Experimental Investigations of Double Refraction from Huygens to Malus

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Communicated by I. BERNARD COHEN

To Stillman Drake who taught me to prefer the clown's habit to the philosopher's dress

Contents

Introduction

Before the early part of the nineteenth century only three quantitative laws were known in optics: the law of reflection, SNEL's law of refraction, and HUYGENS' construction for the peculiar "double refraction" (now termed birefringence) by the crystal Iceland spar. Although HUYGENS' construction is now known to be quite as accurate as the other two laws, nevertheless for over a hundred years it was not accepted. The phenomenon itself, however, was well known, and during the eighteenth century various alternative laws were offered for it. This widespread rejection of an accurate law – HUYGENS' construction – in favor of less accurate laws poses an interesting problem for the historian of optics. Was it simply the case that, during the eighteenth century, scientists rejected the construction on the sole basis that it was an implication of wave theory, which few among them accepted? It is certainly true that several among them regarded the construction as uniquely an implication of HUYGENS' wave theory. Or were there in fact difficulties of calculation, experimental technique and the interpretation of observations which, in this period, were sufficiently acute to cast reasonable doubt on the construction's empirical adequacy?

My purpose in this article is not to examine theoretical discussions of double refraction – of which there were, in any case, almost none except HUYGENS' before the early nineteenth century – but rather to examine in some detail the experiments performed and laws offered for the phenomenon between 1690, the year of the publication of HUYGENS' *Traité de la Lumière*[1], and 1807, when MALUS confirmed the construction beyond all doubt in a series of extremely accurate experiments. The history of experiments on double refraction is particularly interesting for three major reasons; taken together, these reasons demonstrate that contemporary antagonism towards HUYGENS' construction based on theoretical reasons was not the main, or perhaps even a very important, factor in its rejection before 1807. First, experimental investigations of double refraction provide striking examples of the way in which a law, however accurate it may eventually prove to be, can remain problematic for lengthy periods during which experimental techniques are not sufficiently advanced to provide incontrovertible evidence for it. Second, the mathematical language in which the law is expressed – geometry in the case of HUYGENS' construction – may not in fact be readily applicable to contemporary experiments, and this can – and in HUYGENS' case did – preclude testing the law in its full generality until it is expressed in a different language (*e.g.* analysis). Finally, we will see that HUYGENS' construction, when tested according to one common procedure of the time, *does* in fact conflict with experiment for reasons which concern the particular structure of the observational technique. Convincing confirmation for the construction could be obtained *only* with new techniques. Yet we shall also see that, when one such technique did become available early in the nineteenth century, it was incorrectly applied and could not actually be used to confirm every aspect of the construction. It was precisely this fact that was realised by MALUS and led him to his highly accurate experiments.

Previous histories of double refraction have for the most part emphasised the importance of the phenomenon for HUYGENS' wave theory, which was un-

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[1] The sole English translation is that of S. P. THOMPSON (*Treatise On Light*, 2ed. Chicago, 1950; hereafter references are to the THOMPSON translation in this edition). The *Traité* is reproduced in its original French in HUYGENS, *Œuvres complètes* **19** (The Hague, 1937), pp. 457–537.

questionable in view of the principle of secondary waves[2], or they have related it to the development of corpuscular optics in France in the early 1800's (see notes 49 and 51). My intention here is to provide for the first time a discussion of the empirical context in which theory developed. Theory, after all, does not come to be in an empirical void nor does it develop in isolation from observational technique. In fact, full historical understanding, in the case of double refraction at least, requires not merely that we be concerned with experiment but also that, where possible, we replicate the observations that were made. Only by doing so can we hope to understand the difficulties of calculation and observation which persisted for more than a century and which, we shall see, were substantially responsible for the long rejection of HUYGENS' construction, that most accurate of laws. I have accordingly replicated—and in this instance it is possible to repeat many of the experiments with historical precision—both HUYGENS' measurements and several of the tests that were made of his construction. Where I could not do so for lack of precise historical apparatus I have instead carried out calculations which reveal a number of historically significant characteristics of the experiments.

1. Huygens' calculation of the crystallographic angles

In his 61-page *Experimenta*[3] ERASMUS BARTHOLIN described the basic structure of Iceland spar, measured the facet angles, and qualitatively discussed the refractions. HUYGENS was fully acquainted with the *Experimenta*, but from it he took mainly the gross fact of double refraction. To understand precisely what HUYGENS claimed, and how he confirmed his claims, we shall begin by considering how he determined the crystallographic angles. Instead of BARTHOLIN's measures of 101°, 79° for the facet angles, HUYGENS found 101°52′, 78°8′. Unlike BARTHOLIN, he did not measure either angle directly; he calculated them, and every other crystallographic angle, from a single measurement that is much more exact than any measurement of a facet angle can be[4].

In figure 1 (which is given on p. 100 of THOMPSON's translation; like every other figure in the translation it is a direct copy of the corresponding one in HUYGENS' *Traité*), the crystal facets are as they usually occur in nature, *i.e.* as parallelograms which are not equilateral. In the upper facet $AD'BC$, $<ACB$ and $<AD'B$ are obtuse, so that $<D'AC$ and $<D'BC$ are acute. The solid obtuse angle whose vertex is C is diagonally opposite the other solid obtuse angle, vertex E. To find the facet angles, one can try to measure them directly, and BARTHOLIN did, but, as HUYGENS remarked, this is "difficult to do with ultimate exactitude because the edges such as CA, CB, in this figure, are generally worn, and not quite straight"[5]. But one can very accurately measure

[2] A. SHAPIRO, 'Kinematic Optics: A Study of the Wave Theory of Light in the Seventeenth Century', *Archive for History of Exact Sciences*, **11** (1971), pp. 134–226.

[3] ERASMUS BARTHOLIN, *Experimenta Crystalli Islandici Disdiaclastici quibus Mira & Insolita Refractio detegitur* (Hafniae [Copenhagen], 1669).

[4] *Treatise*, pp. 99–100.

[5] *Ibid.*, p. 99.

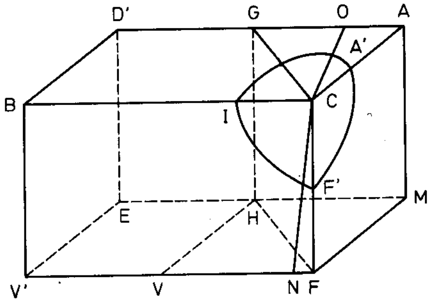

Fig. 1

the obtuse angle in which facets $CBD'A$, $CBV'F$ meet, because the facets are generally quite smooth. That was HUYGENS' single crystallographic measurement; he found the angle, $<OCN$ in the figure, to be 105°. To find the values of the other angles from this one, HUYGENS described a sphere with vertex C as center and which is cut by $ACBD'$ in the arc $A'I$, by $ACFM$ in the arc $A'F'$, and by $BCFV'$ in the arc $F'I$. Each arc is a segment of a great circle since the three cutting planes intersect at C, so that $A'F'I$ is a spherical triangle. Applying spherical trigonometry HUYGENS then found:

obtuse facet angle$=101°52'$, $<CFH=70°57'$, and $<GCH=45°20'$.

(CH is the intersection of the three planes – called principal sections – which are respectively perpendicular to the three facets which meet at C and which bisect the obtuse facet angles.) These three angles were calculated from a single measurement ($\angle OCN$) coupled with the fact that the facets are equally inclined to one another. They are the first observational parameters of the theory. Having briefly considered how HUYGENS dealt with the crystal's structure, let us now turn to his description of the most readily observable optical phenomena which it produces.

2. Preliminary results

HUYGENS began his discussion with a qualitative explanation of the images of luminous or marked points seen through the crystal (figure 2)[6]. For example, covering the surface $ABCD$, leaving only a small hole at K, HUYGENS remarked that two rays are formed from a ray incident normally along IK. One ray, KL, goes straight through the crystal without deviation, and the other or 'extraordinary' ray, KM, lies in the principal section and is inclined to KL within the crystal; at emergence it is refracted back along the facet normal. The angle between KL and KM is, HUYGENS remarked, 6°40'; we will consider below how he deduced optical angles from observation. From this it is simple to understand the appearance of a point marked at L on the base of the crystal. L will be seen, with the eye at I, by the extraordinarily refracted ray RI and by the ordinary ray KI. Observations of this sort, HUYGENS noted, led him to conclude that there

[6] *Treatise*, section 7, chap. 5.

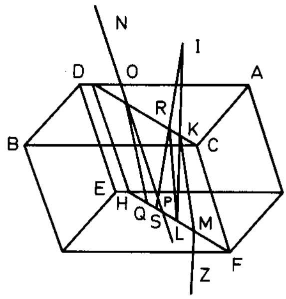

Fig. 2

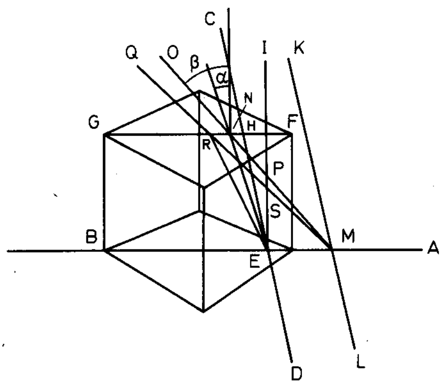

Fig. 3

are two refractions, one of which follows the ordinary rule, though the other does not. That is, one of the rays obeys SNEL's law, but the other cannot be fit to the law even if it is assigned a different index of refraction.

HUYGENS never measured an optical angle directly. It evidently did not occur to him to construct a device akin to a vertical protractor with sights (though, as we shall see, MALUS did precisely that in the early 1800's) or else he was satisfied with the accuracy of his simple measuring technique. Instead, HUYGENS developed a technique that permits angles to be deduced by using only ruler, pen, and paper (and a table of trigonometric functions). Though HUYGENS did not specify the divisions on his ruler, they could not have been much smaller than a millimeter, nor could they have been much larger considering the accuracy he attained. The technique is central to the contemporary observational basis of HUYGENS' theory, and it has also enabled me to replicate HUYGENS' experiments using crystals 17.5 mm and 39 mm in height kindly provided by Professor STILLMAN DRAKE.

On a "thoroughly flat table" (figure 3) a leaf of paper is fixed, and a line AB is drawn upon it[7]. Two other lines, CD and KL, are drawn perpendicular to AB at a small distance from one another. The crystal is so placed that AB either

[7] *Treatise*, section 12, chap. 5. I have added angles α and β.

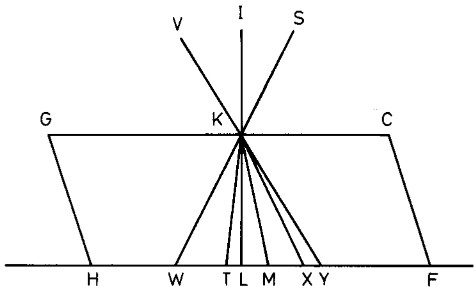

Fig. 4

bisects the obtuse angle of the lower surface or is parallel to its bisector; the crystal's right vertical edge lies between CD and KL. To measure the index of refraction of the ray that obeys SNEL's law – the 'ordinary' ray – HUYGENS first placed one eye (the other closed) in the plane of the principal section. He then moved it, always remaining in this plane, until the ordinary refraction of the line CD (which is distinguished by the fact that it remains stationary as the crystal is rotated) became collinear with the segments of CD which lie outside the crystal. This put his eye directly above point E, along the line EI. A point H on the upper facet coincident with the image of E was then marked. Keeping the eye in the principal section, HUYGENS moved it towards G until the ordinary refraction of CD became collinear with the segments of KL which are visible outside the crystal. The point N on the surface coincident with the image of E was marked. By direct observation HUYGENS therefore had the distances EM, NH, and he could measure the height EH of the crystal.

On a separate sheet of paper draw the line AB with E and M marked off, and draw EH normal to AB at E. Draw also the line MN, which intersects EH at P, and connect points N and E. The angle α is equal to the angle of refraction, $<NEP$, of ray ON, and β, the angle of incidence, is equal to $<NPH$. Consequently the index of refraction, $\sin\beta/\sin\alpha$, is equal to $\frac{NH}{NP}\Big/\frac{NH}{EN}=\frac{EN}{NP}$. The accuracy of the ratio depends on the height of the crystal since, the higher it is, the greater NH will be, and the less will inaccuracies affect the ratio. Thus with my small crystal (17.5 mm) I find that EN/NP is about 4.9/3; with my large crystal (39 mm) I find 5/3, as did HUYGENS, whose own crystal was about 40 mm in height[8].

Applying the same measuring technique to the extraordinary refraction (here the ray RE is extraordinary), HUYGENS found that ER/RS, the measure of its index, is not constant but varies with the angle of incidence, so that this ray does not obey SNEL's law. However, the simplicity of the technique enabled HUYGENS to discover a law that applies to the extraordinary refractions of any two rays, incident in the principal section at the same point, and equally but oppositely inclined to the normal (figure 4)[9]. In the figure, $GCFH$ is the

[8] HUYGENS possessed a piece of crystal weighing half a pound.

[9] *Treatise*, section 16, chap. 5. I have omitted one line from the diagram and added points W and Y. I shall call what follows the law of equal deviations.

principal section, and C is the vertex of the upper solid obtuse angle. If VK, SK are rays of equal but opposite inclinations to the normal IK, then the respective points X, T at which each strikes the base of the crystal are equidistant from the point M of the base which is intersected by the extraordinary refraction of a normally incident ray IK. KM is of course inclined towards the obtuse angle $\angle GCF$ of the principal section. This was the first law HUYGENS gave for the extraordinary refraction, and he later deduced it from his general construction.

Having determined some of the basic optical properties, HUYGENS explained that it occurred to him that there might be two systems of waves within the crystal, one of which is spherical, and which therefore possesses the properties of ordinary refraction, though the other is 'elliptical' or 'spheroidal'. These last, he reasoned, would spread through both the ethereal matter within the crystal and through the particles of which the crystal proper is composed:

> It seemed to me that the disposition or regular arrangement of these [crystal] particles could contribute to form spheroidal waves (nothing more being required for this than that the successive movement of light should spread a little more quickly in one direction than in the other) and I scarcely doubted that there were in this crystal such an arrangement of equal and similar particles because of its figure and of its angles with their determinate and invariable measure[10].

In order to locate precisely what HUYGENS did, and what he did not, confirm, we must extract from his analysis the laws that enable the refractions to be calculated. Since these laws depend directly upon the proportions of the spheroid, I shall first examine how he deduced the parameters of that surface. Following this, I shall discuss his first, limited law for the refraction of a ray incident in the principal section; I will not, however, examine his proof of the law since we are presently concerned solely with the empirical status of the theory. With this law in hand, we will be able to understand the significance of HUYGENS' first (and only) stated numerical confirmation of the theory.

3. The proportions of the spheroid

From the outset HUYGENS assumed that the spheroid was a solid of revolution. The problem was to determine the orientation and proportions of the axes. HUYGENS knew that, for a given angle of incidence, the extraordinary refraction is precisely the same if the plane of incidence contains or is parallel to any of the three principal sections of the crystal. His intention was to use the spheroid just as he had previously used a sphere in isotropic bodies; that is, he intended to determine the extraordinary refractions by the plane sections of the spheroid (see Appendix I). Consequently the only way in which the three principal sections could produce the same refractions was if each sectioned the same curve in the spheroid. For that to be possible, the spheroid's axis of revolution had to lie in all three principal sections, *i.e.* it had to be CH in

[10] *Treatise*, p. 62.

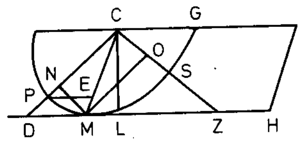

Fig. 5

figure 1. Hence the axis of revolution is equally inclined to each of the edges of the crystal. The orientation of the axis is therefore entirely determined by the fact that the refractions are identical in the principal sections.

Figure 5[11] is a principal section; CS is the axis of revolution, hereafter called the optic axis, for reasons which will presently be clear. The generating ellipse is so constructed that MH, parallel to the crystal facet, is tangent to the ellipse at M, where $<MCL$ is 6°40′. This is obviously required by the extraordinary deviation of a normally incident ray. HUYGENS set CM equal to 100000 as a reference for calculating CS, the length of the semi-axis along the optic axis, and CP, its conjugate.

To calculate the proportions of the spheroid, HUYGENS relied on two simple properties of the conjugate diameters of an ellipse, both of which can easily be deduced from theorems in APOLLONIUS' *Conic Sections*[12]. Here CM, marking the refraction of a normal ray, is conjugate to the facet diameter CG. Given the angle between CM and the normal CL to CG (6°40′), one can then determine CS, CP and CG in proportion to CM. On setting $CM=100000$ as unit, there results following his procedure:

$$\text{semi-major axis } CP = 105032,$$
$$\text{semi-minor axis } CS = 93420,$$
$$\text{and } CG = 98779.$$

Thus in determining the proportions of the spheroid, HUYGENS used precisely one crystallographic measurement, one optical measurement, and a property of the refraction for which no number is necessary. That is, he has used the (crystallographic) measurement of the interfacial angle, the (optical) measurement of the normal deviation, and the property that the extraordinary refraction is the same in all principal sections.

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11 Figure 5 is a composite of the figures on pp. 67 and 103 of the *Traité* (trans.).

12 Namely, Book I, Propositions 21 and 36. Following HUYGENS (see figure 5):

1. To find CP, from the center C draw a line that intersects the tangent at M in the point D and the ellipse in the point P, and from M draw a line parallel to an axis (here CS) of the ellipse to intersect CD in N: then by conic geometry $CP=\sqrt{CD\times CN}$. Similarly we can find $CS=\sqrt{CZ\times CO}$.

2. To find CG, first from P (the end-point of CP) draw PE parallel to DM, the tangent at M, meeting CM in E; then by conic geometry $CG=PE\times MC/\sqrt{MC^2-CE^2}$, where MC, CE, PE are readily found.

4. Refraction in the principal section

Having given the orientation and the size of the spheroid, HUYGENS turned to the refraction of a ray incident in a plane that is, or is parallel to, the principal section. Here he applied the principle of secondary waves in precisely the same way he had before, when he deduced SNEL's law, and he obtained a law governing the extraordinary refraction.

In figure 6, line gK is the intersection of the principal section with the surface of the crystal; the upper solid obtuse angle is towards g. Consider an incident ray RC, and produce CO, the intersection of the wave front with the plane of incidence, until OK, the normal to CO that intersects the surface gk at K, is equal to N, the distance traveled by light in air in unit time. Draw the ellipse $GSPg$ with center C as it appears after unit time. Then, HUYGENS proved, in this plane of incidence, and only in this plane, the refracted ray lies in the principal section; it lies along CI, where I is the point at which a line from K touches the ellipse.

Thus far we have only drawn the tangent; we have not as yet determined its orientation as a function of the position of point K. That is simple, since it is a property of the ellipse that, given a diameter CG, the tangent from a point K on the extension of CG will touch the ellipse at a point I where a line DI, drawn parallel to the conjugate CM to CG, intersects the ellipse, the length CD being a third proportional to CK, CG *i.e.* CK/CG is equal to CG/CD[13].

We now have the orientation and parameters of the surface, though only in proportion to CM as a reference of 100000. To develop a set of absolute parameters that can be used to calculate a refraction from a given incidence, HUYGENS had to determine the value of N in proportion to CM. This determination will incidentally reveal the relationship between the radii of the spheroid and the radius of the sphere that governs the ordinary refraction, since we already know the ordinary index, which is equal to the ratio of the radius of the sphere in air to the radius of the sphere in the medium.

HUYGENS had experimentally to determine the ratio of OK (that is, N) to some radius of the spheroid, say the facet radius CG. The ratio CK/CG $= CG/CD$ does not however immediately give OK. One can nevertheless easily

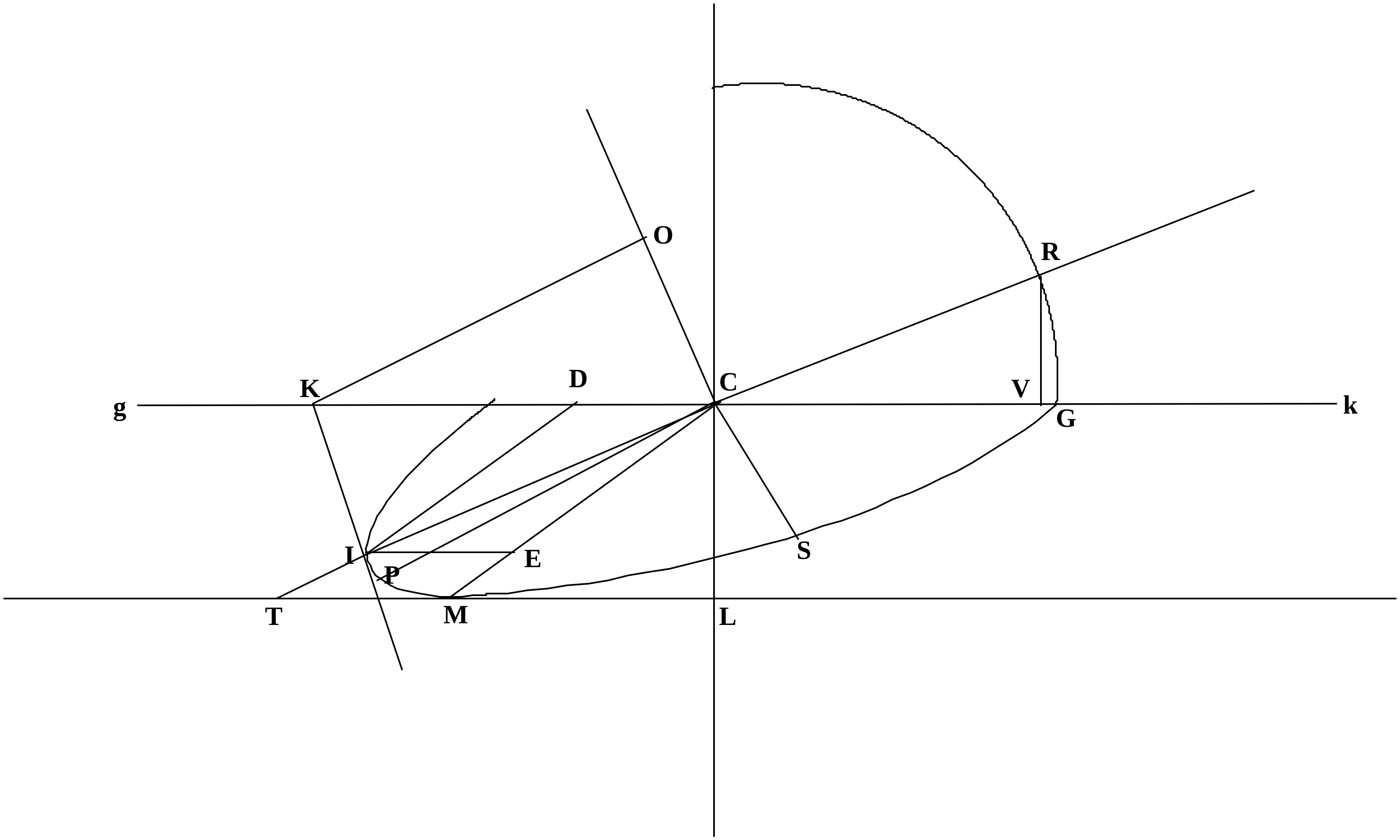

FIGURE 6

use this latter proportion to find OK, which HUYGENS accordingly did. In figure 6, with incident ray RC, and C as center, draw a circle gFG with radius CG, cutting RC in R. Drop RV, the perpendicular on gG, and mark a point D on CG such that OK/CG is equal to CV/CD. Draw DI parallel to CM, cutting the ellipse in I. Then, by use of the relation $CK/CG = CG/CD$, it is simple to show that CI is the extraordinary refraction. We shall call the relation $OK/CG = CV/CD$ the "law of proportions"[14].

The ratio OK/CG can now be determined by experiment. HUYGENS merely gave its value, but we can easily reconstruct how he might have obtained it. The problem is to measure CD and $CV = CG \sin \widehat{RCF}$. To measure this angle $<RCF$ of incidence HUYGENS would, no doubt, have used the technique described above. Finding CD is more difficult, but it could have been done either by reversing the calculation described below in section 5, or by using the equation for the ellipse, as described in Appendix III: either way CD is determined by the angle of refraction, which can be measured using HUYGENS' technique. Probably proceeding in one or the other of these two ways, HUYGENS found that CV/CD is slightly less than 8/5. Since CG is 98779, the law of proportions then gives $OK = 156962$ and so OK/CS is less than 5/3 by about 1/41. This value is so close to the ordinary index 5/3 that HUYGENS felt justified in concluding that CS "may be exactly" the radius of the ordinary sphere (in which case along the optic axis the ordinary and extraordinary rays have the same velocity). It must be understood that this posited equality was a result *not* of theory but of experiment. Had the two not been equal, HUYGENS would not have been worried.

5. The first 'confirmation'

As yet HUYGENS had not provided any measurements to confirm the theory. However, the equality between the radius of the ordinary sphere and the semi-minor axis of the spheroid implies that there is one ray in the principal section that will not be divided into two on entry, the single refraction being along the optic axis. Yet HUYGENS did not test this implication with the natural crystal. The reason he did not is quite simple: no ray can in fact be refracted along the optic axis when the plane of separation is a natural facet. The maximum

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[14] The proof is simple. By similar triangles, $CK/OK = RC/CV$, which is equal to CG/CV by construction. If CI is the refracted ray, then $CK/CG = CV/CD$. Consequently CK/CG is equal to both $OK|CV$ and $CG|CD$, so that $OK/CG = CV/CD$. This is equivalent to the following analytical law of refraction (HUYGENS of course did not give the law analytically):

$$\tan r_e = \tan \delta + \frac{CG^2 \sin i}{N \cdot CM \cos \delta \sqrt{1-(CG/N)^2 \sin^2 i}}.$$

Here δ is the deviation of the normal ray, i is the angle of incidence, and r_e is the angle of refraction. In HUYGENS' parameters (see section 3 above) CM is set to be a unit of 100000 in terms of which N has the experimentally determined value 156962. In WOLLASTON's and STOKES' parameters, where *vice versa* N is taken to be the unit, CM is the value to be measured (see Appendix I).

refraction for an ordinary ray occurs when the angle of incidence is 90°, where the angle of refraction is 36°53′. But the inclination of the optic axis to the vertical for a natural facet is 45°40′, or 8°47′ greater than the maximum of refraction. To test the implication, the crystal must be cut; as we shall see, HUYGENS did test cut crystals.

To this point, then, the theory is without confirmation, with the exception of the law of equal deviations, for which HUYGENS gave no sample measurements, and which in any case is independent of the intimate details of the spheroid. The sole measured optical values HUYGENS has thus far given are the ordinary index, the deviation of a normally incident ray, and the claim that the extraordinary refraction of a ray incident at 16°40′ is not deviated. HUYGENS used this last as the first measured 'confirmation' of the theory. It is of sufficient importance to follow HUYGENS' calculations here, since this was the only numerical confirmation he ever gave.

In figure 6, let a ray be so incident that $\angle RCg$ is 73°20′. To find the refraction, CI, HUYGENS used the law of proportions. First find $CV = RC\cos\widehat{RCg} = 28330$. By the law of proportions, $CV/CD = N/CG$, where HUYGENS has previously found – from what experiment he did not say – that N is equal to 156962. Hence CD is about 17828. By the properties of the ellipse, $(CG^2 - CD^2)/DI^2 = CG^2/CM^2$. Hence DI is about 98358. In the figure, $CE/EI = CM/MT$, where $CE = DI$, and $EI = CD$. Hence MT is about 18126. ML is $CM \sin 6°40′ = 11609$, and so $TL = 29736$. Since $LC = CM\cos 6°40′$, we find $LT/LC = \tan\widehat{LCT} = 0.2994$, so that $\angle LCT = 16°40′$, precisely as HUYGENS claimed. The agreement is so striking that it raises some questions which we shall discuss in section 7 below.

In the next section we shall examine HUYGENS' remaining confirmations. To do so it is essential to discuss how he was able to provide a geometrical construction for the refraction of an arbitrarily incident ray. That construction depends essentially upon a geometrical lemma which must be invoked whenever one chooses a new plane of incidence. Consequently we will begin with a discussion of it. Given the lemma, we can investigate HUYGENS' *law* for calculating the refraction when the plane of incidence is normal to the principal section. That will give us two laws – one for the principal section and one for the plane normal to it. (They are actually the same law with different parameters.) With these laws we will be able to understand what are perhaps the most startling confirmations of the theory. Finally, we shall examine the general construction itself to see whether it can be easily used to calculate an arbitrary refraction. It is essential to keep in mind the distinction between a construction for a ray and a law for calculating it: even though a geometrical construction may completely determine the ray's course, it may be so complicated that one cannot readily thereby calculate the path. And if calculation is impossible, then confirmation is out of the question.

6. Refraction outside of the principal section

In figure 7 $AEBFH$ is a piece of crystal whose upper facet $AEFH$ forms an equilateral parallelogram. The section of the spheroid $QGqgM$ by the facet is

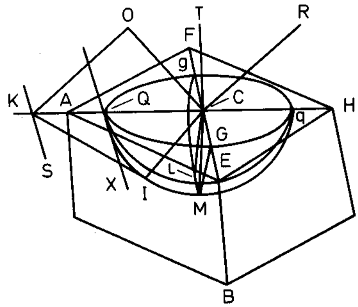

Fig. 7

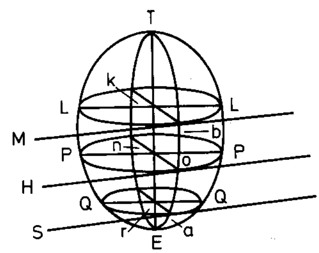

Fig. 8

QgqG. Point *E* is the vertex of the upper solid obtuse angle. Although the refraction lies in the plane of incidence when the latter is parallel to the principal section, as we shall see this is no longer true in any other plane. To deduce the refraction for other planes of incidence, HUYGENS found that he needed the following lemma, for which he provided a proof at the end of the chapter[15]: if a spheroid (figure 8) is touched by a line at a given point, and if, at two other points, the spheroid is touched by planes parallel to this line but not to each other, then the three points of contact (*b*, *o*, *a*) all lie on a single section (*ToE*) made in the spheroid by a plane that passes through its center. The proof cites proposition 15 of ARCHIMEDES' *Conoids and Spheroids* (in RIVAULT's edition of 1615: in the modern HEIBERG edition the relevant propositions are 13 and 14, corollary 2[16]).

Using the lemma HUYGENS deduced the law of refraction for a plane of incidence perpendicular to the principal section. In figure 7, the incident ray is *RC*, where *C* is the intersection of *AH* and *FE*. As before, we draw *OC*, the trace of the incident (plane) front in the plane of incidence, and *OK*, normal to *OC* and meeting the plane of separation at *K*, equal to the distance traveled by light in unit time in air.

Let *CL* be normal to the facet at *C*, with *L* lying on the spheroid, and let *CM* be the radius of the spheroid that lies along the extraordinary refraction of

[15] *Treatise*, pp. 103–5.

[16] See T.L. HEATH, *Works of Archimedes* (Cambridge, 1897; reprinted New York, n.d.), pp. 124–5. I am indebted to ALAN SHAPIRO for pointing this out to me.

a ray normally incident at C. Through CM and the line KH which bisects the acute facet angles draw a plane; that plane sections the ellipse QMg, and the angle, $\angle MCL$, between it and the normal CL is 6°40′. Through point K draw a line KS parallel to Cg. We know that the tangent plane which contains KS is parallel to QX, and that QX is parallel to the tangent plane at M. Consequently, the tangent plane containing KS, and the tangent plane at M, though not parallel to each other, are nevertheless parallel to the line tangent at Q to the ellipse sectioned by the facet. By the lemma, the point of contact I of the tangent plane containing KS lies on the ellipse QMq.

Knowing the plane of the refracted ray, CI, we use the same method as before for finding the position of I in that plane, *viz.* the requirement that, if KI is a tangent to the ellipse at I, and if K lies on the produced diameter qQ, then the point of contact I is the intersection of a line DI parallel to CM with the ellipse. Note that we can now easily prove the law of proportions for this plane of incidence in the same way as before, only here the ratio CV/CD is equal to N/CQ, not N/CG, so that, as HUYGENS remarked, "the proportion of the refraction for this section of the crystal" is less than the corresponding proportion in the principal section[17]. HUYGENS claimed successfully to have tested this conclusion, but again he provided no data.

With that claim, HUYGENS had, thus far, three distinct confirmations of the theory though detailed data were given only for the second:

1. The law of equal deviations was confirmed; no data were given.
2. The angle of incidence of the undeviated ray was, *to the minute*, what theory predicted.
3. The refraction was found to be less in the plane normal to the principal section (as plane of incidence) than in that section; neither data nor measurement technique were given.

Although HUYGENS did not describe how he carried out the third confirmation, the experiment is easily performed. In Appendix III I show how HUYGENS probably carried out the test, and what ratio he would have measured (the "proportion of the refraction"). The fact remains, however, that he provided no data, nor did he indicate how to perform the experiment. What we have provided is a reconstruction based on his measuring technique.

The elevation of the images

Seeking further confirmation, HUYGENS described a series of experiments designed to measure the visual heights of the images, and which therefore require both eyes for observation. These experiments are significant because, with the exception of his experiments with cut crystals (see below), they are the only ones which provide evidence for the construction without requiring calculation of angles of incidence for set angles of refraction. Consider (figure 9) the refractions of the point P at the surface Qq; the refracted rays join each eye to the points C, C' of Qq, where the rays PC, PC' from P emerge at equal but

[17] *Treatise*, p. 80.

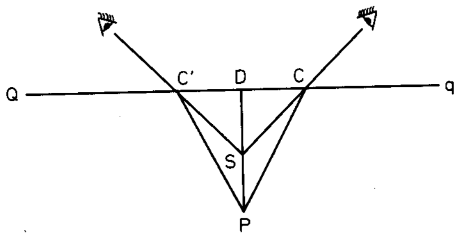

Fig. 9

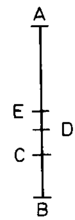

Fig. 10

opposite inclinations to the surface normal, *i.e.* the distances DC, DC' from a point D directly above P are equal. The image of P will appear above it at S, where the prolongations of the emergent rays intersect. In double refraction the extraordinary ray is, for the natural crystal, always less refracted than the ordinary ray, which implies that the equal distances DC, DC' are greater for the extraordinary than for the ordinary ray (for a given P and fixed positions of the eyes). Consequently the lines from C, C' to the eyes are less inclined to the normals in the former than in the latter case, and their intersection lies below the point S of elevation in ordinary refraction, *i.e.* the extraordinary image is always lower than the ordinary image.

One can easily use the law of proportions to calculate the elevations if the plane of incidence is either parallel or perpendicular to the principal section, but *only* for these two planes. Mark off a length AB equal to the height of the crystal (figure 10), and divide it at E, D, C such that AB/AE, AB/AD, AB/AC are, respectively, 5/3, 99324/70283, and 99324/66163. Then the law of proportions implies:

> ... by placing the eyes above the plane which cuts the crystal according to the shorter diameter of the rhombus, the regular refraction will lift up the letters to E; and one will see the bottom, and the letters over which it is placed, lifted up to D by the irregular refraction. But by placing the eyes above in the plane which cuts the crystal according to the longer diameter of the rhombus, the regular refraction will make them, at the same time, appear lifted up only to C; and in such a way that the interval CE will be quadruple the interval ED, which one previously saw[18].

Since HUYGENS' crystal was probably about 40 mm in height, the distances AB, AE, AC, AD would be, respectively, 40 mm, 24 mm, 28.3 mm, 25 mm, so that he had to distinguish points whose heights differ by as little as 1 mm. I have unsuccessfully tried to do so with both crystals. At best the experiment is extremely difficult to perform and highly unreliable. Nonetheless we have here a fourth claimed confirmation of the theory, though, again, no experimental data were supplied.

To this point HUYGENS had not given a general law for the refractions produced in any plane of incidence. At the conclusion of the section on image heights, however, he mentioned that, if the eyes are in any plane except the

[18] *Treatise*, p. 85.

principal section or the plane normal to it, then the height of the extraordinary image always lies between C and D. That assertion raises the question of what the generalized law might be and how the refractions may generally be constructed. This HUYGENS went on to show.

Refraction when the plane of incidence is arbitrary

The plane of refraction has in effect already been found for an arbitrary plane of incidence, since its position follows, as before, from HUYGENS' lemma (here by plane of refraction I mean that plane in which the refraction always lies for a given plane of incidence; this plane is normal to the crystal surface only when it is parallel to the principal section). In figure 11 (which appears on p. 87 of the THOMPSON translation), the ellipse HDE is the section of the ellipsoid by the crystal facet. RC is the incident ray, coming in in that plane, normal to the facet, which intersects the facet in BK. OC is normal to RC, and OK is the distance traveled by light in air in unit time. To determine the plane of refraction, draw a line HF that is parallel to the normal KT to KB in the facet and that touches HDE at H. Join CH and produce it to KT. Consider the plane that contains KT and that is tangent to the spheroid. Both KT, which lies in that plane, and HF, which does not, are parallel to the tangent plane to the spheroid at M. By the lemma, the plane that contains CM, CH is that plane in which refractions from the given incident plane lie.

But to determine the position of the refracted ray in the plane of refraction is not as easy here as it was before. In the two limiting cases (in which the plane of incidence is the principal section or is normal to it), the point H where the perpendicular KB touches the ellipse sectioned by the facet also lies along the line KB joining K to the center of that ellipse. For any other plane of incidence this is no longer true. Thus HF, parallel to KT, does not touch the ellipse at F but at H. Consequently one cannot find the position of I by constructing the tangent line from K to the ellipse sectioned by the plane of refraction. What one does not therefore have is an analog of the law of proportions. HUYGENS could not therefore test his construction in an arbitrary plane of incidence.

He accordingly concluded his tests with a discussion of phenomena for which the law of proportions remains valid and which occur when the crystal is cut. These are fifth and final confirmations of the theory. Like all but one of the first four confirmations, they are not supported by experimental data. In

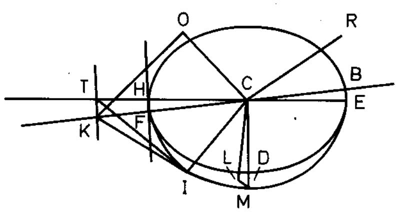

Fig. 11

summary, we have found that HUYGENS had accomplished the following in his analysis of double refraction:

(A) From two optical and one crystallographic measurements he deduced the parameters of the spheroid.

(B) He deduced the law of proportions for *calculating* the refractions in two limiting cases.

(C) He deduced the law of equal deviations, a law which possibly holds in all planes of incidence, but which is limited to two equally but oppositely inclined rays.

(D) He had shown how to *construct* the refraction of a ray for any plane of incidence, but he had not been able to deduce a general law for *calculating* it.

(E) He had provided six confirmations, only one of which (and, as we shall see in the next section, that one is suspect) involves experimental data:

(1) He had tested the law of equal deviations; no data given.

(2) He had confirmed the prediction of the angle of incidence for the non-deviated ray; the angle was given.

(3) He had confirmed that the refraction is not as great when the plane of incidence is normal to the principal section as when it is parallel to the section; no data given.

(4) He had tested the law of proportions in both planes for which it holds by calculating image heights; no data given.

(5) He claimed that the image heights in other planes of incidence lie between the two limiting cases, and that theory predicts the fact; neither data nor deduction are given.

(6) He had cut the crystal at various angles and had confirmed the predicted behavior of normally incident rays; no data given.

Of the five confirmations (1), (3), (4), (5) and (6) for which no details were supplied, data are *not* required in (6), because this implicitly embraces the prediction that a normally incident ray is not deviated for certain crystal shapes, but the remaining four do require data. If HUYGENS had clearly emphasized the laws of proportion and of equal deviations by removing them from the depths of his wave theoretical analysis, and if he had provided the data and the details of the measuring technique in all cases, then his contemporaries, and the scientists of the eighteenth century, would have been able to replicate his experiments and at least to test the *special* implications of the general construction. But they would not have been able to test the construction in all cases because HUYGENS had not been able to deduce a general analog of the law of proportions.

Appendix I: Snel's law and wave theory

A full comprehension of HUYGENS' construction for double refraction requires an understanding of how refractions are determined by means of wave surfaces. Consider ordinary refraction, which, we know, obeys SNEL's law. To construct a refraction one must first suppose with HUYGENS that each point on an expanding wave front is the source of an expanding spherical secondary 'wavelet'. The common tangent to the series of wavelets is, at any instant, the wave front. When the front encounters a refracting interface secondary wavelets are also generated there; these wavelets, however, expand more slowly in the refracting medium than outside it. One can easily demonstrate, as HUYGENS did, that the common tangent to these wavelets within the medium of refraction – the refracted front – obeys SNEL's law. Moreover, the demonstration, which proceeds by construction of the common tangent, leads directly to the following simple method for determining refractions. In the figure (figure AI) an incoming ray RC strikes the interface at C. To find its refraction first draw the spherical wave front in the refracting medium which originates at C as it appears after unit time. Then erect a normal CO in the plane of incidence to the ray RC. To

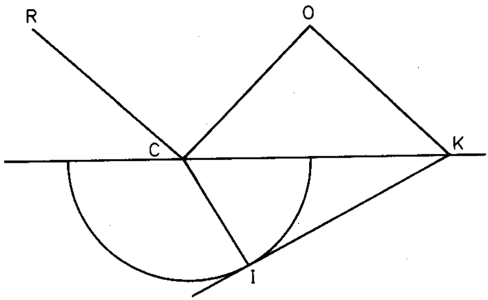

Fig. AI

CO erect a perpendicular OK, also in the plane of incidence, such that OK touches the interface at K and is equal in length to the distance traveled by light in unit time in the medium of incidence. Through point K, and again in the plane of incidence, draw a line KI which is tangent to the front at I. Then CI is the refraction of RC, and it is simple to show that the ratio $\sin i/\sin r$, where r is the angle of refraction, and i is the angle of incidence, is independent of incidence (SNEL's law). As we shall see, this method can be generalised to wave surfaces which are not spherical.

Appendix II: Experimental parameters

	STOKES	WOLLASTON	HUYGENS	MALUS
Index of the ordinary ray	4.962/3	4.97/3	5/3	4.96899/3
Deviation of the normal ray	6°12′	6°16′	6°40′	6°12′38″
Ratios of spheroid radii to N:				
semi-major generator CP	0.67431	0.67204	0.6692	0.674172
semi-minor generator CS	0.604595	0.6035	0.5951	0.604487
normal refraction CM	0.64371	0.6421	0.6371	0.643581
semi-minor facet radius CG	0.63708	0.6365	0.6293	0.636957
Interfacial angle	105°5′	105°5′	105°	105°5′
Inclination of optic axis	45°23′25″	45°23′25″	45°20′	45°23′25″

Note: STOKES' values for the radii of the spheroid are calculated from the fact that, for a plane of incidence normal to the optic axis, and therefore cutting a circle in the spheroid of radius CP, the index is a constant 1.483 for the extraordinary ray (the modern value for calcite is 1.486); with N taken as 1, CP is then the inverse of 1.483. For a plane of separation normal to the optic axis, STOKES found an extraordinary index of $1/CS=1.654$ at 0° incidence (the modern value for calcite is 1.658). STOKES' values were never published, but he gave them in his lectures on optics at Cambridge. JOHN AMBROSE FLEMING attended the lectures as a student in October 1878, and his notes are preserved (University College Library, London, Misc. Add. 122.34, pp. 187–229). I have preferred STOKES' measures to the modern values for calcite since we can be certain that STOKES used Iceland spar, and not a variant form of calcite. In any case the difference between STOKES' values and the modern ones produces less than a minute's difference in the refractions.

Appendix III: An example of how the construction might be tested for a plane of incidence normal to the principal section

1. The experiment

To begin, draw two mutually perpendicular lines $F'E'$, $A'H'$ intersecting at L (figure A III). Place the crystal on the intersection such that the image of L seen *via* a perpendicular ordinary ray shall coincide with point C of the upper facet

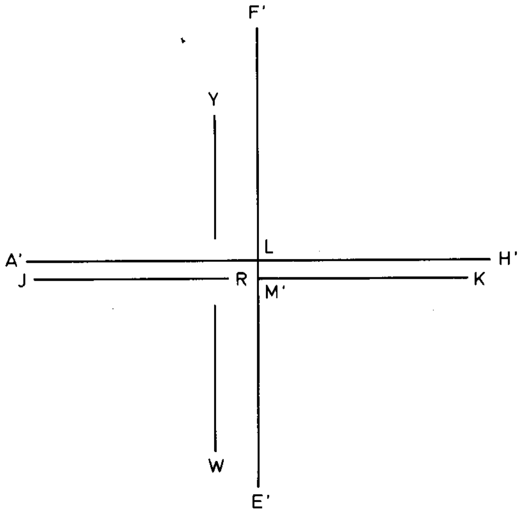

Fig. AIII

(figure 7). Rotate the crystal about L until, on its upper facet, FE—the bisector of the obtuse angles—is parallel to $F'E'$ (with F on the side of M towards F'), and so AH is parallel to $A'H'$. In this alignment a ray normally incident at C will be extraordinarily refracted to the point M' on the base of the crystal; whence M' is seen in coincidence with the ordinary image of L, *i.e.* at C.

From the construction for an incident plane $AHA'H'$, any point on the line JK, parallel to $A'H'$ and through M', can be seen in $AHA'H'$ by looking through C at some given, calculable angle to the plane; for a ray incident at C and in $AHA'H'$ is refracted into the plane $AHJK$, which is inclined to $AHA'H'$ at 6°40′ (according to HUYGENS). Mark a point R on JK, and align the eye, keeping it always in $AHA'H'$, until the extraordinary image of R is coincident with C. Draw the broken line YW, parallel to $F'H'$, which is seen without refraction and along which the image of R appears to lie. Then the tangent of the angle of incidence is the ratio of the distance from YW to $F'E'$ to the height of the crystal.

To test the law of proportions, *viz* $CV/CD = N/CQ$, for this section, we need to determine CV and CD. Of these, CV is equal to $CQ\sin 24°34'$, or 43667, while CD can be found in two ways, one of which was mentioned in section 4 above; simpler, however, is to find CI, the radius of the spheroid along which the refraction lies, and the angle β between CI and CM, and then to compute $CD = CI\sin\beta$.

With my own small crystal 17.5 mm high, I marked the point M' at the distance $LM' = 17.5\tan 6°40' \approx 2$ mm from $A'H$; then along JK I marked R about $5\frac{1}{2}$ mm from M'. When R is seen in plane $AHA'H'$ in coincidence with C, the tangent of the angle of incidence is very nearly 8/17.5, since the distance from YW to $F'E'$ is about 8 mm; whence the angle itself is about 24°34′. Since its tangent is $RM'/CM' = 0.31225$, where $CM' = \sqrt{LM'^2 + 17.5^2}$, the angle β is about 17°20′, while the equation $CI^2\cos^2\beta/CM^2 + CI^2\sin^2\beta/CQ^2 = 1$ yields CI

$=1/\sqrt{\cos^2\beta/CM^2+\sin^2\beta/CQ^2}$ to be about 100418. Hence CD is 19918. CV/CD is therefore in the ratio of 2.919/2. Since N is 156962, CV/CD should be 2.989/2 according to theory, which is quite close to the measured value considering the size of the crystal. Using my large, 39 mm crystal, I find a ratio of 2.93/2. A more complicated derivation of CD – but the one HUYGENS probably used – is to reverse the calculation described in section 5 above. That derivation bypasses the 'Cartesian' equation of the ellipse.