Puzzles

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January 28, 2019

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1 Coin Weighing

A lot of these problems involve a *balance*, which is a device that takes two sets of objects and compares their weights. The use of a balance has three possible outcomes: ‘greater than’, ‘equal’ or ‘less than’.

1.1 100 Coins

Problem (2010, from Al Barboy):
You have 100 coins, four of which are counterfeit. Counterfeit coins are known to be lighter than real coins. Using a balance twice, identify at least one coin that is definitely not counterfeit.

1.2 12 Coins

Problem (2014, from Eitan Glinert):
You have 12 coins, one of which is counterfeit. Counterfeit coins are known to have a different weight than real coins (either lighter or heavier). Using a balance thrice, identify the counterfeit coin.

1.3 12 Coins - Take 2.

Problem (2018, from Luis Brandao):
You have 12 coins, one or zero of which are counterfeit. Counterfeit coins are known to have a different weight than real coins (either lighter or heavier). Using a balance thrice, identify the counterfeit coin if there is one, and identify whether it is lighter or heavier.

1.4 How Much Gold Do I Have?

Problem (2014, from Eitan Glinert):
You have a chunk of gold of integer weight between 1 and 40, inclusive. You would like to be able to determine the exact weight of your gold, and you’re allowed to use four chunks of silver to help you do so. What weights would the chunks of silver need to have to allow you to do so?

1.5 Are All the Coins the Same?

Problem (2016, from Seth Toplosky):
You have 8 coins, each of which has one of two possible weights: A or B. Using a balance thrice, ascertain that they all have the same weight.
Variant:
Instead of 8 coins, you now have 10 coins.

2 Hats

2.1 Three Hats

Problem (2012):
Three prisoners are given hats. Each hat is black with probability 1/2, and white with probability 1/2. Each prisoner sees everyone else's hats, but not their own hat. On the count of three, each prisoner guesses his or her hat color, or abstains, simultaneously. If either everyone abstains or anyone guesses wrong, all three prisoners are executed. What is a strategy that maximizes the chances of their survival?

2.2 Line of Hats

Problem (2016):
n prisoners are arranged in a line. Each prisoner is given a hat. Each hat is black with probability 1/2, and white with probability 1/2. Each prisoner sees, and knows the color of, the hats of those in front of them, but not their own hat or the hats of those behind them. One at a time, starting from the back of the line, each prisoner must announce a guess at their hat color. Each prisoner who is wrong is executed; each prisoner who is correct lives. They cannot communicate other than before the start of this game, and through their guesses. What strategy can they use to minimize the number of executions?

2.3 Room of Hats

Problem (2016):
n prisoners are each given a hat. Each hat has a number between 0 and $n-1$ (inclusive) on it. The number on each hat is independently random; it is i

for $i \in [0, \ldots, n-1]$ with probability $1/n$. Each prisoner sees, and knows the numbers on, the hats of everyone else, but not their own hat. They must all simultaneously announce a guess at their number. If everyone guesses wrong, they are all executed. If at least one prisoner guesses correctly, they all live. They cannot communicate other than before the start of this game. What strategy can they use to guarantee that they all live?

2.4 Four Hats

Problem (2019, from Gene Itkis):
Now, let there be exactly four prisoners, and exactly three possible hat numbers $(0, 1, 2)$. However, the complication is that the prisoners are arranged in a square around a column, so that each prisoner can see their two neighbors (and the hats they are wearing), but not the prisoner directly across from them. The problem is the same as before; the prisoners must simultaneously guess their own numbers, and if no-one is correct, they are all executed. How can they guarantee their survival?

2.5 Hats with Timesteps

Problem (2016):
n people are given hats. $w < n$ hats are white, and the rest are black. Each person sees, and knows the color of, the hats of everyone else, but not their own hat. At midnight on day 0, the people are told that there is at least one white hat. At midnight on days 1, 2, etc, each person who has figured out their hat color that day leaves the room. How long will it take everyone to leave the room?
Variant:
What if the announcement on day 0 is that there is at least one hat *of each color*?

2.6 Colorful Hats with Timesteps

Problem (2016, from Neal Wagner):
31 people are given hats; all they know about the colors is that there are at least two hats of each color. Each person sees, and knows the color of, the hats of everyone else, but not their own hat. At midnight on each day, each person who has figured out their hat color that day leaves the room. Given

the following sequence of events, how many days does it take for everyone to leave?

1. At midnight on day 1, 4 people leave.
2. At midnight on day 2, everyone wearing a red hat (and no-one else) leaves.
3. At midnight on day 3, no-one leaves.
4. At midnight on day 4, at least one person leaves.
5. At some point on day 5 or later, two people with different hats leave simultaneously, but others remain.

3 Liars and Truth Tellers

3.1 Buying One of Three Computers

Problem (2012):
There are three identical-looking computers at the store. One always lies, one always tells the truth, and one answers at random. You want to buy any consistent computer (either of the first two). Before making your purchase, you may ask one computer one question. What should that question be?

3.2 Labeling the Three Computers

Problem (2018, from Anca Nitulescu):
Now, let's say that instead of being a customer, you are the computer store proprietor. You want to figure out which computer always tells the truth, which computer always lies and which computer is random so that you can label them for the convenience of your customers. How can you do this, by asking three questions?

3.3 Labeling the Three Computers . . . With a Twist!

Problem (2012, from Alina Griner):
As before, you are the computer store proprietor, trying to label your three

computers. However, there is an added complication: instead of answering boolean questions with "yes" or "no", the computers respond with "0" and "1", and you don't know which bit corresponds to "yes" and which to "no"! You may ask three questions, each addressed to a single computer, to determine which computer is which. How can you do it?

3.4 1000 Witches

Problem (2016, from an RSM Contest):
There are 1000 witches sitting around a circular table. Each witch either always tells the truth, or always lies. Each witch turns to her immediate neighbors to the right and left, and says, "without saying anything about the three of us, everyone else at this table is a liar." How many witches at the table are truth tellers and how many are liars?
Variant:
Imagine instead that two of them say, "both of my neighbors are liars!", while the other eight say, "both of my neighbors are truth-tellers."

4 Lightbulbs

4.1 100 Prisoners

Problem (2012):
An evil king has 100 prisoners, and decides to have a little fun. He lets the prisoners talk amongst themselves, then takes them all to individual cells deep in his dungeon, and from thereon out does not allow them to communicate. Every day, he picks one prisoner at random and takes that prisoner to a special room, the only feature of which is a lightbulb with a switch. Upon being taken to the special room, a prisoner can do one of three things: nothing, flip the state of the light bulb, or declare that every single prisoner has been in the room. If a prisoner makes the declaration and is correct, the king will release all of the prisoners. If he is wrong, the king will execute all of the prisoners.

a If the lightbulb is known to have started out in the 'off' state, what strategy can these prisoners use to ensure their release?

b What if the initial state of the lightbulb is not known?

4.2 Train

Problem (2012):
There is a train consisting of identical cars, connected in a circle. Each car has a light bulb. Alice is in one of the cars. She can walk from car to car, and change the light bulb state (from on to off, or from off to on) if she wishes. How can she determine the number of cars in the train?

5 Tilings

5.1 Chessboard with 1×2 Dominoes

Problem (2010):
Can an 8×8 chessboard with two diagonally opposite corners removed be tiled with 1×2 dominoes?

5.2 Chessboard with 1×3 Dominoes

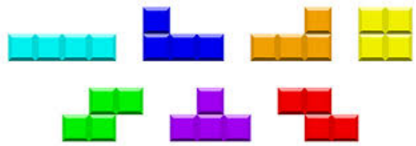
Problem (2012):
When tiling an 8×8 chessboard with 1×3 dominoes, which 1×1 squares on the chessboard can remain as the only untiled square?

5.3 Chessboard with Changing Colors

Problem (2012, from Moscow Olympiad):
We have a chess board. At each step we pick a 2×2 square and change the colors of the (individual) squares as follows: white to green, green to black, and black to white. Is it possible that at the end all white squares become black and vice versa?

5.4 7×4 Board with Tetris Pieces

Problem (2014, from Eitan Glinert):
Can a 7×4 rectangle be tiled using each of the tetris pieces depicted below exactly once?



6 River-Crossings

6.1 Two-Person Bridge

Problem (2016, from Neal Wagner, statement verbatim from `http://wiki.xkcd.com/irc/Puzzles`):

4 people need to traverse a bridge. The bridge is old and only two people can use it at the same time. It is night and to traverse the bridge a flashlight is needed. The group only has one flashlight. Each person traverses the bridge at different speeds and when 2 go together the faster one needs to adapt for the slower one (otherwise they can't share the light). The first person (A) needs 10 minutes to cross the bridge. The second (B) 5; the third (C) 2, and the fastest one (D) only 1 minute. How long does it take the group to cross the bridge?
Example (not the most efficient one):

- A+D → // 10 min (let's assume the slowest and the fastest go first)
- D ← // 1 min (D needs to bring back the flashlight)
- B+D → // 5 min
- D ← // 1 min
- C+D → // 2 min (and the group has crossed the bridge)

This takes a total of 19 minutes. There is a better way!

6.2 Three Swindlers

Problem (2016, from Tanya Khavanova's site):
Three swindlers have two suitcases each. They approach a river they wish

to cross. There is one boat that can carry three objects, where a person or a suitcase counts as one object. No swindler can trust his suitcase to his swindler friends when he is away, but each swindler doesn't mind his suitcases left alone at the river shore. Can they cross the river?

7 Other Puzzles

7.1 100 Prisoners

Problem (2010, from Professor Richard Stanley):
An evil king has 100 prisoners, and decides to have a little fun. He writes each prisoner's name on a piece of paper, and puts each name in a box. The boxes are labeled one through 100, and each contains exactly one name. The king then makes each prisoner (one at a time) look inside exactly 99 of the 100 boxes. If no prisoner sees his or her own name, he releases them all. If at least one prisoner sees his or her own name, he executes them all. What strategy can these prisoners use to maximize their chances of release? At a maximum, what are those chances?

7.2 Wine Tasters

Problem (2014, from Luis Brandao):
There are 200 cups of wine, one of which is poisoned. You have five wine tasters. The king's banquet is in two hours, and since the expected head-count at the banquet is 199, you must identify the poisoned cup of wine by then. However, the wine tasters only get two chances to taste wine mixtures - once now, and once an hour from now. If a wine taster consumes poisoned wine (or a mixture containing poisoned wine) during the first round of tasting, he cannot participate in the second round of tasting. How do you go about identifying the poisoned cup of wine?

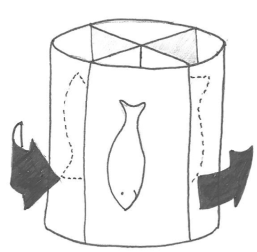
7.3 Splitting Booty

Problem (2014, from Nick Hwang):
There are five monkeys - the president, the vice president, the treasurer, the vice minion and the minion. There is a pile of five bananas. First, the president proposes a way to split up the bananas among the five monkeys. All

five monkeys vote, and if the president's proposal doesn't have the support of half or more of the monkeys (in this case, at least three monkeys), the president is killed, and the vice president gets a chance to make a proposal. Again, all four remaining monkeys vote, and if the vice-president's proposal doesn't have the support of half or more (in this case, at least two monkeys), he is killed as well. This continues until the bananas have been successfully split up. If all the monkeys are much smarter about this than we can actually expect monkeys to be, what will happen? (What happens when there are more than five monkeys?)

7.4 Directional Fish Lock

Problem (2014, almost verbatim from Sam Steingold):
A lady is locked alone in a dungeon, while her captor the vile monster is getting firewood. When the monster comes back in 15 minutes, he plans to cook and eat the lady. The dungeon is locked with a very special lock, consisting of a cylindrical aquarium divided into four compartments, as depicted below. In each compartment, there is a fish. Each fish is facing either up or down. Unlike the aquarium in the picture, though, the lock's aquarium is not transparent. The lady cannot see the fish, or which way they are oriented. The lock unlocks when all of the fish face the same direction. The lady can try to unlock the lock by reaching into the aquarium and feeling (and, if she wants, switching) the orientation of two of the fish. If as a result all of the fish are facing the same way, the lock unlocks and she is free to flee. If not, the aquarium rotates quickly for one minute, so that she no longer knows which fish she just touched; after it stops, she gets to try again. Can she escape? How?



7.5 Handshakes at Mr. and Mrs. Kent's Party

Problem (2016, from Neal Wagner):
Mr. and Mrs. Kent throw a party. They invite four couples (which makes for 10 people total, including themselves). Any pair of people who don't know each other shake hands. Mrs. Kent then asks everyone how many hands they shook, and everyone gives different answers. How many hands did Mr. Kent shake?

7.6 Measuring Time with Ropes

Problem (2016, from Neal Wagner):
Imagine you have a rope which takes one hour to burn, but it is not uniform. That is, the first 9/10 of its length might be gone after the first minute, and the last 1/10 might take the remaining 59 minutes to burn. How can you measure exactly half an hour using that one rope?
Variant:
How can you measure exactly $1/n$ of an hour using that one rope, for any n?

7.7 Three Barburs

Problem (2012, from Nina Dubinsky):
There are 3 types of barburs. When two barburs of different type meet, they transform to ONE barbur of the remaining type. If 20 A, 21 B and 22 C type barburs transform to a single barbur, what type of barbur is it?

7.8 Six Spirits

Problem (2017, from David Wilson):
There are six people standing in a row. One of them is possessed by a spirit. You wan to exorcise the spirit. You do not know the spirit's initial position. During each time step:

1. The spirit moves (invisibly) to an adjacent person. (Note: The the spirit always moves; it never stays still!)

2. You can exorcise any one of the six people. If the spirit is in the person you exorcise, you have accomplished your goal.

Question: Is there a finite sequence of exorcisms you can perform that will guarantee that you will eventually accomplish your goal? If not, show that this is the case; if so, what is the shortest such sequence?

7.9 Using Unfair Coins

Problem (2017, from Jacob Leemaster):
Imagine you have an unfair coin, which, when flipped, shows heads with probability p and tails with probability $1 - p$, where you don't know the value of p. How can you use this coin (and only this coin) to help you make a decision when you want to choose each of "yes" and "no" with probability exactly .5?

7.10 Six Dots

Problem (2018, from Ilya Lifshits):
Imagine you have six dots, in two groups, three dots in each group. Can you complete a bipartite graph between the two groups in such a way that edges do not intersect?

7.11 Chameleons

Problem (2018, from David Wilson, originally heard as a moving target OS problem):
Imagine that 13 chameleons are blue, 15 are green and 17 are red. Whenever two chameleons see each other, if they are of different colors, they both assume the third color. Will all 45 chameleons ever have the same color?

7.12 Splitting Pies

Problem (2018, from Luis Brandao):
Imagine that Alice and Bob want to share a pie fairly, but they don't have any measurement implements. What should they do?
Variant:
Now, imagine that Charlie joins Alice and Bob for a second pie. Assuming all three of them want a fair share, how can they split the pie in such a way that each of them is satisfied with their share, even if the other two collude?

7.13 Dropping Eggs

Problem (2018, from Luis Brandao):
Imagine that you are in a 100-story building, and you want to determine which floor is the lowest floor such that when you drop an egg from it, the egg breaks. You have two identical eggs. What is the smallest number of drops (in the worst case) that you can do this in? (If in a dropped egg doesn't break then it retains its original strength.)

7.14 Bigger or Smaller?

Problem (2018, from Jesper Buus Nielsen):
Two people — Alice and Bob — play a game. Alice chooses two distinct numbers, and reveals a random one of the two to Bob. Bob then has to guess whether the revealed number is larger or smaller than the other one. How can Bob succeed with probability greater than $\frac{1}{2}$, no matter which two numbers Alice chooses?

7.15 Synchronized Sequence Guessing

Problem (2018, from Krzysztof Pietrzak, who heard it from Yevgeniy Dodis):
Two people — Alice and Bob — team up to play a game against an opponent, Charlie. The game consists of n rounds, where in each round, all three players simultaneously play a color: black or white. If all three players play the same color, Alice and Bob win the round; if both white and black is played, then Charlie wins the round. Alice has an advantage — she is psychic, and knows exactly what sequence of colors Charlie will play throughout all of the rounds before the game even starts. However, Bob is not psychic, and Alice doesn't get the chance to tell him what Charlie's strategy is. Instead, the only way she can communicate information to Bob is through the colors she plays. How can Alice and Bob guarantee that they will win at least $\lfloor\frac{2(n-1)}{3}\rfloor$ rounds?

7.16 Spreading Colors

Problem (2018, from Anca Nitulescu):
Imagine a grid of size $n \times n$ of squares, where each square is colored either

white or black. Black spreads as follows: at each step, all the white squares which have at least two black neighbors (where neighbors must share a side — they can not be diagonal neighbors) become black. What is the minimum number of black squares needed at the beginning for the grid to be completely black at some point?