the status quo, the investor's expected utility is,

$$\bar{U}_0 \equiv E_x \left[U \left(w_0 + \boldsymbol{\alpha}^* \mathbf{x}^{(s)} \right) \right] \quad (12)$$

For a fixed known buying price b of a private lottery Y , the investor will reallocate his portfolio to conform to that price. Namely, he will maximize his portfolio selection $\boldsymbol{\alpha}$ for given Y and b :

$$\begin{aligned} \max_{\boldsymbol{\alpha}} E_{x, Y} \left[U \left(w_0 + \boldsymbol{\alpha} \mathbf{x}^{(s)} + Y - b \right) \right] \\ \text{s.t. } \boldsymbol{\alpha} \mathbf{P}^{(0)} + b = w_0 \end{aligned} \quad (13)$$

At the optimum ($\boldsymbol{\alpha}^b$) expected utility is

$$\bar{U}(b) \equiv E_{x, Y} \left[U \left(w_0 + \boldsymbol{\alpha}^b \mathbf{x}^{(s)} + Y - b \right) \right] \quad (14)$$

The **buying price** b_{max} is then defined to be

$$b \quad \text{s.t.} \quad \bar{U}(b) = \bar{U}_0 \quad (15)$$

Similarly the selling price of a lottery is the minimum price the investor is willing to accept in exchange for his lottery. Let s_{min} be the selling price of the lottery Y . Note in this case, the status quo, the investor owns the private lottery Y .

At the status quo, if the investor owns Y , his expected utility is,

$$\begin{aligned} \max_{\boldsymbol{\alpha}} E_{x, Y} \left[U \left(w_0 + \boldsymbol{\alpha} \mathbf{x}^{(s)} + Y \right) \right] \\ \text{s.t. } \boldsymbol{\alpha} \mathbf{P}^{(0)} = w_0 \end{aligned} \quad (16)$$

Define

$$\bar{U}_Y \equiv E_{X,Y} \left[U \left(w_0 + \boldsymbol{\alpha}^M \mathbf{x}^{(s)} + Y \right) \right] \quad (17)$$

For a fixed known selling price s , the investor will reallocate his portfolio to conform to that price:

$$\begin{aligned} \max_{\boldsymbol{\alpha}} E_X \left[U \left(w_0 + \boldsymbol{\alpha} \mathbf{x}^{(s)} + s \right) \right] \\ \text{s.t. } \boldsymbol{\alpha} \mathbf{P}^{(0)} - s = w_0 \end{aligned} \quad (18)$$

At the optimum portfolio rebalance $\boldsymbol{\alpha}^s$, given selling price s , the investor's expected utility is

$$\bar{U}(s) \equiv E_X \left[U \left(w_0 + \boldsymbol{\alpha}^s \mathbf{x}^{(s)} + s \right) \right] \quad (19)$$

The **selling price** s_{min} for Y is defined to be

$$s \quad \text{s.t.} \quad \bar{U}(s) = \bar{U}_Y \quad (20)$$

3.4 Buying and Selling Price Effects on Expected Utility

If our investor added to his holdings Y at cost b , then his expected utility is

$$E_{X,Y} \left[U \left(w_0 + \boldsymbol{\alpha} \mathbf{x}^{(s)} + Y - b \right) \right] \quad (21)$$

Where $\boldsymbol{\alpha}$ satisfies $w_0 = \boldsymbol{\alpha} \mathbf{P}^{(0)} + b$. That is, his wealth allocate to market securities is diminished by b . If he buys Y , he will re-allocate $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha}^*(b)$ to satisfy

$$\frac{\partial L_i}{\partial \alpha_i} = \sum_{s \in S} \sum_{e \in E} p_s q_e U' \left(w_0 + \boldsymbol{\alpha} \mathbf{x}^{(s)} + y_{se} - b \right) x_{is} = \lambda P_i^{(0)}, \quad i = 1, 2, \dots, N \quad (22)$$

and $w_0 = \boldsymbol{\alpha} \mathbf{P}^{(0)} + b$.

The investor's expected utility at the optimal portfolio becomes:

$$E_{X,Y} \left[U \left(w_0 + \boldsymbol{\alpha}^*(b) \mathbf{x}^{(s)} + Y - b \right) \right] \quad (23)$$

We assume *henceforth* that U is monotone increasing with increasing argument.

Proposition 1: For a risk averse investor, equation (23) is monotonic decreasing in b . That is, if portfolio rebalancing is allowed and the investor rebalancing optimally for each possible buying price b , expected utility is monotone decreasing with increasing b .

If our investor **shorts** his holdings Y at ‘price’ b , then his expected utility is

$$E_{X,Y} = \left[U \left(w_0 + \boldsymbol{\alpha} \mathbf{x}^{(s)} - Y + b \right) \right] \quad (24)$$

with $\boldsymbol{\alpha}$ satisfying $w_0 + b = \boldsymbol{\alpha} \mathbf{P}^{(0)}$. That is, his wealth allocation to market securities is increased by b in return for exposure to $-Y$. If he shorts Y , he will re-allocate $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}^*(b)$ to satisfy

$$\frac{\partial L_i}{\partial \alpha_i} = \sum_{s \in S} \sum_{e \in E} p_s q_e U' (w_0 + \boldsymbol{\alpha} \mathbf{x}^{(s)} - y_{se} + b) x_{is} = \lambda P_i^{(0)}, \quad i = 1, 2, \dots, N \quad (25)$$

and

$$w_0 = \boldsymbol{\alpha} \mathbf{P}^{(0)} - b.$$

The investor’s expected utility at the optimal portfolio becomes:

$$E_{X,Y} \left[U \left(w_0 + \boldsymbol{\alpha}^*(b) \mathbf{x}^{(s)} - Y + b \right) \right] \quad (26)$$

Proposition 2: For a risk averse investor, equation (26) is monotonic increasing in b . That is, if portfolio rebalancing is allowed and the investor rebalancing optimally for each possible selling price b , expected utility is monotone increasing with increasing b .

Similarly, If our investor **sold** his holdings Y at the ‘price’ b , then his expected utility is

$$E_{X,Y} = \left[U \left(w_0 + \boldsymbol{\alpha}^* \mathbf{x}^{(s)} + b \right) \right] \quad (27)$$

and we can prove in the same fashion that the above equation is monotone increasing in b .

3.5 Risk Neutral Interpretation

If our investor adds Y to his holdings at the cost b , then his expected utility is

$$E_{X,Y} = \left[U \left(w_0 + \sum_{i=1}^N \alpha_i \mathbf{x}_i + Y - b \right) \right] \quad (28)$$

where $\boldsymbol{\alpha}$ satisfies $w_0 = \boldsymbol{\alpha} \mathbf{P}^{(0)} + b$. That is, his market security wealth allocation is diminished by b . If he buys Y , he will re-allocate $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}^*$ to satisfy

$$\frac{\partial L_i}{\partial \alpha_i} = \sum_{s \in S} \sum_{e \in E} p_s q_e U'(w_0 + \alpha \mathbf{x}^{(s)} + y_{se} - b) x_{is} = \lambda P_i^{(0)}, \quad i = 1, 2, \dots, N \quad (29)$$

and $w_0 = \alpha \mathbf{P}^{(0)} + b. \quad (30)$

The above system has a unique solution $\alpha = \alpha^{**}$ (N+1 unknowns and N+1 equations). Because we have introduced a risk free market asset

$$\lambda = \sum_{s \in S} \sum_{e \in E} p_s q_e U'(w_0 + \alpha \mathbf{x}^{(s)} + y_{se} - b) \quad (31)$$

we can calculate, at $\alpha = \alpha^{**}$, risk neutral probabilities

$$\pi_{es} = \frac{1}{r} \frac{p_s q_e U'(w_0 + \alpha \mathbf{x}^{(s)} + y_{se} - b)}{\sum_{s \in S} \sum_{e \in E} p_s q_e U'(w_0 + \alpha \mathbf{x}^{(s)} + y_{se} - b)} \quad (32)$$

In terms of the NM risk neutral probabilities $(\pi_{es})^8$, we have

$$\sum_{s \in S} \sum_{e \in E} \pi_{es} x_{is} = P_i^{(0)}, \quad i = 1, 2, \dots, N \quad (33)$$

and

$$\sum_{s \in S} \sum_{e \in E} \pi_{es} y_{es} = b \quad (34)$$

As x_{is} and e are functionally independent, upon setting $\pi_s = \sum_{e \in E} \pi_{es}$,

$$\sum_{s \in S} \pi_s x_{is} = P_i^{(0)} \quad (35)$$

and we recover the conditions of the complete market case.

Now decompose π_{es} into π_s and $\pi_{e|s} \equiv$ the risk neutral probability that e obtains conditional upon market state s . Define

$$\sum_{e \in E} \pi_{e|s} y_{es} = \bar{y}_s \quad (36)$$

⁸ If we relax the assumption that events in E and S are *probabilistically independent*. Equation (34) still hold with π_{se} defined as follows:

$$\pi_{es} = \frac{1}{r} \frac{p_{se} U'(w_0 + \alpha \mathbf{x}^{(s)} + y_{se} - b)}{\sum_{s \in S} \sum_{e \in E} p_{se} U'(w_0 + \alpha \mathbf{x}^{(s)} + y_{se} - b)}$$

Where p_{se} is the probability of the joint event $s \cap e$.

a risk neutral conditional expectation of payoff from Y given market state s . then (34) can be represented as

$$\sum_{s \in S} \pi_s \bar{y}_s = b \quad (37)$$

Setting $\bar{\mathbf{y}} = (\bar{y}_1, \dots, \bar{y}_N)$, optimality conditions become

$$\begin{bmatrix} \mathbf{X} \\ \bar{\mathbf{y}} \end{bmatrix} \boldsymbol{\pi}^{(s)} = \begin{bmatrix} \mathbf{P}^{(0)} \\ b \end{bmatrix}. \quad (38)$$

Because we have chose \mathbf{X} to be $(N \times N)$ and non-singular $\boldsymbol{\pi}^{(s)} = \mathbf{X}^{-1} \mathbf{P}^{(0)}$ as in the absence of Y – the market is complete – but we have something new:

Proposition 3: If the investor can purchase the private risk lottery Y at cost b , then

(1) There exist NM risk-neutral probabilities

$$\boldsymbol{\Pi} \equiv [(\pi_{se})]$$

$(N \times M)$

for which $\sum_{e \in E} \pi_{es} = \pi_s$, $s \in S$ and $\sum_{s \in S} \pi_{es} = \pi_e$, $e \in E$,

and satisfies:

$$\sum_{s \in S} \sum_{e \in E} \pi_{es} y_{es} = b$$

(2) At the optimal choice $\boldsymbol{\alpha}^{**}$ for *this* investor, Y is equivalent to a unique market portfolio represented by $\boldsymbol{\theta} (N \times 1)$ satisfying:

$$\boldsymbol{\theta} \mathbf{X} = \bar{\mathbf{y}} \quad \text{and} \quad \boldsymbol{\theta} \mathbf{P}^{(0)} = b$$

where $\bar{\mathbf{y}}$ is a risk neutral conditional expectation of payoff from Y .

Here the private risk lottery can be reinterpreted as a quasi-market security: fix the state s and calculate the conditional expectation of Y (over private lottery outcomes e given a fixed market state s). This conditional expectation is then weighted by market risk neutral probabilities π_s . Hence it behaves like a quasi-security with market value "Expectation of Y given s " when market state s obtains

3.6 Illustration

An investor considers investing in market securities for one time period. There are two traded securities, a risk-free security and a risky security. Possible outcomes of returns are described in the table below:

	<i>Probability</i>	<i>Payoff</i>	<i>Return</i>
Risky Security	0.3	3	200%
	0.4	1	0
	0.3	0.5	-50%
Risk free	1	1.05	5%

Best portfolio allocation in the status quo: The investor problem is to select an optimal portfolio allocation between the risky security α_1 and the risk free α_2 . Assume the investor's utility, for market returns, is $U(x) = \ln(x)$, and no short selling is allowed. If the investor's initial wealth is $w_0 = 10.0$, he wishes to choose α_1 and α_2 to achieve

$$\begin{aligned} & \max_{\alpha_1, \alpha_2} E[\ln(w_0 + \alpha_1 x_1 + \alpha_2 x_2)] \\ \text{s.t.} \quad & \alpha_1 P_1 + \alpha_2 P_2 = w_0 . \end{aligned}$$

Given the numerical specifications of returns in the table above, his portfolio allocation problem is

$$\begin{aligned} & \max_{\alpha_1, \alpha_2} [0.3 \ln(10 + 2\alpha_1 + 0.05\alpha_2) + 0.4 \ln(10 + 0.05\alpha_2) + 0.3 \ln(10 - 0.5\alpha_1 + 0.05\alpha_2)] \\ \text{s.t.} \quad & \alpha_1 + \alpha_2 = 10 . \end{aligned}$$

The optimal allocation α^* is $\alpha_1 = 6.364$, $\alpha_2 = 3.636$. In other words, the investor should invest 63.64% of his wealth in the risky security. At the optimal solution expected utility is

$$\bar{U}_0 \equiv E_x \left[U \left(w_0 + \alpha^* \mathbf{x}^{(s)} \right) \right] = E \left[\ln \left(w_0 + \alpha_1^* x_1 + \alpha_2^* x_2 \right) \right].$$

The certainty equivalent of the optimal portfolio is $\bar{U}_0 = 2.4515$

Buying price: Assume the investor is considering investing in a venture whose payoffs are probabilistically independent of market returns – a private (unique) risk. This venture payoffs are 4 with probability 0.5 and 1 with probability 0.5.

For a fixed known cost b (investment in the venture) the investor wishes to choose α^* to satisfy

$$\begin{aligned} \max_{\alpha_1, \alpha_2} E_{x, Y} \left[U \left(w_0 + \alpha_1 x_1 + \alpha_2 x_2 + Y - b \right) \right] \\ \text{s.t. } \alpha_1 P_1 + \alpha_2 P_2 + b = w_0 \end{aligned}$$

At the optimum, $(\alpha_1^b + \alpha_2^b)$, expected utility is

$$\bar{U}(b) \equiv E_{x, Y} \left[\ln \left(w_0 + \alpha_1^b x_1 + \alpha_2^b x_2 + Y - b \right) \right] =$$

$$\begin{aligned} 0.5 * \left[0.3 \ln(10 + 2\alpha_1^b + 0.05\alpha_2^b + 4 - b) + 0.4 \ln(10 + 0.05\alpha_2^b + 4 - b) + 0.3 \ln(10 - 0.5\alpha_1^b + 0.05\alpha_2^b + 4 - b) \right] \\ + 0.5 * \left[0.3 \ln(10 + 2\alpha_1^b + 0.05\alpha_2^b + 1 - b) + 0.4 \ln(10 + 0.05\alpha_2^b + 1 - b) + 0.3 \ln(10 - 0.5\alpha_1^b + 0.05\alpha_2^b + 1 - b) \right] \end{aligned}$$

In order to find the buying price of this private lottery - the maximum price (b_{\max}) that the investor is willing to pay for it if portfolio rebalancing is allowed - is the value of b for which the investor is indifferent between investing in the venture and the status quo.



Figure 1 - Expected utility is a decreasing function of the venture buying price

Here b_{max} is defined as the solution to

$$\begin{aligned} \max \quad & b \\ \text{s.t.} \quad & \bar{U}(b) = \bar{U}_0 = 2.4515 \\ & \alpha_1 + \alpha_2 + b = 10 \end{aligned}$$

The buying price of the private lottery Y is $\mathbf{b}_{max} = 2.2636$ and the optimal portfolio rebalancing is $\alpha_1 = 6.1950$, $\alpha_2 = 1.5414$.

Best portfolio allocation if the investor owns the lottery: suppose that the investor owns this venture and is considering selling it. His expected utility of rebalancing is allowed is

$$\begin{aligned} \max_{\alpha_1, \alpha_2} \quad & E_{X,Y} \left[\ln(w_0 + \alpha_1 x_1 + \alpha_2 x_2 + Y) \right] \\ \text{s.t.} \quad & \alpha_1 P_1 + \alpha_2 P_2 = w_0. \end{aligned}$$

We find $\alpha_1 = 7.6785$, $\alpha_2 = 2.3215$. At optimal rebalancing expected utility is

$$\bar{U}_M \equiv E_{X,Y} \left[U(w_0 + \alpha^M x_1 + \alpha^M x_2 + Y) \right] = E \left[\ln(w_0 + \alpha_1^M x_1 + \alpha_2^M x_2 + Y) \right] = 2.6573$$

Selling price: for a fixed known selling price s the problem is

$$\begin{aligned} \max_{\alpha_1, \alpha_2} \quad & E_x \left[U(w_0 + \alpha_1 x_1 + \alpha_2 x_2 + s) \right] \\ \text{s.t.} \quad & \alpha_1 P_1 + \alpha_2 P_2 - s = w_0 \end{aligned}$$

At the optimum $(\alpha_1^s + \alpha_2^s)$, expected utility is

$$\bar{U}(s) \equiv E \left[\ln(w_0 + \alpha_1^b x_1 + \alpha_2^b x_2 + s) \right]$$



Figure 2 - Expected utility is an increasing function of the selling price

The selling price, the minimum he should accept with portfolio rebalancing is

$$\begin{aligned}
 & \min \quad s \\
 & \text{s.t.} \quad \bar{U}(s) = \bar{U}_M = 2.6573 \\
 & \quad \quad \alpha_1 + \alpha_2 - s = 10
 \end{aligned}$$

The results are $\alpha_1 = 7.8177$, $\alpha_2 = 4.4673$ and $s_{\min} = 2.2849$.

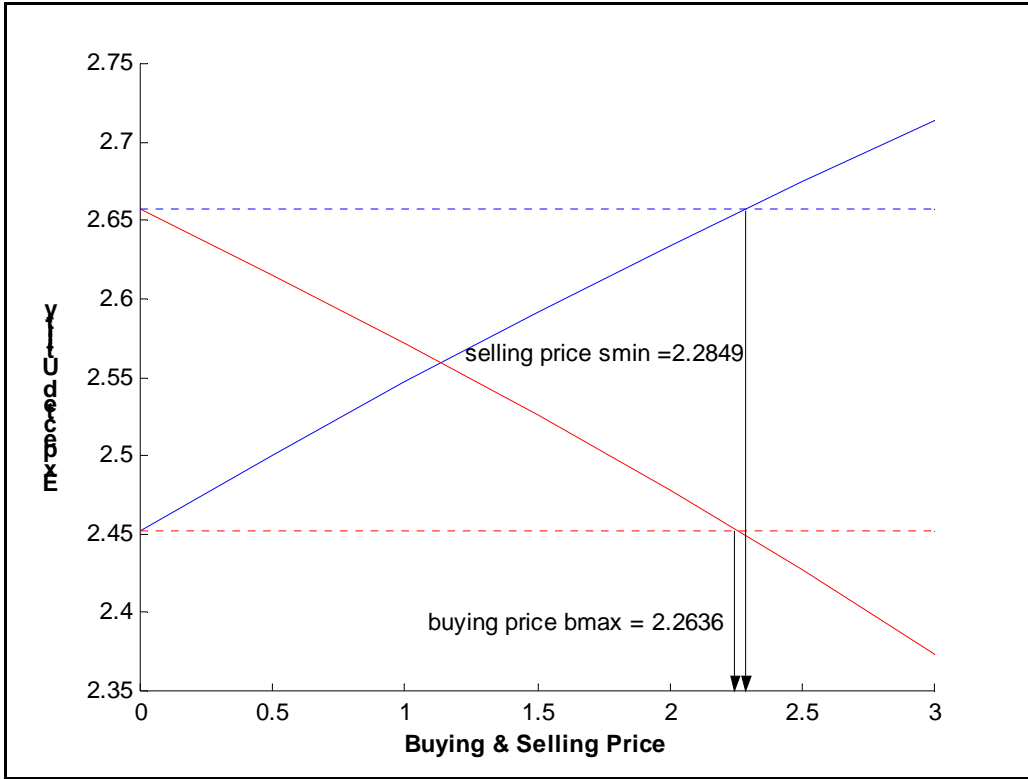


Figure 3 - Selling and buying price

4 Portfolio Rebalancing Effects

4.1 Valuation of Private Risk

The value of a private risk lottery is its buying price (selling price) i.e. the maximum an investor will pay for Y, equal to a price such that the investor is indifferent between the status quo and purchasing Y at that price. Any price below the maximum buying price makes the investor better off; i.e. the expected utility of purchasing Y is larger. (Proposition 1: For a (non-constantly) risk averse investor, expected utility of Y with portfolio rebalancing decreases with increasing buying price). Failing to rebalance can lead to a sub-optimal outcome. For a risk averse investor we have:

Proposition 4: For a (non-constantly) risk averse investor, the buying price with portfolio rebalancing is larger than the buying price without rebalancing.

Proposition 5: For a (non-constantly) risk averse investor, the minimum selling price with portfolio rebalancing is smaller than the minimum selling price without rebalancing.

4.2 Illustration II

An investor is considering investing in the market for one period.. There exist only two traded securities, the risk-free security and a risky security. Possible outcomes are shown below:

	<i>Probability</i>	<i>Payoff</i>	<i>Return</i>
Risky Security	0.3	3	200%
	0.4	1	0
	0.3	0.5	-50%
Risk free	1	1.05	5%

Best portfolio allocation in the status quo: The investor problem is to select the optimal portfolio allocation between the risky and the risk-free security. The investor's utility is $U(x) = \ln(x)$, and no short selling is allowed. Assume $w_0 = 10$. Let α be the proportion invested the risky security (with return r). The investor's best allocation satisfies

$$\max_{\alpha} E(U(w_1)) = \max_{\alpha} E\left[\ln\left(w_0\left(r_f + \alpha(r - r_f)\right)\right)\right]$$

or

$$\max_{\alpha} \left[0.3\ln(10(1.05 + \alpha(1.95))) + 0.4\ln(10(1.05 + \alpha(-0.05))) + 0.3\ln(10(1.05 + \alpha(-0.55)))\right],$$

yielding $\alpha_0 = 0.6364$. In other words, the investor should invest 63.64% of his wealth in the risky security. At the optimal solution his expected utility equals $\bar{U}_0 = 2.4515$

Buying price: Assume that the investor considers investing in a unique risk venture; i.e. the outcome Y this investment entails is uncorrelated with the market. Payoffs are 4 with probability 0.5, and 1 with probability 0.5. For a fixed known cost of purchase b the problem the investor must solve is

$$\bar{U}(b) = \max_{\alpha} E\left[\ln\left((w_0 - b)\left(r_f + \alpha(r - r_f)\right)\right) + Y\right].$$

This investor's buying price b_{max} is

$$\begin{aligned} \max \quad & b \\ \text{s.t.} \quad & \bar{U}(b) = \bar{U}_0 = 2.4515 \end{aligned}$$

The solution is $\alpha_b = 0.8008$ and $b_{max} = 2.2636$

Best portfolio allocation if the investor owns the lottery: If the investor already owns this venture and is considering selling it, the initial state of his portfolio is determined by

$$\bar{U}_M = \max_{\alpha} E\left[\ln\left((w_0)\left(r_f + \alpha(r - r_f)\right)\right) + Y\right].$$

The solution is $\alpha_M = 0.7679$. In other words, the investor should invest 76.85% of his wealth in the risky security for expected utility $\bar{U}_M = 2.6573$

Selling price: For a fixed known selling cost s the problem is to find α satisfying

$$\bar{U}(s) = \max_{\alpha} E \left[\ln \left((w_0 + s) (r_f + \alpha (r - r_f)) \right) \right]$$

The minimum selling price he should accept is

$$\begin{aligned} \min \quad & s \\ \text{s.t.} \quad & \bar{U}(s) = \bar{U}_M = 2.6573 \end{aligned}$$

The solution is $\alpha_s = 0.6364$ $s_{\min} = 2.2849$.

Without rebalancing: If the investor does not rebalance his portfolio but adds the private lottery, the buying price is

$$\begin{aligned} \max \quad & b \\ \text{s.t.} \quad & \bar{U}_{unb}(b) = \bar{U}_0 = 2.4515 \end{aligned}$$

where

$$\bar{U}_{unb}(b) = E \left[\ln \left((w_0 - b) (r_f + \alpha_0 (r - r_f)) \right) + Y \right].$$

Here $b_{\max}^{unb} = 2.2294$. Similarly the selling price is

$$\begin{aligned} \min \quad & s \\ \text{s.t.} \quad & \bar{U}_{unb}(s) = \bar{U}_M = 2.6573 \end{aligned}$$

where

$$\bar{U}_{unb}(s) = \max_{\alpha} E \left[\ln \left((w_0 + s) (r_f + \alpha_M (r - r_f)) \right) \right]$$

and $s_{\min}^{unb} = 2.3278$.

The differences are economically significant: the differences in buying and selling prices look relatively small. However, the difference in asset allocation appears large.

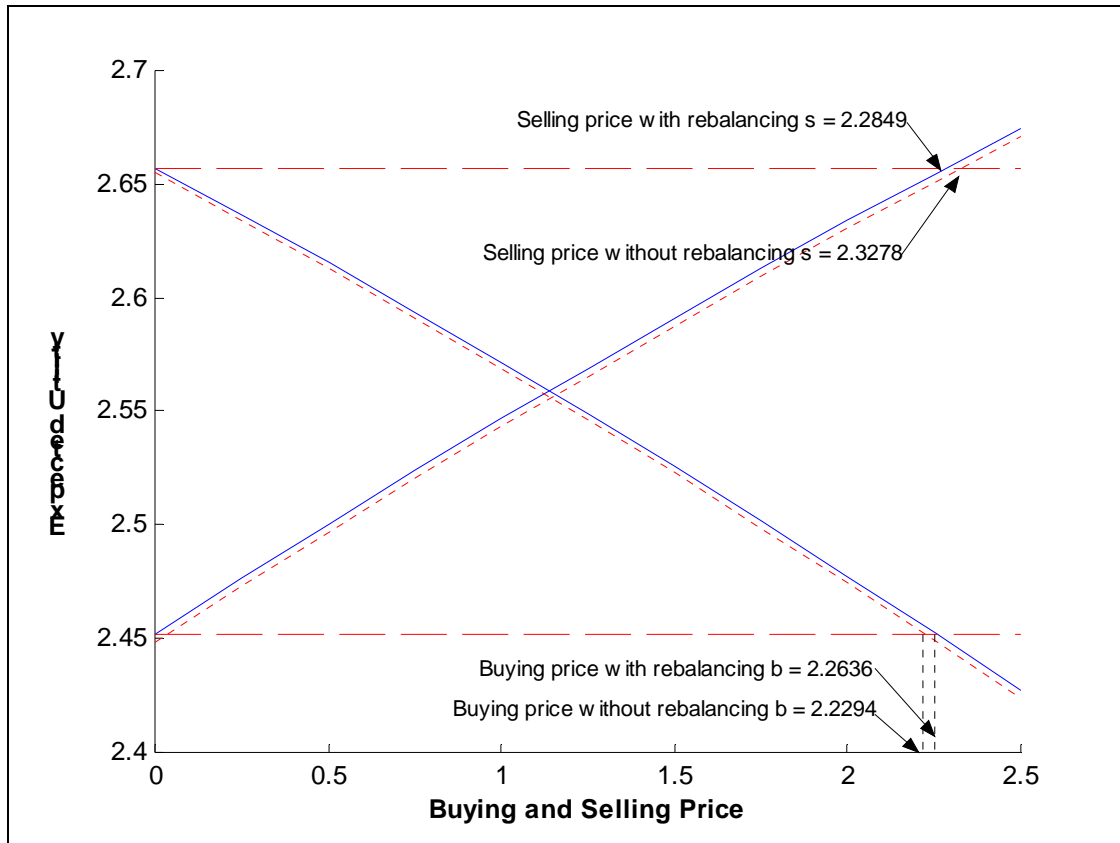


Figure 4 - Effect of Rebalancing on Buying and Selling Prices. The difference between buying and selling prices with and without portfolio rebalancing is economically significant.

4.3 Distributive Bargaining

The buyer's reservation price is the maximum he will pay for Y ; the seller's reservation price is the minimum he will accept for Y . In a distributive bargain between an owner of a private risk lottery Y and a potential buyer, the zone of possible agreement (ZOPA) is the intersection of the buyer's bargaining range and the seller's bargaining range. The buyer's bargaining zone is the region between the buyer's target point (lower bound) and reservation price (upper bound). Similarly the seller bargaining zone is the region between its reservation price (lower point) and its target point (upper bound).

Proposition 6: If (a) both buyer and seller are risk averse and at least one of them is non-constantly risk averse and (b) buyer or seller or both are allowed to portfolio rebalance, then the ZOPA is increased in range relative to not allowing portfolio rebalancing.

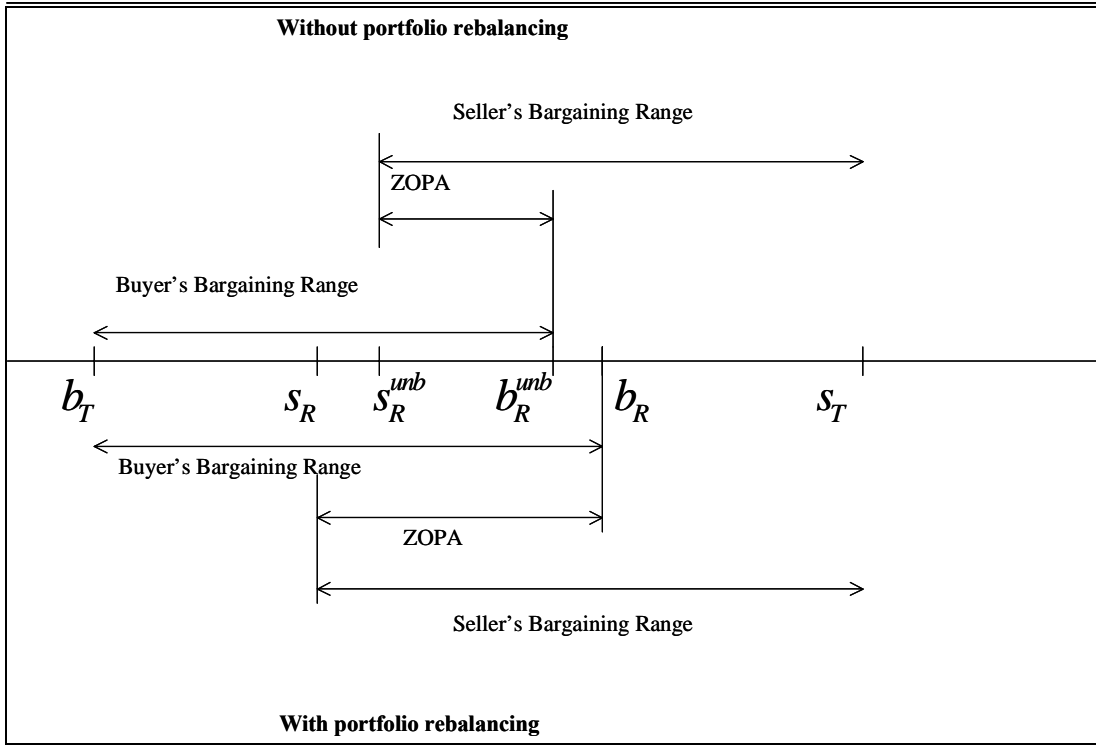


Figure 5 - Effect of simultaneous portfolio rebalancing on ZOPA

Appendix 1

Hoff Results modified to our notation and assumptions about U.

Complete Markets: In the complete market case Y is spanned by market securities. Then Y can be represented as a linear combination of market securities. In this case $\hat{w}_{se} = w_{se}$ and the investor need not rebalance his portfolio i.e. $(\Delta w + y_e - b) = 0$. Then

$$\psi = \frac{1}{r} \frac{\nabla \mathbf{u}}{\nabla \mathbf{u} \cdot \mathbf{1}} = \frac{1}{r} \frac{E[U'(\mathbf{w})]}{E[U'(\mathbf{w})] \cdot \mathbf{1}} \quad (39)$$

which is exactly the market's risk neutral probability (assuming no arbitrage). Then the price of Y becomes $b = \psi \cdot \mathbf{y}$.

Incomplete Markets: According to Hoff's second assumption, in incomplete markets the investor can trade exclusively in the risk free asset. When markets are incomplete the proof goes as follows: By definition the buying price is the price that makes the investor indifferent between the status quo and buying the lottery:

$$E_{X,Y} [U(\hat{w})] = E_{X,Y} [U(w)] \quad (40)$$

where

$$\hat{w} = w + \Delta w + Y - b \quad (41)$$

and since the investor can only modify his investment in the risk free asset,

$$E_{X,Y} [U(\hat{w})] = E_Y [U((w-b)r \cdot \mathbf{1} + \mathbf{y})] \quad (42)$$

and $E_{X,Y} [U(\mathbf{w})] = U(wr)$. (43)

Then

$$\begin{aligned} E_Y [U((w-b)r \cdot \mathbf{1} + \mathbf{y})] &= U(wr) \\ \frac{E_Y [U((w-b)r \cdot \mathbf{1} + \mathbf{y})] - U(wr)}{(y - br \cdot \mathbf{1})} (y - br \cdot \mathbf{1}) &= 0 \end{aligned} \quad (44)$$

$$(r \cdot \nabla \mathbf{u} \cdot \mathbf{1})b = \nabla \mathbf{u} \cdot \mathbf{y}$$

$$b = \frac{1}{r} \frac{\nabla \mathbf{u}}{\nabla \mathbf{u} \cdot \mathbf{1}} \cdot \mathbf{y}$$

as in the second part of (3).

Appendix 2

Proof of Proposition 1: Assume that $u(y)$ is continuous in y , monotone increasing in y and (at least) twice differentiable.

Let $f_\alpha(b) = E_{X,Y} [u(w_0 + \alpha \mathbf{x}^{(s)} + Y - b)] \quad \forall$ fixed value of α .

Then $\frac{d}{db} f_\alpha(b) = -E_{X,Y} [U'(w_0 + \alpha \mathbf{x}^{(s)} + Y - b)]$.

As $U' > 0$, $\frac{d}{db} f_\alpha(b) < 0$ i.e. $\forall \alpha$ fixed $f_\alpha(b)$ is monotone decreasing as a function of b .

Suppose $b_1 > b_2$. Then $\forall \alpha$, $f_\alpha(b_1) < f_\alpha(b_2)$

Let α_1 = the value of α that maximizes $f_\alpha(b_1)$ at fixed $b = b_1$.

Let α_2 = the value of α that maximizes $f_\alpha(b_2)$ at fixed $b = b_2$.

We know $f_{\alpha_1}(b_1) < f_{\alpha_1}(b_2)$ and $f_{\alpha_1}(b_2) \leq f_{\alpha_2}(b_2)$ since α_2 is the value of α that maximizes $f_\alpha(b_2)$. Consequently $f_{\alpha_1}(b_1) \leq f_{\alpha_2}(b_2)$ and equation (23) is monotonic decreasing in b .

Proof of Proposition 2: Let $f_\alpha(b) = E_{X,Y} [u(w_0 + \alpha \mathbf{x}^{(s)} - Y + b)] \quad \forall$ fixed value of α

Then

$$\frac{d}{db} f_\alpha(b) = E_{X,Y} [U'(w_0 + \alpha \mathbf{x}^{(s)} - Y + b)].$$

As $U' > 0$, $\frac{d}{db} f_\alpha(b) > 0$ i.e. $\forall \alpha$ fixed $f_\alpha(b)$ is monotone increasing as a function of b .

Suppose $b_1 < b_2$. Then $\forall \alpha$, $f_\alpha(b_1) < f_\alpha(b_2)$

Let α_1 = the value of α that maximizes $f_\alpha(b_1)$ at fixed $b = b_1$.

Let α_2 = the value of α that maximizes $f_\alpha(b_2)$ at fixed $b = b_2$.

We know $f_{\alpha_1}(b_1) < f_{\alpha_1}(b_2)$ and $f_{\alpha_1}(b_2) \leq f_{\alpha_2}(b_2)$ since α_1 is the value of α that maximizes $f_\alpha(b_1)$. Consequently $f_{\alpha_2}(b_2) \geq f_{\alpha_1}(b_1)$ so that equation (26) is monotonic increasing in b . \square

Proof of Proposition 3: Because X ($N \times N$) is non-singular, there exists a unique solution $\theta = \bar{y}X^{-1}$ for any given \bar{y} .

Proof of Proposition 4: At the status quo investor expected utility is $E_X [U(w_0 + \alpha_0 \mathbf{x}^{(s)})] = \bar{U}_0$ where α_0 is the optimal portfolio allocation. If the investor

calculates a maximum buying price without rebalancing the market portfolio then the buying price (b_{\max}^{unb}) is defined to be

$$\begin{aligned} & \max b \\ & \text{s.t. } \bar{U}_{unb}(b) = E_{X,Y} \left[U(w_0 + \alpha_0 \mathbf{x}^{(s)} + Y - b) \right] = \bar{U}_0 \end{aligned}$$

or ,

$$\bar{U}_{unb}(\alpha_0, b_{\max}^{unb}) = \bar{U}_0 .$$

We know that

$$\max_{\alpha} E_{X,Y} \left[U(w_0 + \alpha \mathbf{x}^{(s)} + Y - b) \right] \geq E_{X,Y} \left[U(w_0 + \alpha_0 \mathbf{x}^{(s)} + Y - b) \right], \quad \forall b$$

As a consequence, for $b = b_{\max}^{unb}$ we have $\bar{U}(b_{\max}^{unb}) \geq \bar{U}_{unb}(b_{\max}^{unb})$, which implies $\bar{U}(b_{\max}^{unb}) \geq \bar{U}_0$. On the other hand, we have by definition that $\bar{U}(b_{\max}^{unb}) = \bar{U}_0$, so then $\bar{U}(b_{\max}^{unb}) \geq \bar{U}(b_{\max}^{unb})$. Since $\bar{U}(b)$ decreasing in b (Proposition 1) then $b_{\max}^{unb} \leq b_{\max}$. \square

Proof of Proposition 5: At the status quo the investor own the utility and his expected utility is equal to $E_{X,Y} \left[U(w_0 + \alpha_M \mathbf{x}^{(s)} + Y) \right] = \bar{U}_M$ where α_M is the optimal portfolio allocation. If the investor calculates the minimum selling price without rebalancing the market portfolio then the selling price (s_{\min}^{unb}) is defined to be

$$\begin{aligned} & \min s \\ & \text{s.t. } \bar{U}_{unb}(s) = E_X \left[U(w_0 + \alpha_M \mathbf{x}^{(s)} + s) \right] = \bar{U}_M \end{aligned}$$

or,

$$\bar{U}_{unb}(\alpha_M, s_{\min}^{unb}) = \bar{U}_M .$$

We know that

$$\max_{\alpha} E_X \left[U(w_0 + \alpha \mathbf{x}^{(s)} + s) \right] \geq E_X \left[U(w_0 + \alpha_M \mathbf{x}^{(s)} + s) \right], \quad \forall s$$

Thus, for $s = s_{\min}^{unb}$, we have $\bar{U}(s_{\min}^{unb}) \geq \bar{U}_{unb}(s_{\min}^{unb})$, which implies $\bar{U}(s_{\min}^{unb}) \geq \bar{U}_M$. On the other hand, we have by definition that $\bar{U}(s_{\min}^{unb}) = \bar{U}_M$, so that $\bar{U}(s_{\min}^{unb}) \geq \bar{U}(s_{\min}^{unb})$.

Since $\bar{U}(s)$ is an increasing function in s (Proposition 2) then $s_{\min} \leq s_{\min}^{unb}$. \square

Proof of Proposition 6: By definition the buyer's reservation price is the price that makes him indifferent between reaching an agreement and walking away from the negotiation. In other words, at this level of buying price the utility of the buyer is equal to his utility at status quo $\equiv \bar{U}_0$. If the buyer does not rebalance his portfolio in the course of calculating the maximum buying price, he calculates a reservation price b_R^{umb} . Let b_R be his reservation price if he rebalances his portfolio to an optimal asset allocation for each possible buying price. Proposition 4 tells us that $b_R^{umb} \leq b_R$. \square

Similarly the seller reservation price is the selling price that makes the seller indifferent between the sale transaction and the status quo. At the reservation price the seller expected utility is equal to the his utility in the status quo. If the seller does not rebalance his portfolio in the course of calculating the minimum selling price, he calculates a reservation price s_R^{umb} . Let s_R be his reservation price if he rebalances his portfolio to an optimal asset allocation for each possible selling price. Proposition 5 tells us that $s_R \leq s_R^{umb}$.

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