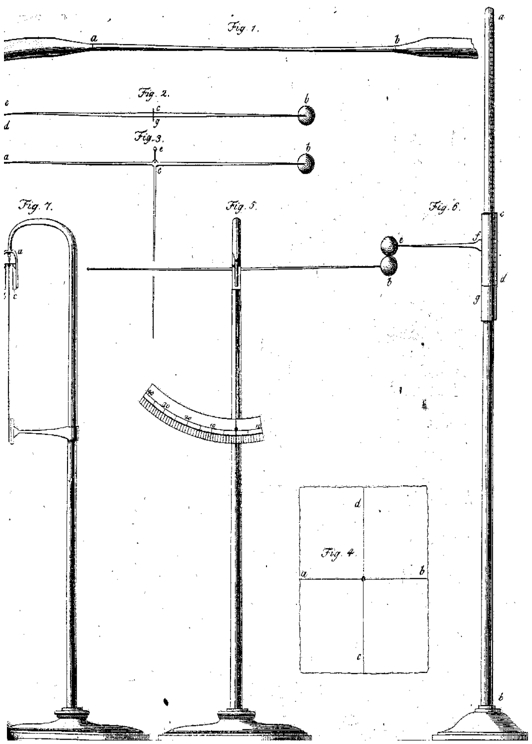
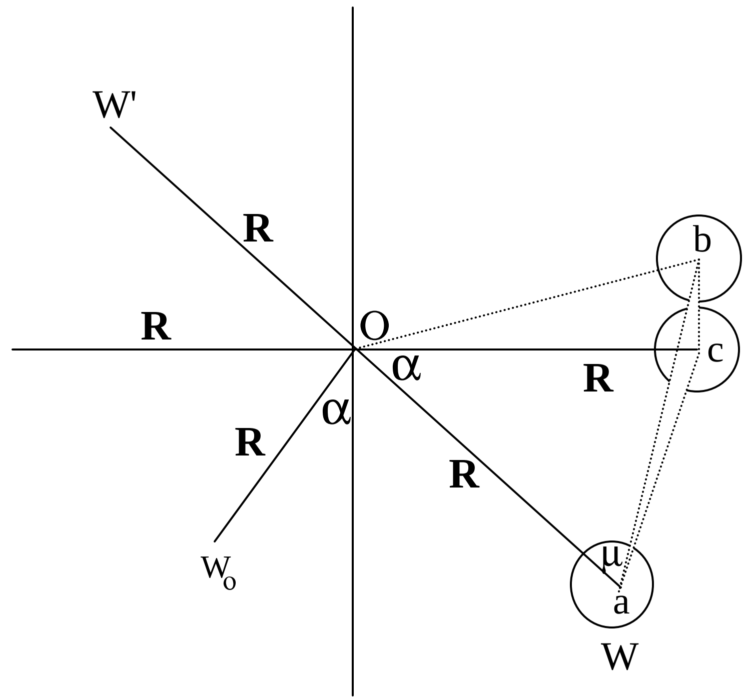
DEBATES OVER THE ELECTRIC FORCE LAW IN EARLY 19th CENTURY GERMANY





In 1807 and 1808 a German civil engineer in Berlin named Simon attempted to develop a device that would be suitable for lecture demonstrations of the law for electric repulsion, finding that Coulomb's torsion balance was much too sensitive for the purpose. Simon constructed one on very different principles, using a balance with a variable weight. His apparatus consists of a beam with arms of length R, to whose center a balancing 'tongue', also of length R, is orthogonally attached. Equal weights W are hung initially from the ends of the beam, and the tongue itself weighs Wo. The tongue moves over a scale marked in degrees, so that its displacement (α) from balance can be read off directly. Point c, at the end of the right-hand arm of the beam, bears a spherical weight W of diameter bc made of elderberry pith pith (**def.** The soft, spongelike, central cylinder of the stems of most flowering plants, composed mainly of parenchyma, here from the elder tree, which is what Coulomb also used.) Simon does not specify whether the weight (initally W) on the lefthand arm is also a pith sphere, but the diagram seems to indicate that it was not - probably Simon just balanced the righthand sphere with the little weights that he used to compensate the electrostatic repulsion.

Simon's length unit is the Zoll ($Zoll := 2.634 \cdot cm$). The length of the moment arm and the diameter of the pith sphere are:

$R := 4$

and

$bc := .4$

Simon gives only the balance formula that he used, and we will consider below his own method of computation. But first we will generate (as a German Gymanisum teacher named Egen did in 1825) an apparently general formula for calculating from experiment the exponent n for a repulsive force $\frac{k}{r^n}$ under the assumption that the force acts between the centers of spherically-symmetric masses (on which see below).

Simon's device makes it possible to avoid measuring the balancing weights W, and requires just one calibation to determine the weight Wo. The experiment consists of a series of observational triads. The fixed ball b and the ball at the end c of the beam are initially in uncharged contact; the lefthand end of the beam carries a weight W. Under these conditions, the beam is in neutral balance. In the first of the two measurements in each experimental triad, the balls b and c are charged by contact with something, after which ball c moves down until a new balance is reached under the combined action of the weights W (on the left), W (on the right), Wo (on the tongue), and the electric repulsion (on c); the tongue now points to an angle $\alpha 1$ on the scale. The distance between the center of the fixed ball b and the center of the moveable ball c increases to ab1, the line ab1 forming an angle $\mu 1$ with the righthand arm of the balance beam. Next, in the second two measurements of an experimental triad, Simon hangs a weight of magnitude δ to the lefthand end of the beam, which increases the weight there from W to W+δ. Ball c now moves upwards towards b until a new balance is achieved, whereupon the tongue now ponts to an angle $\alpha 2$, and the distance between the centers of balls b and c has decreased to ab2, with the angle $\mu 1$ changing to $\mu 2$.

Under these conditions, and assuming that the repulsion acts between the centers of the balls b and c, balance in the two measurements respectively requires:

$$(Wo)(R)\sin(\alpha 1)+(W)(R)\cos(\alpha 1)=(W)(R)\cos(\alpha 1)+\frac{k}{(ab1)^n}R\sin(\mu 1)$$

and

GENERAL BALANCE EQUATIONS

$$(Wo)(R)\sin(\alpha 2)+(W+\delta)(R)\cos(\alpha 2)=(W)(R)\cos(\alpha 2)+\frac{k}{(ab2)^n}R\sin(\mu 2)$$

Simon, and Egen later, both assume that the deflecting angles α are sufficiently small for the purposes of these formulae that they may replace their sines, and their cosines may be set to one. In addition, both assume that the distance between the balls may be replaced by an angular measure. The distance ab between the centers of the balls may accordingly be calculated approximately (we will see just how good below) as bc+Rα. In addition, we assume with Simon and Egen that ab remains nearly perpendicular to the the balance arm. Our two equations thereby become:

$$(Wo)(\alpha 1)=\frac{k}{(bc+R\alpha_1)^n}$$

and

$$(Wo)(\alpha 2)+(\delta)=\frac{k}{(bc+R\alpha_2)^n}$$

These combine to yield Egen's approximate (and, in general, theoretically problematic - see below) expression for the exponent n:

$$n := \frac{\left(\ln\left(\alpha_2 \cdot rad + \frac{\delta}{W_o}\right) - \ln(\alpha_1 \cdot rad)\right)}{\ln\left(\alpha_1 \cdot rad + \frac{bc}{R}\right) - \ln\left(\alpha_2 \cdot rad + \frac{bc}{R}\right)}$$

In order to determine n, the experiment requires measuring the fixed weight Wo on the end of the tongue, as well as the ratio of the ball diamater to the beam arm. As for the latter, Egen in 1825 simply took the ratio to be sufficiently small that the diameter bc can be replaced by the product R<bOc, which therefore gives a constant value (in angular measure) of $\frac{bc}{R} = 5.73\,deg$. In order to find Wo, Simon simply measured the weight necessary to produce a unit angular deflection on an uncharged balance. Again assuming small angles, our belance equation in the absence of electric repulsion immediately yields Wo equal to the deflection. Simon used the Gran as a weight unit, where $Gran \equiv .8 \cdot gm$. He found that 1/250 of a Gran weight produces a deflection of one degree. He accordingly replaced $\frac{\delta}{W_o}$ by the number of 1/250 Grans in δ – that is, Simon gave his tabular weights in degrees, which therefore convert to mg through the factor $\left(\frac{1Gran}{250}\right) = 3.2\,mg$ per degree. Simon needed to perform a single calibrating measurement for the fixed weight hung on the tongue of the device, and Egen needed to know as well the ratio of the ball diameter to the beam arm (Simon, we will see, did not add the diameter to his angular distances).

The folowing are exact expressions for the distances ab and the angles μ as functions of the deflections x:

$$ab(x) := \left[4 \cdot R^2 \cdot \left(\sin\left(\frac{x}{2}\right)\right)^2 + bc^2 + 4 \cdot R \cdot bc \cdot \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)\right]^{\frac{1}{2}}$$

$$\sin\mu(x) := \frac{\left[2 \cdot R \cdot \cos\left(\frac{x}{2}\right) \cdot \sin\left(\frac{x}{2}\right) + bc \cdot \left[\left(\cos\left(\frac{x}{2}\right)\right)^2 - \left(\sin\left(\frac{x}{2}\right)\right)^2\right]\right]}{ab(x)}$$

Simon's experiments provide triplets of numbers for each time he charged the balls: a triplet consists of the intial deflection, the deflection after weight is added to the left arm of the balance beam, and the added weight. The equations below provide, respectively, arrays for the initial (1) and final (2) distances and angles, as well as an exact formula for n in terms of them (although, as we will see, the formula is not, in general, theoretically appropriate). The δi are the number of 1/250 Gran units in the weight added to the left arm of the beam:

$ab1_i := ab\left[\left(\alpha^{\langle 1\rangle}\right)_i \cdot deg\right]$ and $sin\mu1_i := sin\mu\left[\left(\alpha^{\langle 1\rangle}\right)_i \cdot deg\right]$ from the measured initial angles $\left(\alpha^{\langle 1\rangle}\right)_i \cdot deg$

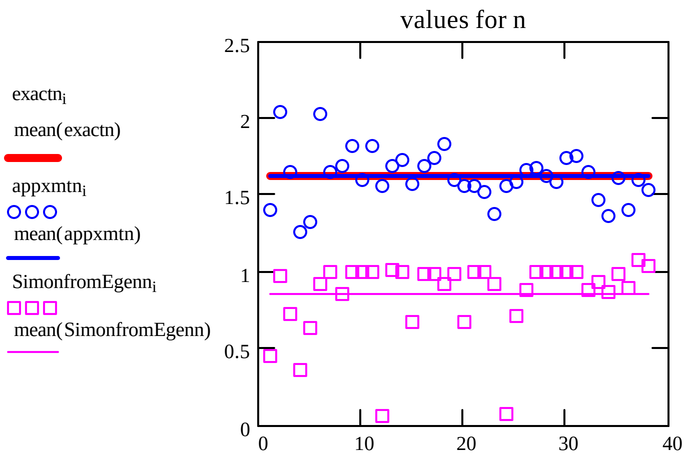
$ab2_i := ab\left[\left(\alpha^{\langle 2\rangle}\right)_i \cdot deg\right]$ and $sin\mu2_i := sin\mu\left[\left(\alpha^{\langle 2\rangle}\right)_i \cdot deg\right]$ from the measured second angles $\left(\alpha^{\langle 2\rangle}\right)_i \cdot deg$

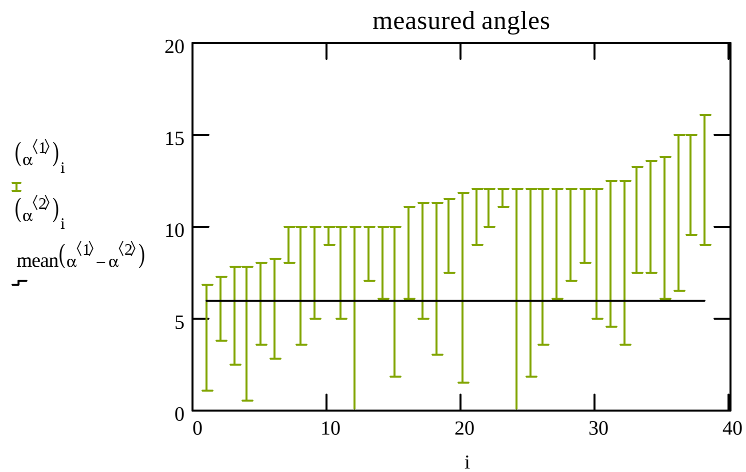
$$exactn_i := \ln\left(\frac{sin\mu1_i}{sin\mu2_i}\right) + \frac{\ln\left[\frac{\left[\sin\left[\left(\alpha^{\langle 2\rangle}\right)_i \cdot deg\right] + (\delta)_i \cdot deg \cdot \cos\left[\left(\alpha^{\langle 2\rangle}\right)_i \cdot deg\right]\right]}{\sin\left[\left(\alpha^{\langle 1\rangle}\right)_i \cdot deg\right]}\right]}{\ln\left(\frac{ab1_i}{ab2_i}\right)}$$

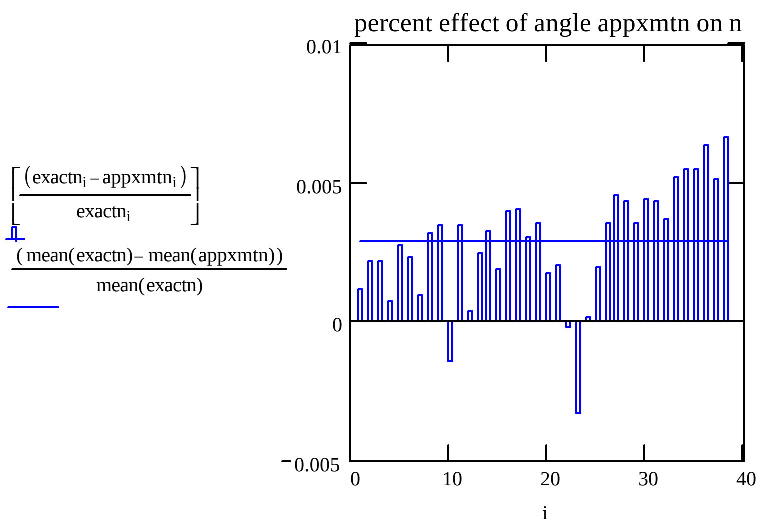
where δ_i is the array of added weights. The expression for the approximate values of the exponents (wherein distances are replaced by angles) is then:

$$appxmtn_i := \frac{\left[\ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \delta_i\right] - \ln\left[\left(\alpha^{\langle 1\rangle}\right)_i\right]\right]}{\ln\left[\left(\alpha^{\langle 1\rangle}\right)_i + degs\left(\frac{bc}{R}\right)\right] - \ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + degs\left(\frac{bc}{R}\right)\right]}$$

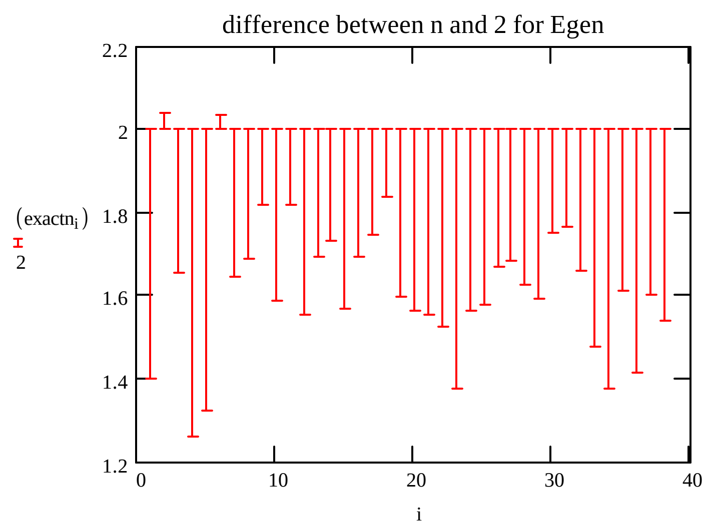
Simon ignored the diameter of the ball. In terms of Egen's formula (but not his) Simon's procedure therefore yields the following expression for the array of indexes: $SimonfromEgenn_i := \dfrac{\left[\ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \delta_i\right] - \ln\left[\left(\alpha^{\langle 1\rangle}\right)_i\right]\right]}{\ln\left[\left(\alpha^{\langle 1\rangle}\right)_i + nodiam\right] - \ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + nodiam\right]}$







mean(2 − exactn) = 0.379



$\text{mean(exactn)} = 1.621$ $\text{mean(appxmtn)} = 1.616$ $\text{mean(SimonfromEgenn)} = 0.852$ $\text{ntheoryforinvsqr} := 2$

$\text{stdev(exactn)} = 0.167$ $\text{stdev(appxmtn)} = 0.166$ $\text{stdev(SimonfromEgenn)} = 0.244$

$$\text{Egenexpmnttotheoryinvsqr} := \left(1 - \frac{\text{mean(appxmtn)}}{\text{ntheoryforinvsqr}}\right)\cdot 100$$

so the mean percentage ratio of experiment to theory for the Coulomb law (where n should be 2) is $\text{Egenexpmnttotheoryinvsqr} = 19.195$

Note that this method of computing the experiment-theory ratio is not as informative as it might be since here we have first calculated n and then compared the result with a theoretical value of 2, for which we required logarithms. Simon himself did not work with logarithms. Instead, he considered only two cases: that n might be 2, as Coulomb had it (in Simon's understanding), or that n might be 1, which allowed him directly to compare predicted and measured weights. We can compare the effects of the difference in computational techniques between Simon and Egen. Here $\delta\text{Coulombcenters}$ is the weight that Simon should have measured, given Coulomb's law between ball centers:

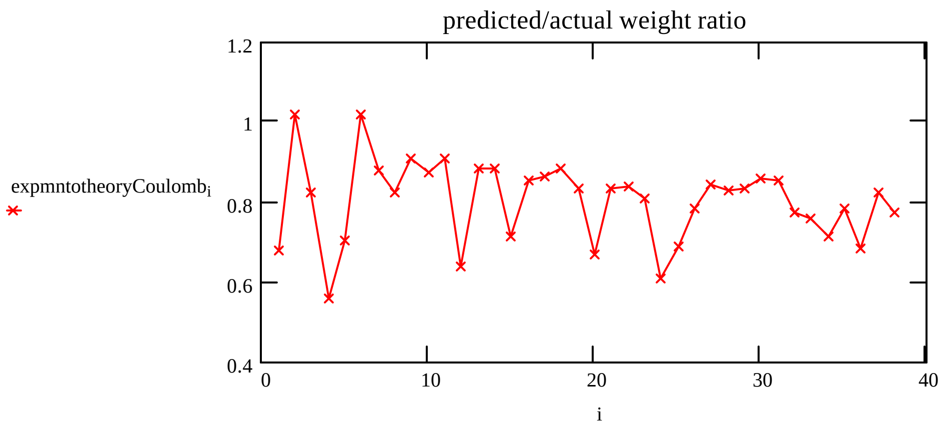
$$\delta\text{Coulombcenters}_i := -\text{degtomg}\left[\left(\alpha^{\langle 2\rangle}\right)_i - \frac{\left[\left(\alpha^{\langle 1\rangle}\right)_i\cdot\left[\left(\alpha^{\langle 1\rangle}\right)_i + \text{degs}\left(\frac{bc}{R}\right)\right]^2\right]}{\left[\left(\alpha^{\langle 2\rangle}\right)_i + \text{degs}\left(\frac{bc}{R}\right)\right]^2}\right]$$

$$\text{expmntotheoryCoulomb}_i := \left(\frac{\text{actualweightadded}_i}{\delta\text{Coulombcenters}_i}\right)$$

actualweightadded·mg =

	1
1	48
2	32
3	48
4	64
5	32
6	64
7	14.4
8	67.2
9	48
10	6.72
11	48
12	153.6
13	23.36
14	33.92
15	96.96
16	44.8

mg



$\text{mean(expmntotheoryCoulomb)} = 0.802$ $\qquad$ $\text{stdev(expmntotheoryCoulomb)} = 0.1$

so the mean percentage ratio of experiment to theory is $\text{SimonCoulomb} := 100 \cdot (1 - \text{mean(expmntotheoryCoulomb)})$, whence $\text{SimonCoulomb} = 19.777$ (previously, using logarithms, we had $\text{Egenexpmnttotheoryinvsqr} = 19.195$). Consequently there is little difference in computational accuracy between Simon's and Egen's methods. Egen's method computes n, and then finds its difference from 2, whereas Simon's method computes what the added weights should be assuming that n is two. Both computational methods produce about a $\frac{(\text{SimonCoulomb} + \text{Egenexpmnttotheoryinvsqr})}{2} = 19.486$ percent difference from Coulomb, using in both cases the center-center distances.

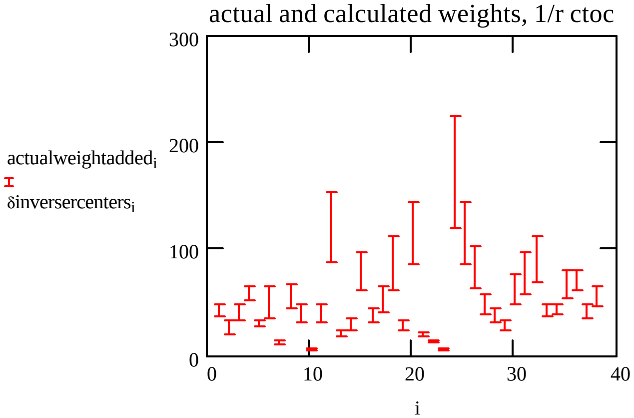
For future use, we calculate, using Simon's method, what the added weight should be *if we assumed that the force acted between the balls' centers but that it varied recipriocally with that distance*, and not with its square (see below). In that case:

$$\delta\text{inversercenters}_i := -\text{degtomg}\left[\left(\alpha^{\langle 2\rangle}\right)_i - \frac{\left[\left(\alpha^{\langle 1\rangle}\right)_i \cdot \left[\left(\alpha^{\langle 1\rangle}\right)_i + \text{degs}\left(\frac{bc}{R}\right)\right]\right]}{\left[\left(\alpha^{\langle 2\rangle}\right)_i + \text{degs}\left(\frac{bc}{R}\right)\right]}\right]$$

$$\text{expmntotheory}_i := \left(\frac{\text{actualweightadded}_i}{\delta\text{inversercenters}_i}\right)$$

$\text{mean(expmntotheory)} = 1.474$ $\quad$ $\text{stdev(expmntotheory)} = 0.192$

$\text{Simonexpmntotheoryinvdistcentocent} := -100 \cdot (1 - \text{mean(expmntotheory)})$



Here the mean percentage ratio of experiment to theory is $\text{Simonexpmntotheoryinvdistcentocent} = 47.357$ which is a difference of $\text{Simonexpmntotheoryinvdistcentocent} - \text{Egenexpmnttotheoryinvsqr} = 28.162$ percent from the inverse square law's. That is, if we use a center-to-center inverse distance law, then we find that it is much worse, given Simon's data, than a center-to-center inverse square law, which is what Egen discovered as well. We will consider below the results with a point-to-point inverse distance law

Simon himself found a much greater difference than we have between experiment and the Coulomb law as he understood it. *He reckons that the appropriate distance to use in calculating the repulsion, even for the Coulomb law, is the distance between the low point on the upper ball and the high point on the lower one* - in other words, the closest distance between the surfaces. Simon accordingly sets the distance bc to zero in his computation, finding as a result that the following relation should hold if "Coulomb's" inverse-square repulsion is correct:

$$\delta\text{Coulombsurfaces}_i := \left(\frac{-1 \cdot 800}{250}\right) \cdot \left[\left(\alpha^{\langle 2\rangle}\right)_i - \frac{\left[\left(\alpha^{\langle 1\rangle}\right)_i \cdot \left[\left(\alpha^{\langle 1\rangle}\right)_i + \text{nodiam}\right]^2\right]}{\left[\left(\alpha^{\langle 2\rangle}\right)_i + \text{nodiam}\right]^2}\right] \qquad \text{expmntotheory}_i := \left(\frac{\text{actualweightadded}_i}{\delta\text{Coulombsurfaces}_i}\right)$$

And here we find that:

$$\text{mean(expmntotheory)} = 0.352 \qquad \text{stdev(expmntotheory)} = 0.185$$

$$\text{Simonexpmntotheoryinvsqrsurftosurf} := 100 \cdot (1 - \text{mean(expmntotheory)}) \qquad \text{Simonexpmntotheoryinvsqrsurftosurf} = 64.776$$

This is obviously an extremely poor result for an inverse-square surface-to-surface law. Simon himself conceived that the force between the surfaces might vary reciprocally with the first power of the distance, instead of its square, which produces the following result according to his method of calculation:

$$\delta\text{Simonsurfaces}_i := -\text{degtomg} \cdot \left[\left(\alpha^{\langle 2\rangle}\right)_i - \frac{\left[\left(\alpha^{\langle 1\rangle}\right)_i \cdot \left[\left(\alpha^{\langle 1\rangle}\right)_i + \text{nodiam}\right]\right]}{\left[\left(\alpha^{\langle 2\rangle}\right)_i + \text{nodiam}\right]}\right]$$

$$\text{expmntotheory}_i := \left(\frac{\text{actualweightadded}_i}{\delta\text{Simonsurfaces}_i}\right)$$

$$\text{mean(expmntotheory)} = 0.828 \qquad \text{stdev(expmntotheory)} = 0.281 \qquad \text{Simonexpmntotheoryinvdistsurftosurf} := 100 \cdot (1 - \text{mean(expmntotheory)})$$

Here we find that the mean percentage difference between Simon's theory and experiment becomes $\mathrm{Simonexpmntotheoryinvdistsurftosurf} = 17.159$, which vastly surpasses the inverse-square surface-to-surface law and in fact compares quite well with the differences that the center-center distance using Coulomb's law produces, namely $\mathrm{SimonCoulomb} = 19.777$. Indeed, Simon's inverse-distance law between surfaces is apparently somewhat better!

What does this mean? Let's suppose that Simon had actually thoroughly understood - as he clearly did not - that the Coulomb law must be calculated between centers. Suppose however that he himself reckoned that repulsive actions should actually occur between nearest surface points - reasoning, e.g., that electric atmospheres surround the surface like balloons, with the action taking place at the points where the balloons come into contact with one another. Such an image leads easily to an inverse-distance 'repulsion' between the surfaces on the (somewhat vague) analogy of Boyle's law. Simon would then have been comparing two significantly different theoretical structures with one another and not, say, the question (which is how Egen understood it) of whether the center-center force varies as the reciprocal of the distance or as its square. Simon could then have argued, on the basis of his measurements, that the atmosphere theory works marginally better than the Coulomb theory, which concentrates all electricity on the surface and works by action at a distance rather than by atmospheric expansion. There is more.

Both theories raise questions concerning instrumental accuracy in respect to the approximations that must be made to facilitate computation - or that must be made because there is simply no known way to calculate deviations. But there is a significant difference here. In Simon's case there was no method for calculating what the effect would be of the balls' electric atmospheres pushing one another aside. Gilbert, clearly thinking in Coulombian terms, had written Simon that the experiment might be affected considerably at small distances by the displacement of the centers of the electric substance from the centers of the balls. In fact, Gilbert (apparently not perceiving that Simon had used the surface-surface distance in his computation) assigned the conflict between experiment and Coulomb repulsion to this effect (which is in fact altogether insignificant in respect to errors of measurement). Simon disagreed, evidently reasoning that the effect would be too small to account for the large discrepancies he had found. However, Simon understood Gilbert as referring to the central points of "electric atmospheres", and so he decided to avoid the effect altogether by doing further experiments (not reported in the body of the paper but only in a remark quoted by Gilbert in a note to Simon's 1808 paper) using flat discs instead of spheres - in which case the 'atmospheres' would presumably extend more or less uniformly over the disks' surfaces, and the distances could be taken always between the surfaces (whereas, with spheres, the atmospheres would deform different amounts at different points of their surfaces, though Simon says nothing about this).

Here we spy a signal, instrumentally-significant difference between Simonian 'atmospheres' and Coulombian repelling fluids. Specifically, Coulomb, like Poisson and other French analysts after him, conceived the electric fluid to form very thin layers near the surface of a conductor. The 'thickness' of a layer at a given point would be a measure of the quantity of charge per unit area there (assuming a uniform density). One could - and this is precisely what Coulomb did - develop an experimental system in which a small disk could be used to pick up charge at an arbitrary point of a conductor, and the resulting force measured for a standard electrometric distance in Coulomb's device. Then ratios of these forces would constitute measures of relative charge densities, thereby permitting an experimental determination of the manor in which electricity distributes over given surfaces under given conditions. This is what Poisson would later calculate using spherical harmonics (in 1811), and it is also the very thing that atmospheric theories have nothing to say about: since they do not localize electricity on surfaces, the very phrase "charge density" has little meaning, and the kinds of things that might thereby be computed (such as the effective net force between a pair of spheres of unequal size with their centers at given distances from one another) remained essentially outside atmosphere theories' abilities.

Egen did not interpret the issue as one between Simon's theory (atmospheres, surface distances) and Coulomb, but rather as the question of whether the force between electric point masses runs as the inverse-square or as the inverse distance. Supposing it to run as the inverse-distance, we do not know without a calculation, not performed by Egen, what the force would be as a function of the distance between the centers of the balls - since only an inverse-square force reduces the action of a spherical distribution to the same result as if the material were concentrated at the center, and the centers acted directly on one another.

Calculation shows that a point-to-point inverse distance action produces the following resultant force at a distance r from the center of a spherical distirbution with diameter d for a total mass M:

$$\text{force} = \left(\frac{M}{r^2 \cdot d}\right) \cdot \left[\left(4 \cdot r^2 - d^2\right) \cdot \ln\left(\frac{2 \cdot r + d}{2 \cdot r - d}\right) + 4 \cdot d \cdot r\right] = Mg(r)$$

In effect, a point-to-point inverse distance force produces net forces between spherical distributions that may be considered to vary with the inverse-square distance from their centers, with the 'mass' being multiplied by a factor that depends on the center-center distance and on the sphere's diameter. This expression leads (as before) to the following relationship that can be used with Simon's data:

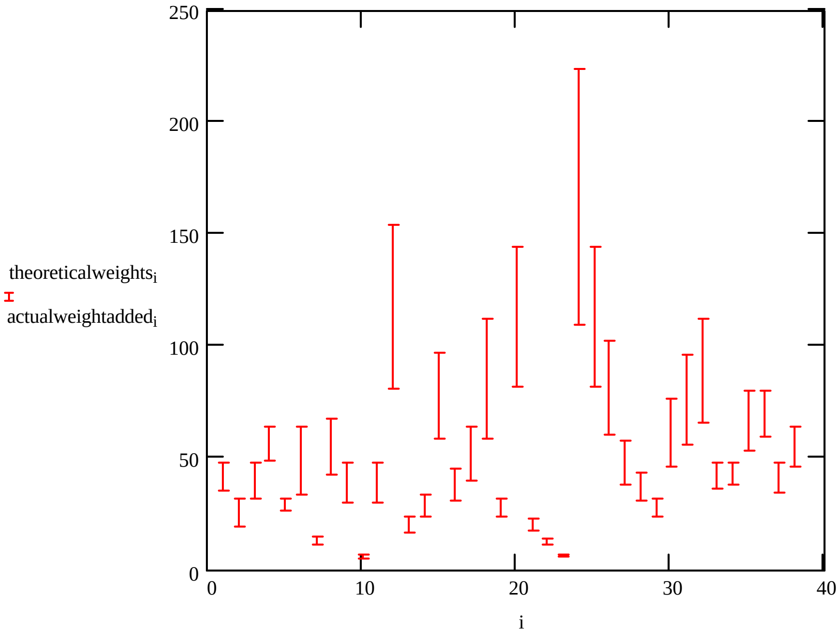
predicted weights = alpha1(g(r2)/g(r1)-alpha2)

We can calculate both sides of the equation from experimental data and see how much they differ.

$$g(x) := \frac{\left[\left(4\cdot x^2 - bc^2\right)\cdot \ln\left(\frac{2\cdot x + bc}{2\cdot x - bc}\right) + 4\cdot bc\cdot x\right]}{x^2\cdot bc}$$

$$theoreticalweights_i := degtomg\left[\left[\frac{g\left[ab\left[\left(\alpha^{\langle 2\rangle}\right)_i\cdot deg\right]\right]}{g\left[ab\left[\left(\alpha^{\langle 1\rangle}\right)_i\cdot deg\right]\right]}\right]\cdot\left[\left(\alpha^{\langle 1\rangle}\right)_i\right] - \left(\alpha^{\langle 2\rangle}\right)_i\right]$$

$$expmntotheory_i := \frac{actualweightadded_i}{theoreticalweights_i}$$



$$|mean(expmntotheory)| = 1.519$$

So the mean percentage ratio of experiment to theory is here:

$$correctexpmntotheoryinvdist := -100\cdot(1 - |mean(expmntotheory)|)$$

$$correctexpmntotheoryinvdist = 51.862$$

Precision demands that we compare this with the inverse-square law.

Egen had assumed that he could use his expression to calculate n from Simon's data. He was incorrect: except in the special case that n is actually 2, Egen's formula fails because it presumes that the appropriate distance to use is always the one between the centers of the spheres. We have just seen that this does not hold in the case of n=1, and in fact it fails for any n other than 2, since the following is the general expression for the inverse nth power force of a spherically-symmetric mass, excepting n=1:

$$\text{force} = \left(\frac{M}{2r^2 \cdot d}\right) \cdot \left[\frac{\left(r^2 - \frac{bc^2}{4}\right)}{1-n} \cdot \left[\left(r + \frac{d}{2}\right)^{1-n} - \left(r - \frac{d}{2}\right)^{1-n}\right] + \left(\frac{1}{3-n}\right) \cdot \left[\left(r + \frac{d}{2}\right)^{3-n} - \left(r - \frac{d}{2}\right)^{3-n}\right]\right] = \text{Mg(r)}$$

We can use this result to compare Simon's data among a possible sequence of n, say for n running from 1.1 to 2 in steps of .1 For each possible value of n, the weights measured (δ) should again be equal to $(-\alpha_2) + \alpha_1 \cdot \frac{g(r_2)}{g(r_1)}$

$l := 1, 2 .. 10$

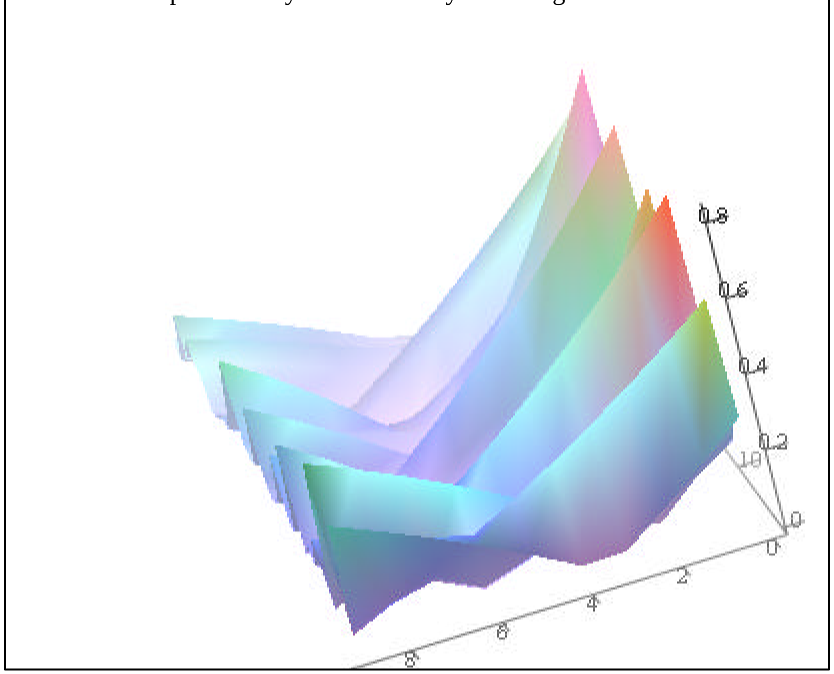
$$g(x,y) := \frac{\left[\frac{\left(x^2 - \frac{bc^2}{4}\right)}{1-y} \cdot \left[\left(x + \frac{bc}{2}\right)^{1-y} - \left(x - \frac{bc}{2}\right)^{1-y}\right] + \left(\frac{1}{3-y}\right) \cdot \left[\left(x + \frac{bc}{2}\right)^{3-y} - \left(x - \frac{bc}{2}\right)^{3-y}\right]\right]}{2 \cdot x^2 \cdot bc}$$

$$\text{theoreticalweights}_{i,l} := \text{degtomg}\left[\left[\frac{g\left[ab\left[\left(\alpha^{\langle 2\rangle}\right)_i \cdot deg\right], 1 + .1 \cdot l\right]}{g\left[ab\left[\left(\alpha^{\langle 1\rangle}\right)_i \cdot deg\right], 1 + .1 \cdot l\right]}\right] \cdot \left[\left(\alpha^{\langle 1\rangle}\right)_i\right] - \left(\alpha^{\langle 2\rangle}\right)_i\right]$$

$$\text{experimentalarray}_{i,l} := \text{actualweightadded}_i$$

$$\text{expmntotheory}_{i,l} := \frac{\text{experimentalarray}_{i,l}}{\text{theoreticalweights}_{i,l}} \qquad \text{difffrom1}_{i,l} := \left|1 - \text{expmntotheory}_{i,l}\right|$$

expmnt/theory diff from unity for a range of indexes



difffrom1

$\mathrm{Simonarrayofexpmnttotheory}_l := \mathrm{mean}\left(\mathrm{expmntotheory}^{\langle l \rangle}\right)$

$\mathrm{differencefrom1array}_l := \left|1 - \mathrm{Simonarrayofexpmnttotheory}_l\right|$

differencefrom1array =

	1
1	0.419
2	0.328
3	0.244
4	0.166
5	0.094
6	0.027
7	0.034
8	0.092
9	0.145
10	0.194

$\mathrm{Simonbestexpmtotheory} := 100\,\mathrm{min}(\mathrm{differencefrom1array})$

$\mathrm{fullformulaexpmntotheoryinvsqr} := 100 \cdot \left(1 - \left|\mathrm{mean}\left(\mathrm{expmntotheory}^{\langle 10 \rangle}\right)\right|\right)$

The graph shows quite clearly that the best fit between theory and experiment for Simon's data occurs in the vicinity of index value 6, corresponding to n of 1.6 (where the ratio is closest overall to one). This is interestingly consistent with our previous computation, in which we used Egen's formula for calculating n (which also produced a mean value from the data of 1.6). Specifically, we find here that the mean difference ratio of experiment to theory for n equal to 1.6 is only $\mathrm{Simonbestexpmtotheory} = 2.75$ percent. This is considerably better than the $\mathrm{correctexpmntotheoryinvdist} = 51.862$ percent that we just found for a point-to-point inverse distance law. Apparently Egen's method (in which he calculated the value of n by assuming that, whatever the force law might be, one could assume the unaltered values of the masses to be located at the centers of the balls) works perfectly well for Simon's apparatus. The appropriateness of the approximation must certainly be a function of the ratio between the ball diameter and the distance between the balls' centers.

Turn now to compare with the inverse-square law. Previously, using Simon's method of computation (comparing predicted and measured weights), we had found, for the inverse-square and the center-to-center inverse distance laws respectively, mean percentage ratios of experiment to theory of $\mathrm{SimonCoulomb} = 19.777$ and $\mathrm{Simonexpmntotheoryinvdistcentocent} = 47.357$, yielding a difference between the two of $\mathrm{Simonexpmntotheoryinvdistcentocent} - \mathrm{SimonCoulomb} = 27.58$ percent. For n equal to 2 (the Coulomb law), we find from the present method of computation a mean percentage ratio between experimental and theoretical values of $\mathrm{fullformulaexpmntotheoryinvsqr} = 19.439$ percent; the comparable value for a point-to-point inverse-distance law is, again, $\mathrm{correctexpmntotheoryinvdist} = 51.862$ percent, yielding now a difference between the two of $\mathrm{correctexpmntotheoryinvdist} - \mathrm{fullformulaexpmntotheoryinvsqr} = 32.423$ percent. The difference between using a point-to-point inverse-distance law and just assuming an action between centers with masses concentrated there is insiginficant, though the inverse-distance theory does becomes marginally worse than before using a point-to-point calculation..

using Egen's (theoretically unsupported) method for calculating n

without a small angle approximation & without ignoring the balls' diameters:

$\text{mean(exactn)} = 1.621$ for n

% diff expmnt and theory is $100\cdot\left(1 - \frac{\text{mean(exactn)}}{2}\right) = 18.961$

with a small angle approximation & without ignoring the balls' diameters:

$\text{mean(appxmtn)} = 1.616$ for n

% diff expmnt and theory is $100\cdot\left(1 - \frac{\text{mean(appxmtn)}}{2}\right) = 19.195$

with a small angle approximation & ignoring the balls' diameters:

$\text{mean(SimonfromEgenn)} = 0.852$ for n

% diff expmnt and theory is $100\cdot(1 - \text{mean(SimonfromEgenn)}) = 14.771$

using Simon's method for comparing predicted and measured weights & ignoring the balls' diameters

for an inverse-distance law the percentage difference between experiment and theory is $\text{Simonexpmntotheoryinvdistsurftosurf} = 17.159$

for an inverse-square distance law the percentage difference between experiment and theory is $\text{Simonexpmntotheoryinvsqrsurftosurf} = 64.776$

using Simon's method for comparing predicted and measured weights & without ignoring the balls' diameter

1.) for n=1 and n=2

for a center-to-center inverse-distance law the percentage difference between experiment and theory is $\text{Simonexpmntotheoryinvdistcentocent} = 47.357$

for a point-to-point inverse-distance law the percentage difference between experiment and theory is $\text{correctexpmntotheoryinvdist} = 51.862$

for an inverse-square distance law the percentage difference between experiment and theory is $\text{fullformulaexpmntotheoryinvsqr} = 19.439$

2.) for n from 1 to 2 in steps of .1 the best fit between theory and experiment yields n equal to 1.6 , where the difference between experiment and theory is $\text{Simonbestexpmtotheory} = 2.75$

ERROR ANALYSES:

$$\mathrm{errorfromalpha1}_{i,j} := \left[\left[\frac{\left[\ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \delta_i\right] - \ln\left(\mathrm{erroralpha1}_{i,j}\right)\right]}{\ln\left(\mathrm{erroralpha1}_{i,j} + \mathrm{degs}\left(\frac{bc}{R}\right)\right) - \ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right]}\right] - \frac{\left[\ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \delta_i\right] - \ln\left[\left(\alpha^{\langle 1\rangle}\right)_i\right]\right]}{\ln\left[\left(\alpha^{\langle 1\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right] - \ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right]}\right]$$

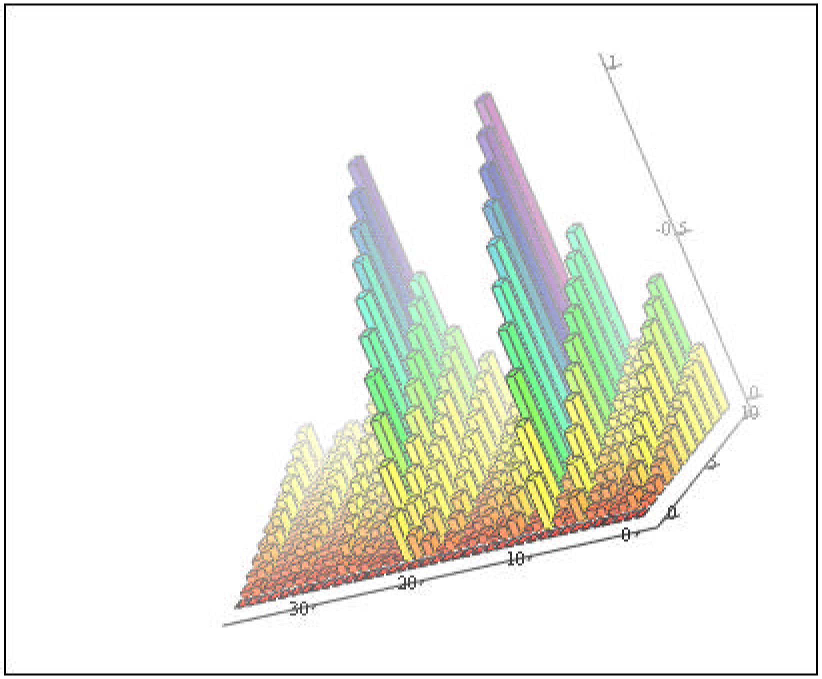
$$\mathrm{errorfromalpha2}_{i,j} := \left[\frac{\left[\ln\left(\mathrm{erroralpha2}_{i,j} + \delta_i\right) - \ln\left[\left(\alpha^{\langle 1\rangle}\right)_i\right]\right]}{\ln\left[\left(\alpha^{\langle 1\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right] - \ln\left(\mathrm{erroralpha2}_{i,j} + \mathrm{degs}\left(\frac{bc}{R}\right)\right)}\right] - \frac{\left[\ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \delta_i\right] - \ln\left[\left(\alpha^{\langle 1\rangle}\right)_i\right]\right]}{\ln\left[\left(\alpha^{\langle 1\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right] - \ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right]}$$

$$\mathrm{errorfromdelta}_{i,k} := \left[\frac{\left[\ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \mathrm{errordelta}_{i,k}\right] - \ln\left[\left(\alpha^{\langle 1\rangle}\right)_i\right]\right]}{\ln\left[\left(\alpha^{\langle 1\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right] - \ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right]}\right] - \frac{\left[\ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \delta_i\right] - \ln\left[\left(\alpha^{\langle 1\rangle}\right)_i\right]\right]}{\ln\left[\left(\alpha^{\langle 1\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right] - \ln\left[\left(\alpha^{\langle 2\rangle}\right)_i + \mathrm{degs}\left(\frac{bc}{R}\right)\right]}$$

$\mathrm{minweighterror} \equiv .1$

$\mathrm{minangerror} \equiv .05$

error in n due to errors in alpha1 ranging from $\mathrm{minangerror} = 0.05$ to $\mathrm{minangerror}\cdot 10 = 0.5$ deg



errorfromalpha1

$\max(\mathrm{errorfromalpha1}) = 0$

$\min(\mathrm{errorfromalpha1}) = -1.016$

$\mathrm{stdev}(\mathrm{errorfromalpha1}) = 0.156$

$\mathrm{mean}(\mathrm{errorfromalpha1}) = -0.139$

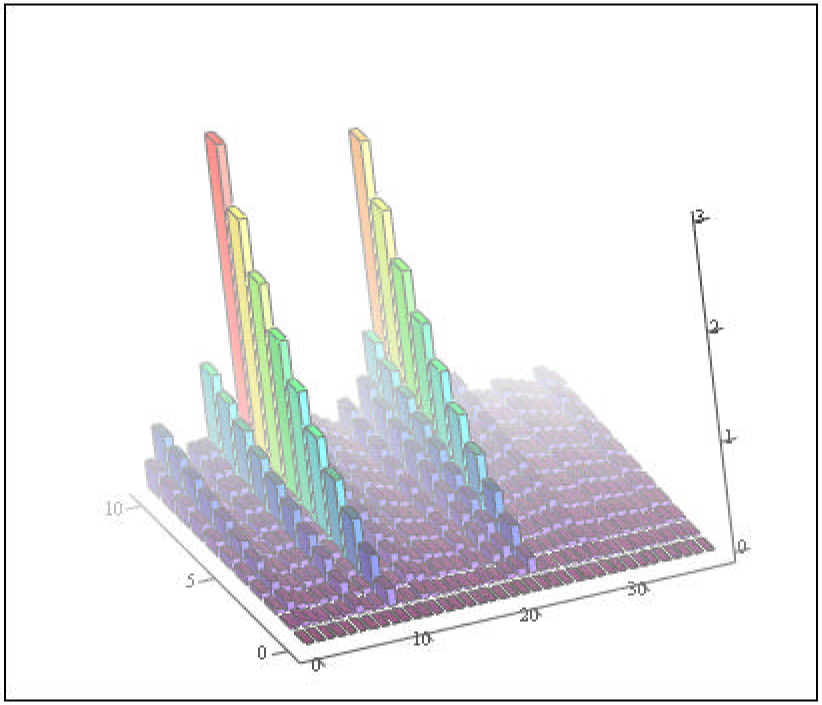
n would run from

a minimum of $\mathrm{mean}(\mathrm{appxmtn}) - \left|\mathrm{mean}(\mathrm{errorfromalpha1})\right| = 1.477$

to

a maximum of $\mathrm{mean}(\mathrm{appxmtn}) + \left|\mathrm{mean}(\mathrm{errorfromalpha1})\right| = 1.755$

error in n due to errors in alpha2 ranging from $\text{minangerror} = 0.05$ to $\text{minangerror} \cdot 10 = 0.5$ deg



errorfromalpha2

$\max(\text{errorfromalpha2}) = 3.006$

$\min(\text{errorfromalpha2}) = 0$

$\text{stdev}(\text{errorfromalpha2}) = 0.334$

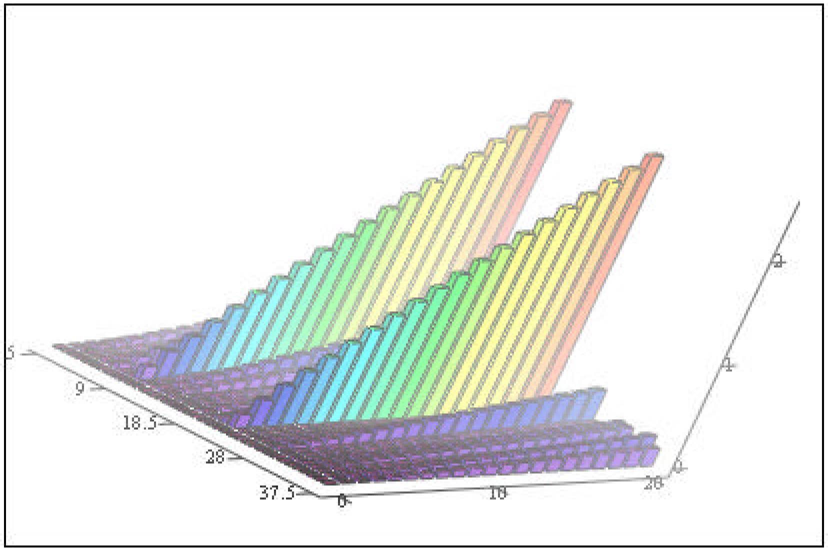
$\text{mean}(\text{errorfromalpha2}) = 0.197$

n would run from

a minimum of $\text{mean}(\text{appxmtn}) - \left|\text{mean}(\text{errorfromalpha2})\right| = 1.419$

a maximum of $\text{mean}(\text{appxmtn}) + \left|\text{mean}(\text{errorfromalpha2})\right| = 1.813$

error in n due to errors in added weights ranging from $\mathrm{minweighterror} = 0.1$ to $\mathrm{minweighterror} \cdot 20 = 2$ in units of $\frac{\mathrm{Gran}}{250} = 3.2\,\mathrm{mg}$



errorfromdelta

$\max(\mathrm{errorfromdelta}) = 2.522$

$\min(\mathrm{errorfromdelta}) = 0$

$\mathrm{stdev}(\mathrm{errorfromdelta}) = 0.349$

$\mathrm{mean}(\mathrm{errorfromdelta}) = 0.195$

n would run from

a minimum of $\mathrm{mean}(\mathrm{appxmtn}) - |\mathrm{mean}(\mathrm{errorfromdelta})| = 1.422$

to

a maximum of $\mathrm{mean}(\mathrm{appxmtn}) + |\mathrm{mean}(\mathrm{errorfromdelta})| = 1.811$

SORTED DATA

column 1 gives initial angles in degrees
column 2 gives second angles in degrees
column 3 gives added weights in degrees

csort(Data1, 1) =

	1	2	3
1	6.75	1	15
2	7.25	3.75	10
3	7.75	0.5	20
4	7.75	2.5	15
5	8	3.5	10
6	8.25	2.75	20
7	10	5	15
8	11	6	14
9	11.25	5	20
10	11.25	3	35
11	11.5	7.5	10
12	11.75	1.5	45
13	12.5	4.5	30
14	12.5	3.5	35
15	13.25	7.5	15
16	13.5	7.5	15

csort(Data2, 1) =

	1	2	3
1	10	7	7.3
2	10	5	15
3	10	6	10.6
4	10	1.75	30.3
5	10	3.5	21
6	10	0	48
7	10	8	4.5
8	10	9	2.1
9	12	6	18
10	12	7	13.5
11	12	8	10
12	12	9	7
13	12	10	4.4
14	12	5	23.8
15	12	0	70
16	12	11	2

UNSORTED DATA

$$\mathrm{Data1} \equiv \begin{pmatrix} 11.5 & 7.5 & 10 \\ 15 & 9.5 & 15 \\ 13.25 & 7.5 & 15 \\ 11.0 & 6 & 14 \\ 13.5 & 7.5 & 15 \\ 7.25 & 3.75 & 10 \\ 16.0 & 9 & 20 \\ 8 & 3.5 & 10 \\ 13.75 & 6 & 25 \\ 15 & 6.5 & 25 \\ 10 & 5 & 15 \\ 11.25 & 5 & 20 \\ 12.5 & 4.5 & 30 \\ 8.25 & 2.75 & 20 \\ 7.75 & 2.5 & 15 \\ 12.5 & 3.5 & 35 \\ 11.25 & 3 & 35 \\ 6.75 & 1 & 15 \\ 11.75 & 1.5 & 45 \\ 7.75 & .5 & 20 \end{pmatrix} \qquad \mathrm{Data2} \equiv \begin{pmatrix} 10 & 9 & 2.1 \\ 10 & 8 & 4.5 \\ 10 & 7 & 7.3 \\ 10 & 6 & 10.6 \\ 10 & 5 & 15 \\ 10 & 3.5 & 21 \\ 10 & 1.75 & 30.3 \\ 10 & 0 & 48 \\ 12 & 11 & 2 \\ 12 & 10 & 4.4 \\ 12 & 9 & 7 \\ 12 & 8 & 10 \\ 12 & 7 & 13.5 \\ 12 & 6 & 18 \\ 12 & 5 & 23.8 \\ 12 & 3.5 & 32 \\ 12 & 1.75 & 45 \\ 12 & 0 & 70 \end{pmatrix}$$

$$\mathrm{degs}(x) \equiv \frac{x \cdot \mathrm{rad}}{\mathrm{deg}} \qquad \mathrm{degtomg}(x) \equiv 800 \cdot \frac{x}{250}$$

$$\mathrm{nodiam} \equiv .0000000001$$

$$i \equiv 1..\ \mathrm{rows}(\mathrm{stack}(\mathrm{Data1}, \mathrm{Data2}))$$

$$\mathrm{SORTED\_DATA} \equiv \mathrm{csort}(\mathrm{stack}(\mathrm{Data1}, \mathrm{Data2}), 1)$$

$$\mathrm{actualweightadded}_i \equiv \mathrm{degtomg}\left[(\mathrm{SORTED\_DATA})_{i,3}\right]$$

$$\alpha^{\langle 1 \rangle} \equiv \mathrm{SORTED\_DATA}^{\langle 1 \rangle} \qquad j \equiv 1..\ 11$$

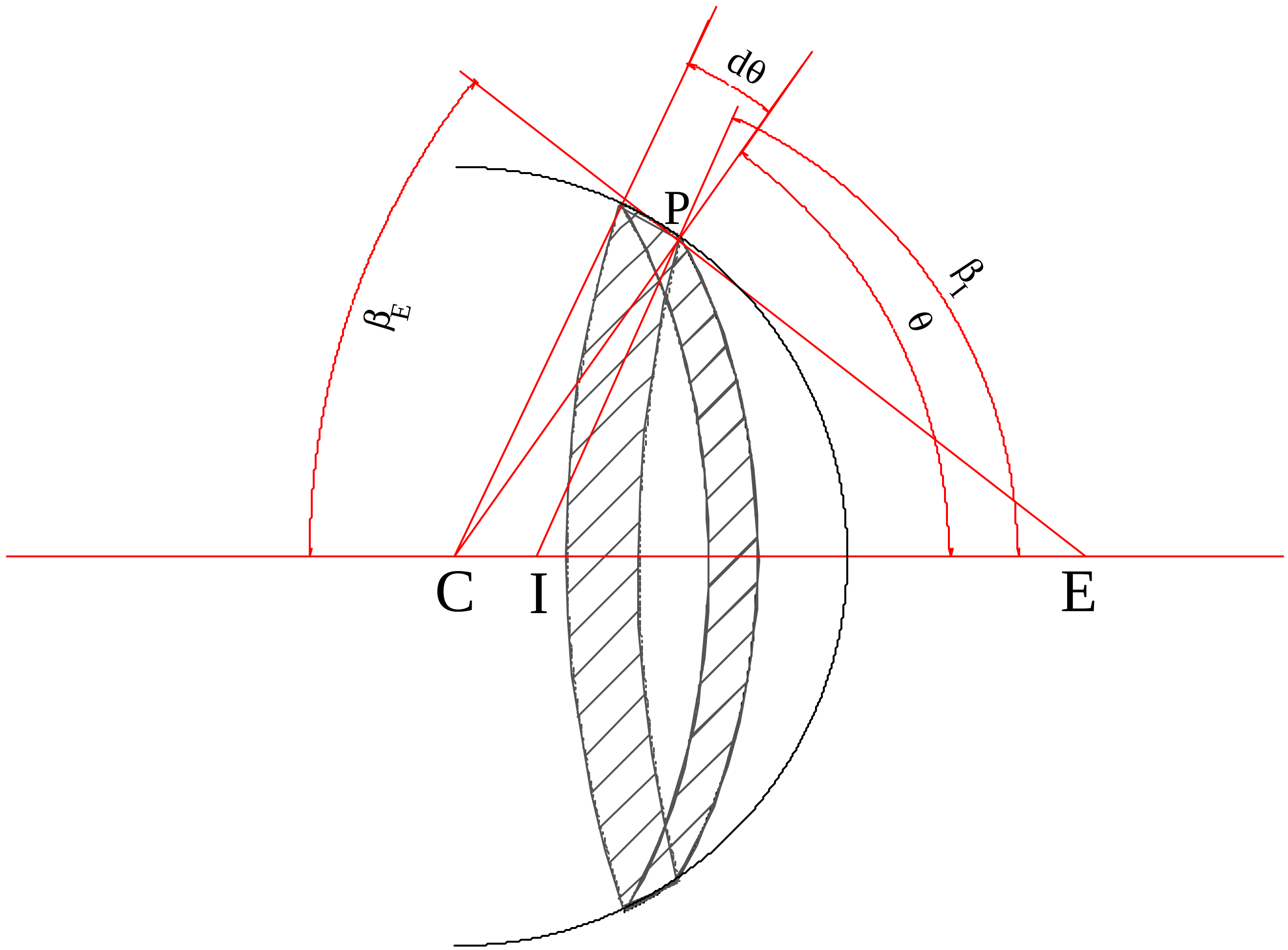
$$\alpha^{\langle 2 \rangle} \equiv \mathrm{SORTED\_DATA}^{\langle 2 \rangle} \qquad \mathrm{erroralpha2}^{\langle j \rangle} \equiv \left[\mathrm{minangerror} \cdot (j-1) + \alpha^{\langle 2 \rangle}\right]$$

$$k \equiv 1..\ 21$$

$$\delta \equiv \mathrm{SORTED\_DATA}^{\langle 3 \rangle} \qquad \mathrm{erroralpha1}^{\langle j \rangle} \equiv \left[\mathrm{minangerror} \cdot (j-1) + \alpha^{\langle 1 \rangle}\right]$$

$$(\mathrm{errordelta})^{\langle k \rangle} \equiv \left[\mathrm{minweighterror} \cdot (k-1) + \delta\right]$$

Remarks on the forces exerted at points within the space enclosed by a spherically-symmetric shell



Let's investigate the force at any point within or outside a spherically-symmetric surface distribution of total quantity Q, radius r, under the assumption that a point at which unit quantity is located exerts a force equal to $1/x^n$ at I (internal) or E (external) located a distance x from the point.

Case 1: Point E external to the spherical shell

let x_E=EP, r=CP, X_E=CE

a. $(x_E)^2 = r^2 + (X_E)^2 - r\,X_E\cos(\theta)$ which yields $\sin(\theta)\,d\theta = \dfrac{(x\cdot dx)}{rX_E}$

b_E. $x_E\cos(\beta_E) = X_E - r\cos\theta$ whence $\cos(\beta_E) = \dfrac{(X_E - r\cdot\cos(\theta))}{x_E}$

c. area dS of slice normal to CE =(rdθ)(2πrsin(θ))= $2\pi r\cdot x_E\cdot\dfrac{dx}{X_E}$

d. dF=$\left[\dfrac{\left(\dfrac{QdS}{4\pi r^2}\right)}{(x_E)^n}\right]\cos(\beta_E)$ because the force components normal to the line CE cancel out by symmetry

e_E. resulting integral is F=$\left[\dfrac{Q}{4r(X_E)^2}\right]\left[\displaystyle\int_{X_E-r}^{X_E+r}\left[\dfrac{1}{(x_E)^n}\right]\cdot\left[(X_E)^2 - r^2 + (x_E)^2\right]dx_E\right]$

f. **If n=1**

$$F=\left[\frac{Q}{4r(X_E)^2}\right]\left[\left[(X_E)^2 - r^2\right]\cdot\ln\left(\frac{X_E + r}{X_E - r}\right) + 2r\cdot X_E\right]$$

For any other value of n

$$F=\left[\frac{Q}{4r(X_E)^2}\right]\left[\left[\frac{(X_E)^2 - r^2}{1-n}\right]\cdot\left[(X_E + r)^{1-n} - (X_E - r)^{1-n}\right] + \left(\frac{1}{3-n}\right)\cdot\left[(X_E + r)^{3-n} - (X_E - r)^{3-n}\right]\right]$$

Case 2: Point I within the area enclosed by the spherical shell

let x_I=CP, r=CP, X_I=CI

From the figure, we see that the only effects are to change the sign of the expression (b) and the limits in the integral (e):

b_I. $X_I + x_I \cdot \cos(\beta_I) = r \cdot \cos(\theta)$ whence $\cos(\beta_I) = \dfrac{(r \cdot \cos(\theta) - X_I)}{x_I}$

e_I. resulting integral is F= $\left[\dfrac{Q}{4r(X_I)^2}\right]\left[\displaystyle\int_{r-X_I}^{r+X_I}\left[\dfrac{1}{(x_I)^n}\right]\cdot\left[r^2 - (X_I)^2 - (x_I)^2\right]dx_I\right]$

f. **If n=1**

$$F=\left[\frac{Q}{4\cdot r\cdot(X_E)^2}\right]\cdot\left[\left[r^2 - (X_I)^2\right]\cdot\ln\left(\frac{r + X_I}{r - X_I}\right) - 2r\cdot X_I\right]$$

For any other value of n

$$F=\left[\frac{Q}{4r(X_I)^2}\right]\left[\frac{r^2 - (X_I)^2}{1 - n}\cdot\left[(r + X_I)^{1-n} - (r - X_I)^{1-n}\right] - \left(\frac{1}{3 - n}\right)\cdot\left[(r + X_I)^{3-n} - (r - X_I)^{3-n}\right]\right]$$

Exploring the force within the space enclosed by the spherical shell for several values of n

For n equal to 2, our expression (f) above at once shows that the force withint the enclosed space vanishes altogether. Putting aside n equal to 1 for a moment, we can explore the enclosed space to see where in it the force will vanish for other values of n. From our expression (f) we see that the force will vanish wherever the following two functions f and g are equal to one another (note that here $r := 4$):

$$f(x,n) := \frac{\left[\left(r^2 - x^2\right)\cdot\left[(r+x)^{1-n} - (r-x)^{1-n}\right]\right]}{1-n} \qquad g(x,n) := \frac{\left[\left[(r+x)^{3-n} - (r-x)^{3-n}\right]\right]}{3-n}$$

$$m := 1, 2 .. 50 \quad p := 1 .. 25 \quad xrun_m := m\cdot\frac{\frac{r}{2}}{50} \qquad nrun_p := 1 + p\cdot\frac{6}{25}$$

$$diffmap_{m,p} := f(xrun_m, nrun_p) - g(xrun_m, nrun_p)$$

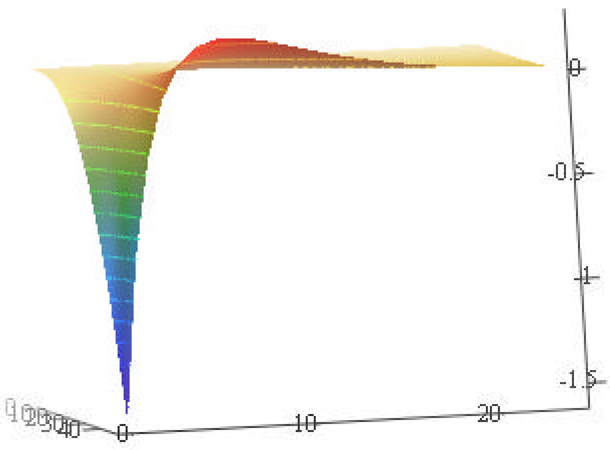
Wherever the surfaces represented below reach zero, the force vanishes.

In the figure below left, the axis that runs more or less horizontally represents values of n running from just above 1 through 7. The axis pointing more or less into the plane of the page in figure left, and parallel to the page in figure right, represents distance, with the 0 point marking the center of the sphere and the other end marking half the radial distance to the surface.

We can see where the line on the surface at n=2 lies in the zero vertical plane, indicating the complete absence of force. All other powers will exert a finite force everywhere within the enclosed space. Note that for regions close to the mid-radial point the force decreases with increasing exponent but then actually *reverses sign* as the exponent grows past two, only to drop again towards zero as the exponent increases further.

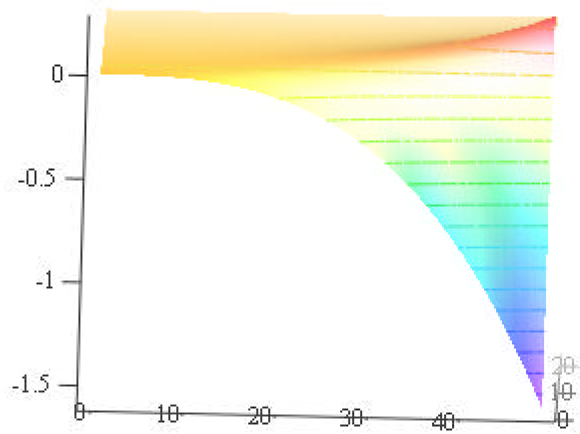
We can (see below) demonstrate that the function f-g must have no roots except at the center of the space unless n=2.

viewed with the exponent axis parallel to the page

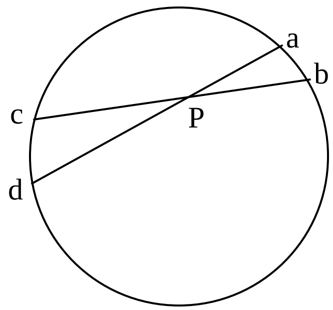


diffmap

viewed with the radial distance axis parallel to the page

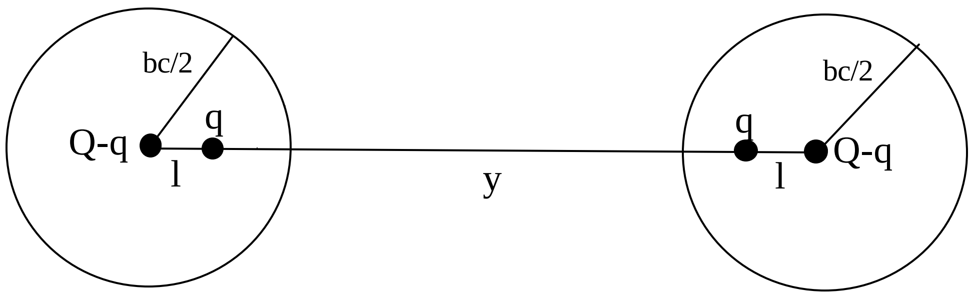


diffmap



If we assume on some physical grounds (e.g. ones given by Egen among others) that there can be no net electric force at any point within the substance of a conductor, then a demonstration first provided by Newton in the *Principia* easily shows that the exponent in the force law must be 2. Specifically (see fig.), it's a property of the circle that the rectangles on intersecting chords are equal among one another, i.e. that aP * dP = cP * bP . Taking the limit of small surface arcs cd, ab, it follows that the triangles cPd and cPb are similar, whence cd/dP = ab/bP. Squaring, we have $\frac{cd^2}{dp^2} = \frac{ab^2}{bP^2}$. In our limit we take cd^2, ab^2 as the surface areas intercepted by the cones that meet at P, and of course dP, bP represent the distances from P to cd, ab respectively. If we assume that the force must be inversely as the nth power of the distance, it follows at once that n must be 2.

Effect of mutual induction between the conducting spheres on the force



Each sphere bears the same charge, say Q, and will therefore distort by induction the charge distribution on the other. To calculate the effect in principle requires a complicated expansion in Legendre ploynomials (given by both Poisson and Maxwell), but we can approximate using images. If the chaarge on the sphere on the right in the figure were concentrated at its center, then we could calculate the force between the concentrated charge and the sphere on the left by replacing the latter with two point charges: one, located at the left sphere's center, would have the charge Q(1+bc/y), where y is the center-center distance and bc the radius; the other point charge would have a magnitude -((bc/2)/y)Q and be located a distance l equal to $\frac{\left(\frac{bc}{2}\right)^2}{y}$ to the right of the sphere's center. If we approximate by assuming that each sphere acts as a charge concentrated at its center in respect to its inductive effect on its neighbor, then we replace the spheres with 4 charges and can recompute the force between them, which will have the ratio G(x,y) to the force computed without taking account of induction, where G(x,y) is given below, with x=bc/2:

remove zeros from alpha2 distances to avoid calculation singularities: $(\mathrm{alpha2nozero})_i := \left(\alpha^{\langle 2\rangle}\right)_i + 10^{-10}$

$$G(x,y) := \left[\left[\frac{\left(1+\frac{x}{y}\right)^2}{y^2}\right] - \left[\frac{\left[\left(2\cdot\frac{x}{y}\right)\cdot\left(1+\frac{x}{y}\right)\right]}{\left(y-\frac{x^2}{y}\right)^2}\right] + \left[\frac{\left(\frac{x^2}{y^2}\right)}{\left(y-2\cdot\frac{x^2}{y}\right)^2}\right]\right]$$

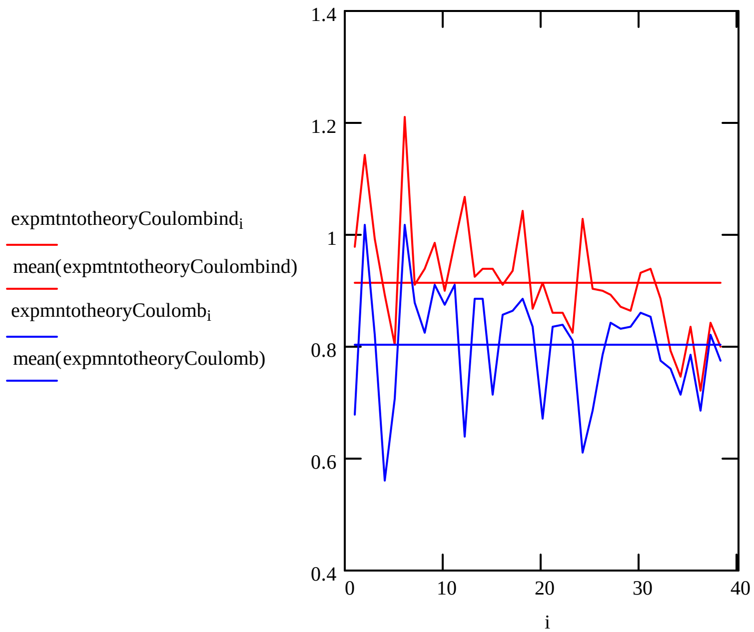
$$\delta\text{Coulombinduction}_i := \text{degtomg}\left[\left[\frac{G\left[\frac{bc}{2}, R(\text{alpha2nozero}_i \cdot \text{deg}) + bc\right]}{G\left[\frac{bc}{2}, R\left[\left(\alpha^{\langle 1\rangle}\right)_i \cdot \text{deg}\right] + bc\right]}\right] \cdot \left[\left(\alpha^{\langle 1\rangle}\right)_i\right] - \text{alpha2nozero}_i\right]$$

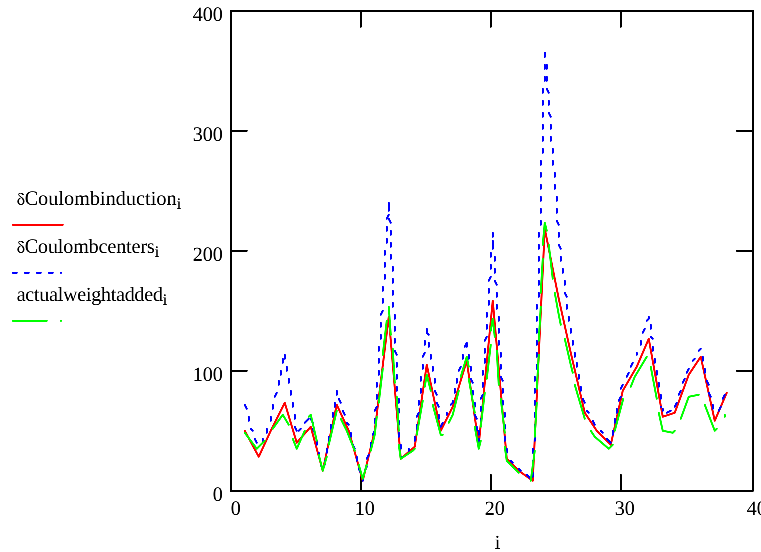
Ratio of experiment to theory with induction taken into account is:

$$\text{expmtntotheoryCoulombind}_i := \frac{\text{actualweightadded}_i}{\delta\text{Coulombinduction}_i}$$

The mean ratio is $\text{mean}(\text{expmtntotheoryCoulombind}) = 0.914$

$\text{SimonCoulombind} := 100 \cdot (1 - \text{mean}(\text{expmtntotheoryCoulombind}))$





We see at once that taking induction into account changes the difference between theory and experiment from $\text{SimonCoulomb} = 19.777$ to $\text{SimonCoulombind} = 8.632$ percent, which is a considerable improvement. Evidently mutual induction does have a significant effect in SImon's experiment, worsening the apparent fit between theory and experiment by $\text{SimonCoulomb} - \text{SimonCoulombind} = 11.145$ percent.

There is no apparent way that Egen could have taken this directly into account, since the method of images was unknown, and Poisson's 1811 calculations give relative charge distributions - not their integral effects, which are of no compelling interest to convinced Coulombians.But Egen was well aware that the effect would be to worsen the apparent agreement, remarking the point explicitly (pg. 299), and even attempting an ad-hoc correction of it by displacing the effective centers of repulsion away from the spheres' geometric centers. That is, since the actual weight added is always less than the amount that it should be, assuming that the centers of repulsion are not displaced from the centers of the spheres', then the repelling force must be less than assumed, in which case the distance between the centers of repulsion must be increased, which is what one would expect since the electric distributions push one another to opposite ends of the spheres. Since the second angle of a triad is always smaller than the first (it being the one produced by adding weight to the balance), we can crudely simulate the effect by increasing the effective distance for this second repulsion, say by 15%, but not for the first angle (which, being - usually - much larger - would be less affected by induction). If we do so then we can compensate for induction, and this is what Egen realized.

$\text{displacefact} := 1.15$

$$\delta\text{Coulombdisplcdcenters}_i := -\text{degtomg}\left[\left(\alpha^{\langle 2\rangle}\right)_i - \left[\left(\alpha^{\langle 1\rangle}\right)_i\right]\cdot\frac{\left[R\cdot\left(\alpha^{\langle 1\rangle}\right)_i\cdot\text{deg} + \text{bc}\right]^2}{\left[\text{displacefact}\cdot R\cdot\left(\alpha^{\langle 2\rangle}\right)_i\cdot\text{deg} + \text{bc}\right]^2}\right]$$