Feynman Diagrams in Quantum Mechanics

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Abstract

We explain the use of Feynman diagrams to do perturbation theory in quantum mechanics. Feynman diagrams are a valuable tool for organizing and understanding calculations. We first work several examples for the 1-dimensional harmonic oscillator, and then proceed to justify our calculations.

1 Introduction

In this paper we introduce the terminology of Feynman diagrams. We first give several examples of the application of Feynman diagrams to perturbative quantum mechanics on the harmonic oscillator. We then explain the interaction picture of quantum mechanics, and Wick's Theorem, culminating in a justification for the Feynman rules used in our examples.

Throughout this paper, we will simplify equations by using the conventions $c = \hbar = m = 1$. One can always add these three factors back into equations at the end of a calculation, since they have linearly independent units, and leaving them out vastly simplifies notation.

Feynman diagrams are useful tools for

- Systematically calculating values associated to scattering processes that can occur in several different ways.

- Algorithmically constructing the exact integrals giving the kth order term of a perturbation series.

- Extracting classical intuition about what event(s) each term of a perturbation series for a complicated interaction represents.

2   Examples with Feynman Rules

We will start with the example of an anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\lambda x^3}{6}$$

and compute first-order correction in the perturbation series of $\langle\Omega| x(0)^3 |\Omega\rangle$, where $|\Omega\rangle$ is the ground state of the perturbed Hamiltonian (we write $x(0)^3$ rather than x^3 because the time associated with these operators is important).

We will do this by constructing certain *Feynman diagrams* for this problem. A diagram is very similar to a graph: it consists of a set of vertices and connecting lines (edges). There are two types of vertices in these diagrams – internal and external vertices. Internal vertices in this case have three lines, corresponding to the perturbation x^3, and are labeled by a parameter t; each internal vertex has a different parameter. External vertices have a single line, and correspond to the x's in the expectation we're computing. External vertices are labeled by the time the x's are evaluated at. A diagram is constructed when the lines are connected together in pairs to form edges of a graph, with no stray lines leaving.

The general procedure is to first figure out which diagrams are relevant to the problem being solved, and then use some Feynman rules to calculate a value for each diagram. The values are then added together, with certain weights, to evaluate the desired expectation. This process may seem a bit arbitrary, but in the second half of the paper we give a natural explanation as to where these diagrams come from. As we go through examples, the Feynman rules describing how to calculate will become apparent; but we will also summarize them in the next section.

For our example problem, we are computing the first-order term of the perturbation, so we have one internal vertex, labeled with time t. We also have 3 external vertices, each labeled with time 0, since we're computing $\langle\Omega| x(0)^3 |\Omega\rangle$. The topologically distinct possibilities are shown below, since the only information relevant to the calculation of the value of a Feynman diagram is the topological structure of the diagram.
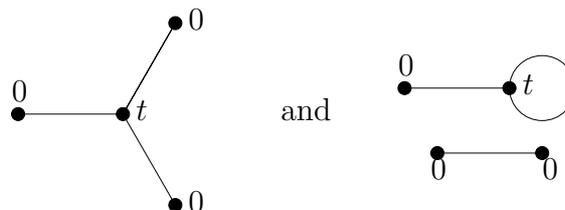
In our figures, we denote vertices by black dots, and the edges (called *propagators*) by lines. We can assign weights to the diagrams, which can be computed as the number of ways the diagram can be constructed, divided 3! (this factor comes from the symmetry of permuting the three edges out of an internal vertex). This gives rise to an equation of the form

$$\langle\Omega|\,x^3\,|\Omega\rangle = \left(\tfrac{6}{6}\right)\!\!\!\!\!\!\!\raisebox{0pt}{\includegraphics}\quad + \quad \left(\tfrac{9}{6}\right)\!\!\!\!\!\raisebox{0pt}{\includegraphics}\quad + O(\lambda^2)$$

Now, we assign a value to each diagram, based on the vertices it possesses and the edges connecting the pairs, using the following *Feynman rules*.

t ● ——— $=$ external vertex $= 1$

———●t $=$ internal vertex $= -i\lambda\int dt$

t ●———● t' $=$ propagator $= D(t,t') = \frac{1}{2\omega}e^{-i\omega|t-t'|}$

So, in this example, the diagrams take on the values shown below.

$$= \quad -i\lambda\int D(t,0)^3 dt = \tfrac{-i\lambda}{8\omega^3}\int e^{-i3\omega|t|}dt$$

$$= \quad -i\lambda\int D(0,0)D(t,0)D(t,t)dt = \int(\tfrac{i\lambda}{8\omega^3})e^{-i\omega|t|}dt$$

Unfortunately, the integrals we use here are a bit complicated. The limits of integration are from $-\infty(1-i\epsilon)$ to $\infty(1-i\epsilon)$ (think of this as integrating along a line slightly tilted off the x-axis in the complex plane). We need those $i\epsilon$'s because without them, the integrals would not converge. However, the result of this integral is in fact independent of ϵ, so we can omit ϵ from our notation. Every integral we write down in this paper will have those limits of integration, unless limits are otherwise explicitly given. With a bit of complex analysis (see [4]), one can derive the following results:

Theorem 1 *For $\omega > 0$,*

$$\lim_{T \to (1-i\epsilon)\infty} \int_{-T}^{T} dt e^{-i\omega|t|} = \frac{2}{i\omega}$$

To aid in the calculation of higher order examples, we use the following theorem, which can be proven using Theorem 1

Theorem 2 *If a, b, c are positive real numbers such that the sum of any two of them is positive, we have that under the same limits as in Theorem 1*

$$\int ds \int dt e^{-ia|s|} e^{-ib|t|} e^{-ic|s-t|} ds dt = \frac{-2}{(a+b)(b+c)} + \frac{-2}{(a+b)(a+c)} + \frac{-2}{(a+c)(b+c)}$$

We can now evaluate the integrals in our example to obtain

$$\langle\Omega| \, x(0)^3 \, |\Omega\rangle = \frac{6}{6} \frac{-i\lambda}{8\omega^3} \frac{2}{3i\omega} + \frac{9}{6} \frac{-i\lambda}{8\omega^3} \frac{2}{i\omega} + O(\lambda^2) = \frac{-\lambda}{8\omega^4} (\frac{2}{3} + 3) + O(\lambda^2) = \frac{-11\lambda}{24\omega^4} + O(\lambda^2) \tag{1}$$

The reader can check that one would get the same answer using standard time-independent perturbation theory. Let's do another example. To compute the first-order correction to the expectation of $x(0)$ in the perturbed ground state, we do the same calculation, but with only one external vertex. So, we consider all possible diagrams containing one degree-3 internal vertex and one degree-1 external vertex. The only possibility is for the internal vertex to be connected to the external vertex, with the remaining two edges connected back to itself. This diagram occurs in three possible ways, and we still have the weighting factor of $\frac{1}{3!}$ from the internal vertex, so this diagram has weight $\frac{3}{6} = \frac{1}{2}$. We can then compute
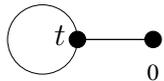
 $= -i\lambda \int D(t,0)D(t,t)dt = \int (\frac{-i\lambda}{4\omega^2})e^{-i\omega|t|}dt = \frac{-i\lambda}{w\omega^2} \frac{2}{i\omega} = \frac{-\lambda}{2\omega^3}$

which implies that (remembering the weight of $\frac{1}{2}$ on this diagram)

$$\langle\Omega| \, x(0) \, |\Omega\rangle = \frac{1}{2} \frac{-\lambda}{2\omega^3} + O(\lambda^2) = \frac{-\lambda}{4\omega^3} + O(\lambda^2) \tag{2}$$

as one would obtain using standard time-independent perturbation theory.

Now, suppose we were instead computing the expectation of x^2 to first-order. In this case we'd have one degree-3 internal vertex and two degree-1 external vertices. On inspecting the situation, the reader should notice that it is impossible to connect these vertices together, since the total degree of any graph is even, but these have a

total degree that is odd. Thus, the first-order term in the perturbation series is 0. Let us proceed to compute the second-order term in this series as well. We now have two degree-3 internal vertices (labeled by times s and t) and two degree-1 external vertices, both labeled by time 0. The possible topologically distinct diagrams are shown below.

$$= D_1 = -\lambda^2 \int dsdt D(s,t)^3 D(0,0)$$

$$= D_2 = -\lambda^2 \int dsdt D(s,t)D(s,s)D(t,t)D(0,0)$$

$$= D_3 = -\lambda^2 \int dsdt D(s,0)^2 D(s,t)D(t,t)$$

$$= D_4 = -\lambda^2 \int dsdt D(s,t)^2 D(0,t)D(0,s)$$

$$= D_5 = -\lambda^2 \int dsdt D(s,0)D(s,s)D(t,0)D(t,t)$$

The values for diagrams composed of two subdiagrams we've seen before (up to topological equivalence) can be calculated immediately by multiplying the previous results. For example, the diagram D_5 above has value

$$D_5 = \left(\frac{-\lambda}{2\omega^3}\right)^2 = \frac{\lambda^2}{4\omega^6}$$

Using Theorem 2 we can calculate

$$D_3 = -\lambda^2 \int ds \int dt D(s,0)^2 D(s,t)D(t,t) \tag{3}$$

$$= -\lambda^2 \frac{1}{(2\omega)^4} \int ds \int dt e^{-i2\omega|s|} e^{-i\omega|s-t|} \tag{4}$$

$$= \frac{-\lambda^2}{16\omega^4}\left(\frac{-2}{2\omega^2} + \frac{-2}{6\omega^2} + \frac{-2}{3\omega^2}\right) = \frac{\lambda^2}{8\omega^6} \tag{5}$$

Similarly, the fourth diagram evaluates to

$$D_4 = -\lambda^2 \int dsdt D(s,t)^2 D(s,0)D(t,0) = \frac{-\lambda^2}{16\omega^4}\left(\frac{-2}{6\omega^2} + \frac{-2}{6\omega^2} + \frac{-2}{9\omega^2}\right) = \frac{\lambda^2}{18\omega^6}$$

The first two diagrams are a bit more difficult to calculate. One will notice that Theorem 2 does not apply for calculating these diagrams. In fact, the integrals involved are not even bounded. They key property of these diagrams is that they have connected components that contain no external vertices. These diagrams are

called *vacuum bubbles* in quantum field theory. This seems disastrous – how can we do anything useful with these quantities that fundamentally diverge?

It turns out that we've thus far been ignoring an important detail when doing these Feynman diagram calculations. The ground state $|\Omega\rangle$ we've been using in these calculations is not actually normalized. However, the first order correction to its normalization is 0, so it has not been a problem in our first order calculations. We can in fact use the Feynman diagram technique to expand the expectation of 1 in the ground state, as shown below. For simplicity, we only show the diagrams, not their weights, but we construct these from having no external vertices and m internal vertices for the order λ^m terms. The fact that there are no diagrams with only one internal vertex confirms our claim that the first-order correction is 0.

$$\langle\Omega|\Omega\rangle = 1 + \bigcirc\!\!-\!\!\bullet\!\!\bullet\!\!-\!\!\bigcirc \; + \; \ominus\!\!-\!\!\bullet \; + \; \ominus\!\!\ominus \; + \; \text{(diagrams)} \; + \; \text{(diagrams)} \; + \cdots$$

You may notice that we did not record the times attached to the various vertices. Since internal vertices always have distinct times attached to them, and the value of that time is then integrated over, many authors leave out the time data for internal vertices. Similarly, because in this paper we will focus only on expectations of powers of $x(0)$, all the external vertices can be assumed to be labeled by 0, as well. These techniques are valid for $x(t)$ for arbitrary times t, but the answers are much messier.

It turns out that the vacuum bubbles we find when calculating $\langle\Omega|\,x^2\,|\Omega\rangle$ exactly cancel the vacuum bubbles in the normalization factor in the denominator, to all orders. In particular, if we were to include the vacuum bubbles, our series would actually calculate $\frac{\langle\Omega|x^2|\Omega\rangle}{\langle\Omega|\Omega\rangle} \times \langle\Omega|\Omega\rangle$, and we'd have to divide by $\langle\Omega|\Omega\rangle$ in order to normalize it. The diagrams containing vacuum bubbles arise from the product of the higher-order terms of the series for $\langle\Omega|\Omega\rangle$ times lower-order terms of the series we actually want to compute. This can be seen in our example by noting that the 0th-order term in this expectation is given by the diagram with a single propagator connecting two external vertices, and each of the vacuum bubbles in our example multiplies that 0th-order term. Thus, we can in fact *ignore* all diagrams containing vacuum bubbles, and get a correct value for the expectation in the true ground state.

We now complete the calculation. The weighting factors for each of the remaining diagrams once again equals the number of ways it could be produced, divided by the internal vertex symmetries of $2! \times (3!)^2 = 72$ (3! for the edges on each of the two vertices, and an additional 2! because we can permute the vertices themselves).

The diagrams D_3 and D_4 can each occur in 36 ways, while D_5 can occur in 18 ways. The other two diagrams we can ignore because they have vacuum bubbles. Thus the weights are $\frac{36}{72}, \frac{36}{72}$, and $\frac{18}{72}$. We can now compute that

$$\langle\Omega|\, x^2\, |\Omega\rangle = \frac{1}{2\omega} + \frac{1}{2}\frac{\lambda^2}{8\omega^6} + \frac{1}{2}\frac{\lambda^2}{18\omega^6} + \frac{1}{4}\frac{\lambda^2}{4\omega^6} + O(\lambda^3) = \frac{1}{2\omega} + \frac{11\lambda^2}{72\omega^6} + O(\lambda^3) \quad (6)$$

Our estimate is in fact correct to order λ^3 as well, because for the same parity reason that the term of order λ equals 0, the term of order λ^3 equals 0.

We can also use Feynman rules with other Hamiltonians. Suppose that we instead had a perturbation of the form

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\lambda x^4}{4!} \quad (7)$$

In this setting, we use the same Feynman rules as before, except that the internal vertices each have degree 4. The diagrams relevant to the computation of $\langle\Omega|\, x(0)^2\, |\Omega\rangle$ up to first order are shown below. We omit the vacuum bubbles, since we know that they will not contribute.

$$\langle\Omega|\, x^2\, |\Omega\rangle = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \;+\; \left(\tfrac{12}{24}\right)\!\!\!\raisebox{-0.5ex}{\includegraphics{diagram}} \;+O(\lambda^2) = \tfrac{1}{2\omega} - i\lambda \int D(t,0)^2 D(t,t)$$

$$\langle\Omega|\, x^2\, |\Omega\rangle = \frac{1}{2\omega} - \frac{i\lambda}{2}\int \frac{dt}{8\omega^3}e^{-i2\omega|t|} + O(\lambda^2) = \frac{1}{2\omega} - \frac{i\lambda}{16\omega^3}\frac{2}{2i\omega} + O(\lambda^2) = \frac{1}{2\omega} - \frac{\lambda}{16\omega^4} + O(\lambda^2)$$

Another interesting feature to notice is that we can detect from these rules that for this Hamiltonian, $\langle\Omega|\, x(0)\, |\Omega\rangle$ has no nonzero terms in its perturbation series, since there will always be an odd number of edges in every diagram. This observation is confirmed by the fact that x is an odd function in an even potential, and thus must have 0 expectation in the ground state!

Exercise. Use Feynman diagrams to demonstrate

$$\langle\Omega|\, x^2\, |\Omega\rangle = \frac{1}{2\omega} - \frac{\lambda}{16\omega^4} + \frac{35\lambda^2}{1536\omega^7} + O(\lambda^3). \quad (8)$$

Only three new diagrams are involved. Check the answer using the perturbation theory we learned in class, and see which technique you prefer.

3   Feynman Rules for the Harmonic Oscillator

In this section we present a complete set of Feynman Rules for perturbations to the Harmonic Oscillator, of the general form

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\lambda x^k}{k!}$$

We wish to calculate the term of order λ^m in the expansion of

$$\langle \Omega | \, x^n \, | \Omega \rangle$$

The Feynman rules are as follows. We have m internal vertices, each with k edges, labeled by times t_i, $i = 1 \ldots m$ and with value $-i \int \lambda dt_i$. We also have n external vertices, each with a single edge of value 1. We connect these together in all possible pairings of edges, and each propagator contributes a value $D(t, t')$, if the vertices connected by it are labeled by t and t' respectively. Note that since we're using the time-ordered product, the order of the arguments here is not important. The integral for an internal vertex has the limits of integration from Theorem 1.

All diagrams that contain vacuum bubbles are given weight 0. We weight each diagram without any vacuum bubbles by the quantity $\frac{1}{k!^m m!}$. This value represents the symmetries inherent in the internal vertices, with a factor of $m!$ for the symmetry of permuting the vertices, and a factor of $k!^m$ for permuting the edges of each internal vertex. In our examples, we saved work by considering all the classes of topologically distinct diagrams, and computed a total weight for each by multiplying $\frac{1}{k!^m m!}$ by the number of diagrams in the class.

These Feynman rules can be extended to handle more complicated systems with two harmonic oscillators in coordinates x and y. For such a system we would have two different types of edges, one corresponding to an x and the other to a y. We then could use these techniques to represent perturbations of the form $x^a y^b$ with a vertex with a x-edges and b y-edges. These techniques further generalize to the case where there are several perturbative terms; there is then more than one type of internal

vertex in the problem. One can see that for perturbations of the harmonic oscillator, Feynman diagrams can be used in a quite general fashion.

The Feynman rules give a nice way to study complicated potentials that look like the harmonic oscillator potential. Harmonic oscillators are ubiquitous because they are solvable exactly, and many potentials are well approximated locally by a quadratic. In quantum field theory, we often represent a complicated particle as a superposition of harmonic oscillators. Since Feynman diagrams can be used to calculate perturbations of harmonic oscillators, we can use them to calculate properties of the elementary particle interactions that quantum field theory describes. For more about this topic, take a course in quantum field theory.

4   Correlation Functions

Now that we have explained how to calculate with Feynman Diagrams, we will proceed to explain why these calculations should be correct. In this section we introduce some of the terminology that will be used through the rest of the paper. The objects in this section are of particular importance when we work in the interaction picture, which we will discuss in the next section.

Define the *N-point correlation function* as

$$\langle x(t_N)x(t_{N-1})\cdots x(t_1)\rangle \tag{9}$$

You can think of this as measuring the correlation between the location of the particle at each of the N moments in time. The quantities we've been computing throughout this paper are constructed from these correlation functions, though it should not yet be apparent.

A special case of particular importance is the 2-point function, or *propagator*

$$\langle x(t_2)x(t_1)\rangle = \langle 0|\, x(t_2)x(t_1)\,|0\rangle \tag{10}$$

We also will find it useful to consider the time-ordered product of operators

$$T(x(t_1)x(t_2)) = \begin{cases} x(t_1)x(t_2), & t_1 > t_2 \\ x(t_2)x(t_1), & t_2 > t_1 \end{cases} \tag{11}$$

Combining the last two constructions, we obtain the time-ordered propagator

$$D(t_2, t_1) = \langle 0|\, x(t_2)x(t_1)\,|0\rangle \tag{12}$$

The time-ordered propagator is the propagator that we have been computing with when doing examples. The time-ordered property is important for preserving causality. By convention, we choose the first argument of the propagator to occur after the second.

In our example of the harmonic oscillator, the time-ordered propagator can be calculated explicitly:

$$D(t_2, t_1) = \langle 0| \, x(t_2)x(t_1) \, |0\rangle \tag{13}$$

$$= \langle 0| \left\{ \begin{array}{ll} (ae^{-i\omega t_1} + a^\dagger e^{i\omega t_1})\frac{1}{\sqrt{2\omega}}(ae^{-i\omega t_2} + a^\dagger e^{i\omega t_2})\frac{1}{\sqrt{2\omega}}, & t_1 > t_2 \\ (ae^{-i\omega t_2} + a^\dagger e^{i\omega t_2})\frac{1}{\sqrt{2\omega}}(ae^{-i\omega t_1} + a^\dagger e^{i\omega t_1})\frac{1}{\sqrt{2\omega}}, & t_2 > t_1 \end{array} \right\} |0\rangle \tag{14}$$

$$= \left\{ \begin{array}{ll} \frac{1}{2\omega}e^{-i\omega t_1 + i\omega t_2}, & t_1 > t_2 \\ \frac{1}{2\omega}e^{-i\omega t_2 + i\omega t_1}, & t_2 > t_1 \end{array} \right\} = \frac{1}{2\omega}e^{-i\omega|t_2 - t_1|} \tag{15}$$

In the third step we use that the $aa^\dagger$ term is the only one that survives the ground state expectation. The value we just computed is exactly the value we used for the propagator in Section 2. Notice that this result is symmetric under swapping t_2 and t_1; this arises because we took the time-ordered product. As you have already seen, the time-ordered propagator appears in the Feynman rules for the Harmonic oscillator, as the value attached to an "edge" in a Feynman diagram.

5   The Interaction Picture

The interaction picture of quantum mechanics is somewhere between the familiar Schrödinger and Heisenberg pictures. The idea is to treat the main term of the Hamiltonian as in the Heisenberg picture, with a constant wavefunction and time-dependent operators, but to handle the small interacting or perturbative term of the Hamiltonian in the style of the Schrödinger picture, where the wavefunction itself moves. This allows us to isolate the effect of the interacting Hamiltonian from the (presumable well-understood) base Hamiltonian. We will use the interaction picture to produce a formula for expectations in the true ground state in terms of certain expectations in the unperturbed ground state. While we will explain things in terms of an abstract Hamiltonian, it may help to keep the harmonic oscillator in mind. Our presentation here is based on [2], p. 83-87.

Suppose that we have a Hamiltonian of the form

$$H = H_0 + H_1 \tag{16}$$

where H_0 is a time-independent Hamiltonian, and H_1 is a perturbative interaction term. Let $|\Omega\rangle$ be the ground state of the interacting theory, and define the interacting wavefunction

$$|\psi(t)\rangle_I = e^{iH_0 t} |\psi(t)\rangle_S$$

Defining the interacting picture interacting Hamiltonian

$$H_I(t) = e^{iH_0 t} H_1 e^{-iH_0 t}$$

we find the differential equation

$$i\frac{\partial}{\partial t} |\psi(t)\rangle_I = -H_0 |\psi(t)\rangle_I + ie^{iH_0}\frac{\partial}{\partial t} |\psi(t)\rangle_S \tag{17}$$

$$= e^{iH_0 t}(-H_0 + H) |\psi(t)\rangle_S \tag{18}$$

$$= e^{iH_0 t} H_1 e^{-iH_0 t} |\psi(t)\rangle_I = H_I |\psi(t)\rangle_I \tag{19}$$

The solution can be presented as a power series for the interaction picture time-evolution operator $U(t, t_0)$.

$$U(t, t_0) = 1 + (-i) \int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \cdots \tag{20}$$

This expression can be checked by explicit differentiation; each term gives the previous term, multiplied by $-iH_I(t)$. Note that we use the initial condition $U(t_0, t_0) = 1$, and that each term is time-ordered. This may be simplified by using the time-ordering operator. For example,

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) = \frac{1}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T(H_I(t_1) H_I(t_2)) \tag{21}$$

where now all the integrals have the same range. The factors of $\frac{1}{n!}$ correct for the overcounting. We may convert this to an exponential by pulling the time-ordering operator out of the integrals:

$$U(t, t_0) = 1 + (-i) \int_{t_0}^{t} dt_1 T(H_I(t_1)) + \frac{1}{2}(-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T(H_I(t_1) H_I(t_2)) + \cdots \tag{22}$$

$$U(t, t_0) = T\left(\exp\left[-i \int_{t_0}^{t} dt' H_I(t')\right]\right) \tag{23}$$

Why did we go through all this work to find the time evolution operator? The goal here is to find the ground state of the interacting (perturbed) Hamiltonian, so

that we can compute expectations in that state. It turns out that if we evolve the original ground state from time $-T$ to time t, we get something from which we can obtain $|\Omega\rangle$, as $T \to \infty$. Let $|n_I\rangle$ be the nth state of the interacting theory, with $|0_I\rangle = |\Omega\rangle$. Then the time evolution of $|0\rangle$ is

$$e^{-iHt}|0\rangle = \sum_n^\infty e^{-iE_n t}|n_I\rangle\langle n_I|0\rangle \tag{24}$$

$$e^{-iHt}|0\rangle = e^{-iE_0 t}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0}^\infty e^{-iE_n t}|n_I\rangle\langle n_I|0\rangle \tag{25}$$

We've not yet used the fact that t is a real-valued variable, so we'll tilt it into the complex plane in order to isolate the ground state. To do this, we need to assume that ground state in the interacting theory $|\Omega\rangle$ is unique. We replace t by $t(1 - i\epsilon)$, and consider the limit as $t \to \infty$ (so that terms of lower order in $e^{-\epsilon t}$ vanish, in particular those with energy greater than E_0). Our expression reduces to

$$e^{-iHt(1-i\epsilon)}|0\rangle = e^{-iE_0 t - E_0\epsilon t}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0}^\infty e^{-iE_n t - \epsilon E_n t}|n_I\rangle\langle n_I|0\rangle \to e^{-iE_0 t(1-i\epsilon)}|\Omega\rangle\langle\Omega|0\rangle \tag{26}$$

We then can obtain the ground state itself as

$$|\Omega\rangle = \lim_{t\to\infty}\left(\frac{e^{iE_0 t(1-i\epsilon)}e^{-iHt(1-i\epsilon)}|0\rangle}{\langle\Omega|0\rangle}\right) = \lim_{t\to(1-i\epsilon)\infty}\left(\frac{e^{iE_0 t}e^{-iHt}|0\rangle}{\langle\Omega|0\rangle}\right) \tag{27}$$

Now, it suffices to find a way to calculate such limits. Note that Ω is not normalized; in fact

$$\langle\Omega|\Omega\rangle = \lim_{t\to(1-i\epsilon)\infty}\left(\left(|\langle 0|\Omega\rangle|^2 e^{-iE_0 2t}\right)^{-1}\langle 0|U(t,0)U(0,-t)|0\rangle\right) \tag{28}$$

With some straightforward algebra, we can then calculate the time-ordered propagator for $t_2 > t_1$

$$\langle\Omega|x(t_2)x(t_1)|\Omega\rangle = \lim_{t\to(1-i\epsilon)\infty}\frac{\langle 0|U(T,t_2)x(t_2)U(t_2,t_1)x(t_1)U(t_1,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} \tag{29}$$

Notice that both sides of this equation are in time-order. Thus we can insert a time-ordered product on both sides, and obtain the final result

$$\langle\Omega|T\left(x(t_2)x(t_1)\right)|\Omega\rangle = \lim_{t\to(1-i\epsilon)\infty}\frac{\langle 0|T\left(x(t_2)x(t_1)\exp[-i\int_{-T}^T dt H_I(t)]\right)|0\rangle}{\langle 0|T\left(\exp[\int_{-T}^T dt H_I(t)]\right)|0\rangle} \tag{30}$$

This expression generalizes to higher correlation functions in the obvious way – simply add extra $x(t_i)$ terms on both sides. Notice that in Equation (30) the RHS contains no terms involving $|\Omega\rangle$, only $|0\rangle$. This formula is an exact expression for the ground state expectation. To do rth order perturbation theory, we must expand the exponential as a power series, and only retain the terms whose degree in λ is at most r. In each term of the power series, we will have a time-ordered correlation function. Wick's Theorem explains how to calculated these correlation functions.

6 Wick's Theorem for the Harmonic Oscillator

Theorem 3 (Wick's Theorem) *Suppose that $t_i > t_{i-1}$, so that the left-hand side product below is time-ordered. Then in a harmonic oscillator potential, we have*

$$\langle x(t_{2N})x(t_{2N-1})\cdots x(1)\rangle = \sum_{i_k>j_k, i_{k+1}>i_k} \langle x(t_{i_N})x(t_{j_N})\rangle \cdots \langle x(t_{i_1})x(t_{j_1})\rangle \qquad (31)$$

where the expectations are taken in the (unperturbed) ground state.

Conceptually, it states that every $2n$-point function is in fact the sum over all possible pairings of the product of the n 2-point functions on the pairs. The $i_k > j_k$ condition ensures that all the 2-point functions are time-ordered. The $i_{k+1} > i_k$ condition breaks the symmetry of the $N!$ possible orderings for the terms in each product.

Example. For $N = 2$, Wick's Theorem states that

$$\langle x(t_4)x(t_3)x(t_2)x(t_1)\rangle = \qquad (32)$$

$$\langle x(t_4)x(t_3)\rangle \times \langle x(t_2)x(t_1)\rangle + \langle x(t_4)x(t_2)\rangle \times \langle x(t_3)x(t_1)\rangle + \langle x(t_4)x(t_1)\rangle \times \langle x(t_3)x(t_2)\rangle \qquad (33)$$

Wick's Theorem is typically stated as a complicated operator identity. We believe that the formulation presented here is much easier to understand and apply, as it is typically the one that is used in practice. This version follows from the standard version by noting the extra terms all have 0 ground state expectation. A proof of the standard version can be found in a standard text on Quantum Field Theory, such as [2] p. 88-90. The result can also be proven directly using a relatively straightforward induction.

Wick's Theorem is an extremely useful result for doing calculations. An immediate consequence is that in order to calculate arbitrary time-ordered $2N$-point functions,

it suffices to be able to calculate the time-ordered propagator. The $(2N + 1)$-point function is always 0 in this situation, since we have an odd number of a and $a^\dagger$ operators on each term. Thus we can in fact calculate any n-point function given only the values of the time-ordered propagators. As we will see in the next section Wick's Theorem gives rise to the Feynman diagram formalism.

7   Why Feynman Diagrams Work

When we combine Wick's Theorem together with the result we derived using the interaction picture, we can justify the Feynman rules we provided in section 3. We have a Hamiltonian of the form $H = \frac{p^2}{2} + \frac{w^2 x^2}{2} + \frac{\lambda x^k}{k!}$. By Equation (30), if we wish to compute the expectation of x^n, we use

$$\langle \Omega | x(0)^n | \Omega \rangle = \lim_{T \to (1-i\epsilon)\infty} \frac{\langle 0 | T \left(x(0)^n \exp[-i \int_{-T}^{T} dt H_I(t)] \right) | 0 \rangle}{\langle 0 | T \left(\exp[\int_{-T}^{T} dt H_I(t)] \right) | 0 \rangle} \tag{34}$$

Let us calculate the numerator (as we've said before, the denominator will cancel with the vacuum bubbles in the numerator. For a proof, see [2], p. 95-99). These integrals have the same limits of integration as those we've used in our Feynman rules previously. Thus, we're going to drop the limit, and write our integrals without limits of integration, to simplify notation. To do perturbation theory, we can expand the exponential as a power series

$$\langle 0 | T \left(x(0)^n \exp[-i \int dt H_I(t)] \right) | 0 \rangle = \tag{35}$$

$$\langle 0 | T \left(x(0)^n \left[1 + \frac{-i\lambda}{k!} \int dt x(t)^k + \frac{1}{2!} \frac{(-i\lambda)^2}{k!^2} \int ds dt x(t)^k x(s)^k + \cdots \right] \right) | 0 \rangle \tag{36}$$

In general, the mth order term of the series is of the form

$$\langle 0 | T \left(\frac{1}{m!} \frac{(-i\lambda)^m}{k!^m} \int dt_1 \ldots dt_m x(t_1)^k \cdots x(t_m)^k x(0)^n \right) | 0 \rangle \tag{37}$$

This can be rearranged to read

$$\frac{1}{m!} \frac{(-i\lambda)^m}{k!^m} \int dt_1 \ldots dt_m \langle 0 | T \left(x(t_1)^k \cdots x(t_m)^k x(0)^n \right) | 0 \rangle \tag{38}$$

We now apply Wick's Theorem. Wick's Theorem says that in order to compute that time-ordered product, we need to pair together the $x(t_i)$'s and $x(0)$'s in all

possible ways, and sum over the resulting pairings. We represent diagramatically an $x(t_i)^k$ by an internal vertex of degree k, and each $x(0)$ by a an external vertex of degree 1. The value attached a Wick pairing of two vertices at times t and t' is exactly the time-ordered propagator $D(t, t')$ – we represent this diagramatically as an edge. Since Wick's Theorem sums over all possible pairings, we have to sum over all possible diagrams that can be constructed by pairing the edges from these vertices. In practice, of course, we sum over the topologically distinct diagrams and then count how many times each occurs.

An internal vertex from $x(t_i)^k$ can be assigned the value $-i\lambda \int dt_i$. Giving external vertices value 1, the only remaining factor we haven't accounted for is the scalar $\frac{1}{m!} \frac{1}{k!^m}$. The integer in this denominator is exactly the number of ways to permute the internal vertices ($m!$) and their edges ($k!$ for each vertex), the weighting factor we discussed in Section 3.

In summary, we've presented the formalism of Feynman diagrams for perturbations of the 1-dimensional harmonic oscillator, and explained enough of their theory to remove some of the mystery as to why these diagrams work.

Acknowledgments

References

[1] R.L. Liboff, *Introductory Quantum Mechanics, 3rd Ed.* (Addison-Wesley, Reading, MA, 1998)

[2] M. Peskin, D. Schroeder, *An Introduction to Quantum Field Theory* (Westview Press, USA, 1995)

[3] I learned many of these methods from taking MIT's course 8.323 from Washington Taylor in Spring 2005.

[4] L. Ahlfors, *Complex Analysis* (McGraw-Hill, 1979)