Mass-point Geometry and Vectors

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In antiquity, the mathematician-physicist Archimedes, who used the methods of physics and math interchangeably, has come up with a method of solving geometrical problems using properties of mass configurations. Mass-point geometry was largely unnoticed until a sort of renaissance in the 20th century on the part of recreational mathematicians. The folks who write ARML took note, and starting with the mid-1970s, problems with nifty mass-point solutions are a frequent occurence. Late 1990s saw a decline in their abundance, but nonetheless the method is a beautiful gem that deserves prime showing in the galleries of mathematics.

Before we formally introduce mass-point geometry, though, we need to review vectors. Many of you have seen these friendly creatures in geometry or physics class, but often they are buried in books instead of being applied to olympiad-type problems.

In plane geometry, a vector $\mathbf{AB}$ is a directed line segment from A to B. This is not the same definition, *per se*, as in linear algebra, but we can show the two definitions to be equivalent in a sense.

Exercise 1: prove that $\mathbf{AB} + \mathbf{BC} = \mathbf{AC}$.

Now, it is useful to introduce an origin. Given an origin, we can, instead of having two-point vectors, just call $\mathbf{OA}$ as $\mathbf{a}$, and then $\mathbf{AB} = \mathbf{b} - \mathbf{a}$. But wait! What if we want to get a point BETWEEN A and B, the midpoint, say? So we are looking for a point X such that $\mathbf{x} - \mathbf{a} = \mathbf{b} - \mathbf{x}$, right? That's because the directed distance we move from X to A is the same as from B to X. Solving for X (as you see, addition and subtraction work for vectors), $\mathbf{x} = (\mathbf{a} + \mathbf{b})/2$. Now, prove this:

Exercise 2: show that the point that's k of the distance from A to B along line AB (midpoint for $k = 1/2$) is given by $k\mathbf{b} + (1 - k)\mathbf{a}$.

In solving the last problem, you probably had to use this fact: if we have two noncollinear vectors in the plane, then there is a unique way to express any other vector as a linear combination of them. The proof of this so-called *vector space property* is as follows. It is a good technique to add to your proof-writing toolbelt. Suppose that there were two ways to write the same vector. Then, $a\mathbf{x} + b\mathbf{y} = c\mathbf{x} + d\mathbf{y}$. But that implies $(a - c)\mathbf{x} = (d - b)\mathbf{y}$. Hence X and Y are multiples, meaning they are collinear. Contradiction.

Now, we introduce the notion of a *dot product*. In fancy jargon, it is called an *inner product*, so in your further studies, be prepared to use that term. Basically, the only definition for a dot product that satisifies the distributive property and certain others is as follows: $\mathbf{a} \cdot \mathbf{b} = ab\cos(\theta)$.

Exercise 3: show that this definition satisfies the distributive property as advertised: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$. Hint: use the Law of Cosines.

Now, we are ready to tackle some mass-points. Suppose we have some points A_1,A_2,...,A_n on the plane. Then, put masses $m_1, m_2, ..., m_n$ at those points, respectively. We then define the *centroid* to be the vector point

$$(m_1 + m_2 + ... + m_n)G = m_1A_1 + m_2A_2 + ... + m_nA_n.$$

This point, designated G, then, has a remarkable property. If we, instead of points A and B with masses m and n, place a point at THEIR centroid, and mass $m+n$, then the centroid of the system is preserved. Let's look at an example. Suppose that we have masses m_1, m_2, and m_3 at the vertices of a triangle A,B,C, respectively. Then, the point on AB with ratio m_2 to m_1 (call it D), has mass $m_1 + m_2$. Thus the centroid of the system, which is the centroid of C and D, lies on CD. Similarly, if E is the centroid of AC, then EB goes through the centroid of the system. Finally, point F, the centroid of BC, has the property that AF, BE, and CD are concurrent.

So, the moral of the story is this: if one knows (or seeks to find) ratios between collinear line segment lengths, it is useful to place masses at the vertices. Using the propety of concurrency of connections between the global centroid and local centroids, we can find ratios of these connecting segments, too (these are called cevians, for a reason you will discover momentarily).

And now, let's do some problems! (Yee-haw!)

1. Prove Ceva's theorem: if triangle ABC has points A', B', C' on BC, AC, and AB, respectively, such that $AB'CA'BC' = B'CA'BC'A$, then AA', BB', and CC' meet at one point.

2. [Vlad, 2000] We have triangle ABC. Point B' is such that C is the midpoint of $B'A$, point A' is such that B is the midpoint of CA', and C' is one-fifth of the way from B to A. Prove that AA', BB', and CC' concur.

3. [Vlad, 2001] In problem 2, call the point of concurrence P. What is PC/CC'?

4. [Traditional, Vlad] Prove Appolonius's theorem using vectors: this theorem says that the length of median from A to BC, m, satisfies $4m^2 = 2b^2 + 2c^2 - a^2$.

5. [Vlad, 2001] We have a triangle ABC. It's tredian points P,Q, and R are defined as follows: P is one-third of the way from A to B, Q is one-third of the way from B to C, and R is one-third of the way from C to A. The lengths of PC, QA, and RB are 7, 8, and 9, respectively. Find the area of ABC.