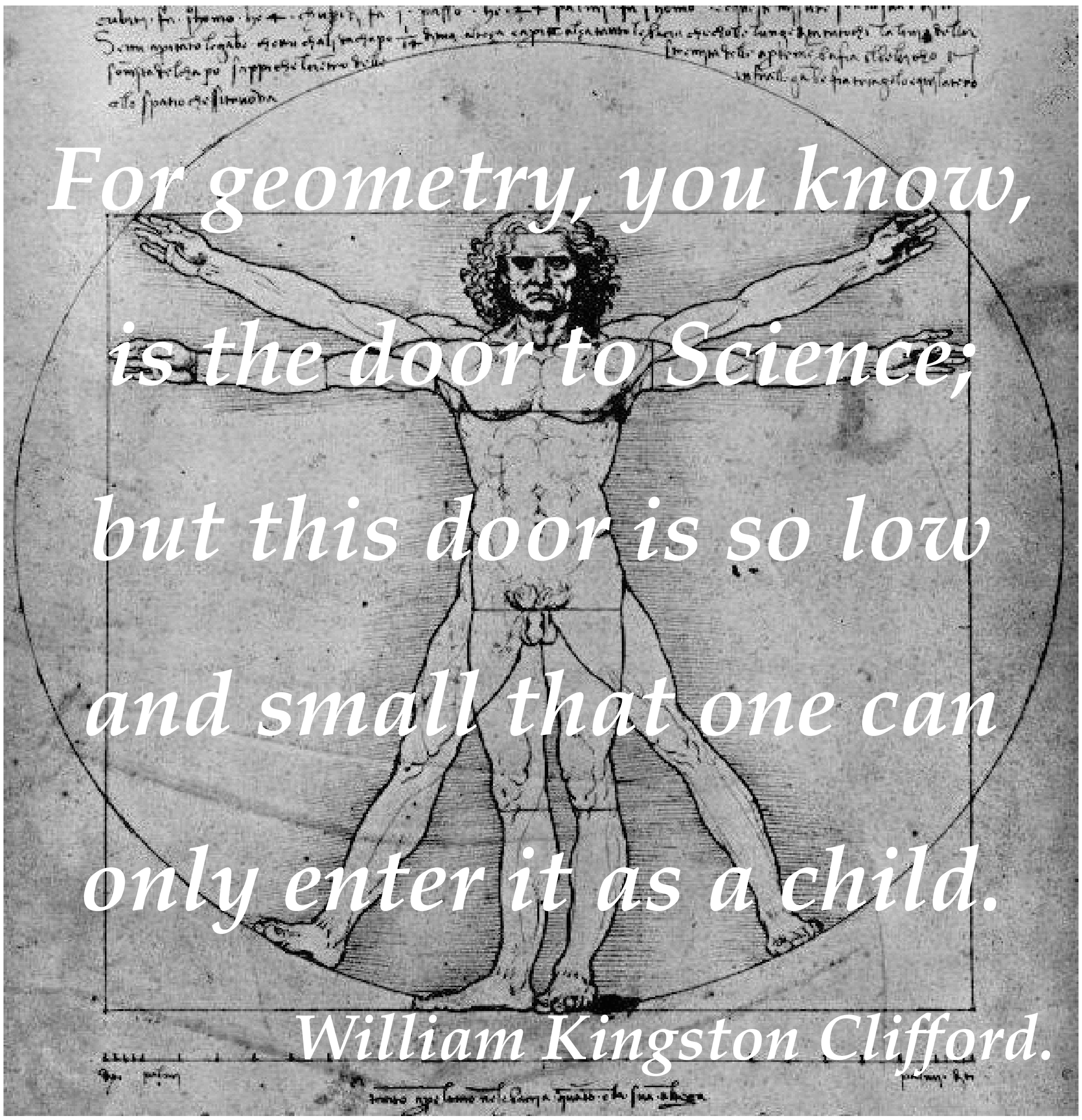
GEOMETRIC ALGEBRA:

Parallel Processing for the Mind

Timothy F. Havel (Nuclear Engineering)

AN IAP-2001 PRODUCTION



THE TIME IS 1844

The place, the rural town of Stettin, in what is now a part of Poland

A second generation gymnasium Lehrer by the name of Hermann Gunther Grassmann has just completed his Magnum Opus, the *Ausdehnungslehre*, or Calculus of Extension. In it, he lays forth a general *theory of forms* which today would be called abstract algebra, and applies it to geometry, mechanics, electro-dynamics and crystal-lography. He sends copies to all the leading mathematicians of his time, including Gauss, Möbius, and Cauchy, none of whom were willing or able to read it.

The following excerpt gives some idea of why:

> ***The primary division in all the sciences is into the real and the formal. The former represents in thought the existent as existing independently of thought, and their truth consists in their correspondence with the existent. The formal sciences on the other hand have as their object what has been produced by thought alone, and their truth consists in the correspondence between the thought processes themselves. Pure mathematics is thus the science of the particular existent which has come to be through thought. The particular existent, viewed in this sense, we name a thought form or simply a form. Thus pure mathematics is the theory of forms.***

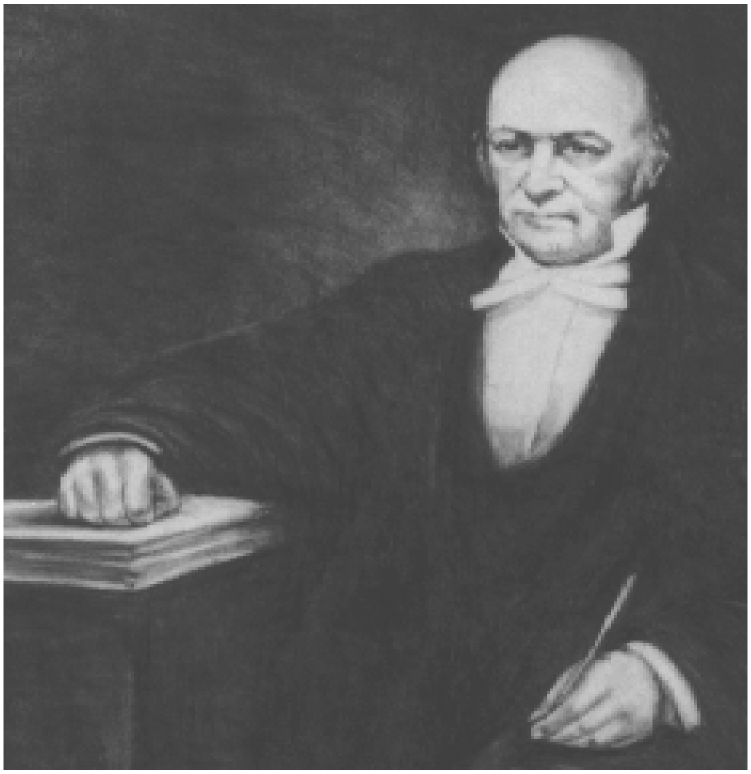
This provoked the following comment from his contemporary Heinrich Baltzer:

> ***It is not possible for me to enter into those thoughts; I become dizzy & see sky-blue before my eyes when I read them.***

Grassmann tried repeatedly but never received a university post (he was rather more successful as a scholar of Sanskrit), and in 1860 the remaining copies of his book were shredded by the publisher. A second edition in 1862 did no better.

THE TIME IS 1843

October 16, to be precise. The place is Brougham Bridge in Dublin, Ireland



The gentleman shown here is William Rowan Hamilton, a poet, philologist *par excellence*, the Royal Astronomer of Ireland since the age of 21, knighted at 28 and now President of the Royal Society of Ireland. Walking to work, he suddenly stops and in great agitation scratches with his pocket knife the following equations on the stonework of Brougham bridge:

$$i^2 = j^2 = k^2 = ijk = -1$$

Quaternions, which Hamilton regards as the *algebra of pure time*, have been born.

Hamilton spent the last 22 years of his life developing Quaternions, publishing his results in 109 papers and a 735 page book *Lectures on Quaternions.* In the process he introduced such now common terms as scalar, vector, real and imaginary. Despite his fame (Hamilton's equations!), his new ideas were only slowly assimilated, as indicated by the following comment from his contemporary J. T. Graves:

> ***There is still something in this system which gravels me. I have not yet any clear views on the extent to which we are at liberty to arbitrarily create new imaginaries, and to endow them with supernatural properties*** **(such as noncommutativity).**

This is also indicated by Hamilton's own remarks:

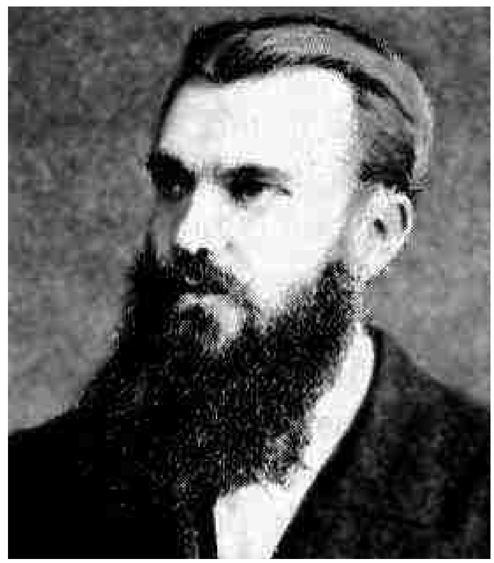
> ***... it required a certain capital of scientific reputation, amassed in former years, to make it other than dangerously imprudent to hazard the publication of a work which has, although at bottom quite conservative, a highly revolutionary air. It was a part of the ordeal through which I had to pass, an episode in the battle of life, to know that even candid and friendly people secretly, or, as it might happen, openly censured or ridiculed me for what appeared to them my monstrous invention.***

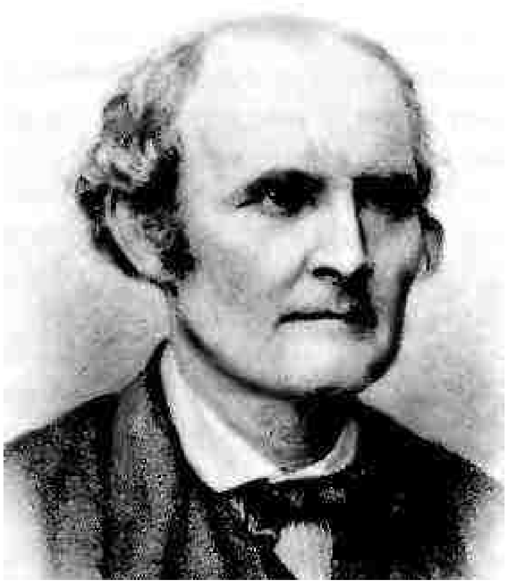
This eventually got to him, for he died of alcoholism at the age of 60, still working on *Elements of Quaternions*.

We Arrive in the Late 19th Century

A Few People Begin to Take Interest

William Kingston Clifford realizes the connections between Grassmann and Hamilton's works, develops biquaternions, applies them to non-Euclidean geometry, and anticipates Einstein's theory of relativity in 1870. But he dies at the age of 35.



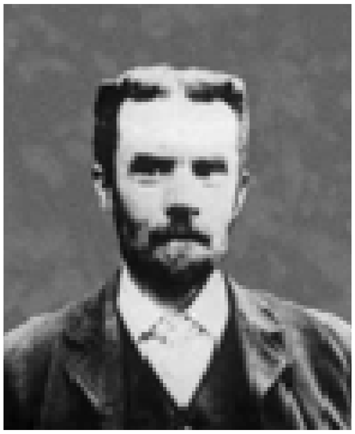


Arthur Cayley, a lawyer before accepting a chair at Cambridge and the supreme master of determinants, discovers in 1885 something that escaped Hamilton and Clifford, namely how to express general 3-D rotations using quaternions. But he finds them unintuitive!

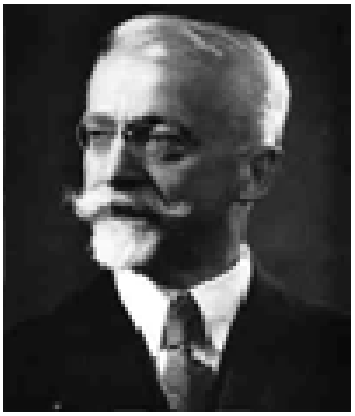
ANOTHER CENTURY PASSES

In it, the Only Good Mathematician is an Abstract Mathematician

Oliver Heaviside & **Josiah Gibbs** succeed in getting a very small part of Grassmann & Hamilton's ideas incorporated into physics and engineering under the name of "vector algebra".



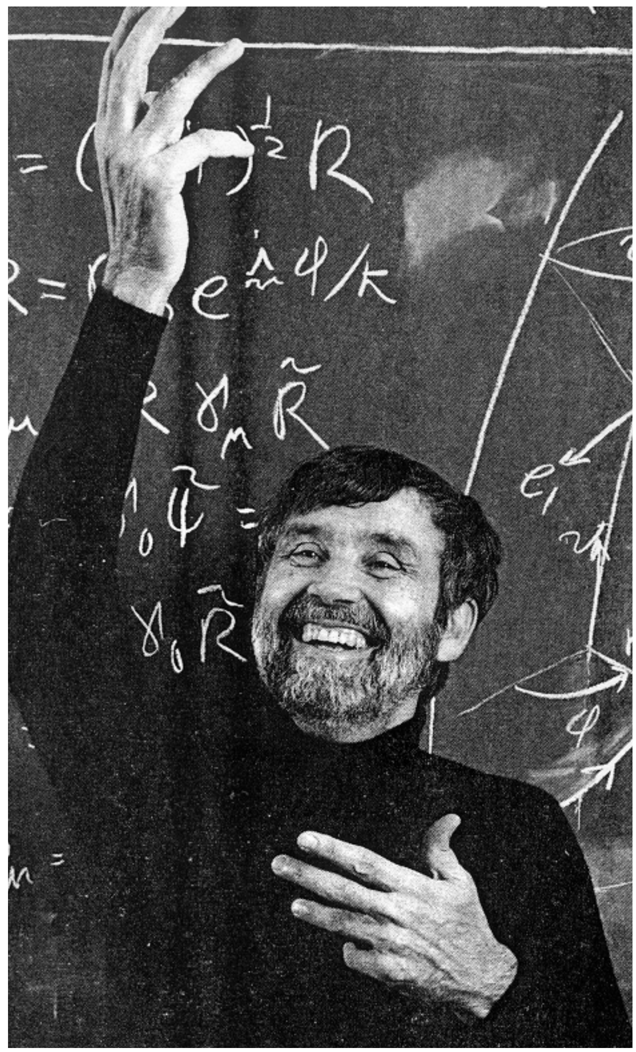
In pure mathematics, pieces of "geometric algebra" are rediscovered by **Marcel Riesz** & **Elie Cartan**, but cloaked in a new notation and vocabulary so as to be completely unrecognizable.



These new names include "tensor algebra", "differential forms" and "spinor calculus".

A New Dawn Comes

In a Way as Unlikely as Anything Else



David Hestenes, the son of a well-known mathematician, a student of the physicist John Archibald Wheeler, and a dedicated physics teacher in at Arizona State University, Tempe, recognizes that all these seemingly different formalisms are, like life itself, pieces of the same thing united by their common lineage. Moreover, he sees that geometric algebra provides a language which, due it is geometric content, captures much of the logic that makes physics what it is. His first book, *Space-Time Algebra*, appears in 1966, and applies geometric algebra to relativity and quantum mechanics.

Together with his first student, **Garret Sobczyk**, he writes a comprehensive treatise on the underlying mathematics, *From Clifford Algebra to Geometric Calculus*. An editor at his first publisher sits on it for several years, until they give up and try another publisher. Unbeknownst to them, the editor takes a new job at that same publisher and continues to delay its publication several more years.

Finally, they send a copy to Gian Carlo Rota in the MIT mathematics department. Rota, who uses higher algebra to study probability and combinatorics, but also has an appreciation for physics through long association with Los Alamos, recognizes the value of the work and gets it published soon thereafter by D. Reidel. Rota's recent & untimely death, however, prevented them from ever meeting each other.

We have arrived in the present. This woefully incomplete history will close with Rota's remarks on Grassmann in his collection of autobiographical sketches, *Indiscrete Thoughts*:

Mathematicians can be divided into two types: problem solvers and theorizers. Most mathematicians are a mixture of the two although it is easy to find extreme examples of both types.

To the problem solver, the supreme achievement in mathematics is the solution of a problem that had been given up as hopeless. It matters little that the solution may be clumsy; all that counts is that it is correct. Once the problem solver finds the solution, he will permanently lose interest in it ... The problem solver is the role model for budding young mathematicians. When we describe to the public the conquests of mathematics, our shining heroes are problem solvers.

To the theorizer, the supreme achievement of mathematics is a theory that sheds sudden light on some incomprehensible phenomenon. Success in mathematics does not lie in solving problems, but in their trivialization. The moment of glory comes with the discovery of a new theory that does not solve any old problems, but renders them irrelevant. To the theorizer, the only mathematics that will survive are the definitions. Great definitions are what mathematics contributes to the world. Theorems are tolerated as a necessary evil since they play an essential role in understanding the definitions.

Grassmann was a theorizer all the way. His great contribution was the definition of geometric algebra. Evil tongues whispered that there was really nothing new in Grassmann's algebra: "What can you prove with it that you can't prove without it?" they asked. Whenever you hear this question, be assured that you are likely to be in the presence of something important.