WHAT'S GOING ON HERE?

First, who am I?

- My name is Tim Havel, and I am a research scientist now working on a quantum computing project in the Nuclear Engineering Dept. at MIT.
- I am however a biophysicist by training, who has worked extensively on a geometric theory of molecular conformation based on an area of mathematics called "distance geometry".
- I first learned about geometric algebra through some of Rota's students who were studying the mathematical aspects of distance geometry, which I subsequently related to some of Hestene's work.

Second, why am I telling you this?

- I am more of a theorizer than a problem solver, and this course will be mainly about definitions. It will also show you some of the neat things you can do with them, and I expect all of you will quickly find new uses of your own.
- The "course" is absolutely informal, and is based on the philosophy that once students get interested in something, they'll quickly teach it to themselves ... and probably you too!

Third, what will these lectures try to cover?

1) Historical introduction (which you've just had!), and then an introduction to the basic notions of geometric algebra in one, two, three and n-dimensional Euclidean space.

2) Geometric calculus, some examples of how one can do classical statics and mechanics with geometric algebra, and how these fields of "physics" can be regarded as geometry.

3) Introduction to the space-time algebra: Special relativity, Maxwell's equations, and multispin quantum mechanics.

4) Nuclear magnetic resonance and quantum computing in the language of geometric algebra.

Fourth, where you can find out (lots) more:

◆ Hestenes' book *New Foundations for Classical Mechanics* (2nd ed., Kluwer, 1999) is a great introduction and exposition of classical mechanics using geometric algebra; his web site also contains most of his papers ready to be downloaded:

`http://ModelingNTS.la.asu.edu`

◆ The geometric algebra group at the Cavendish Labs of Cambridge Univ. also has a great web site for the physics, including the notes and overheads to a much more complete course:

`http://www.mrao.cam.ac.uk/~clifford`

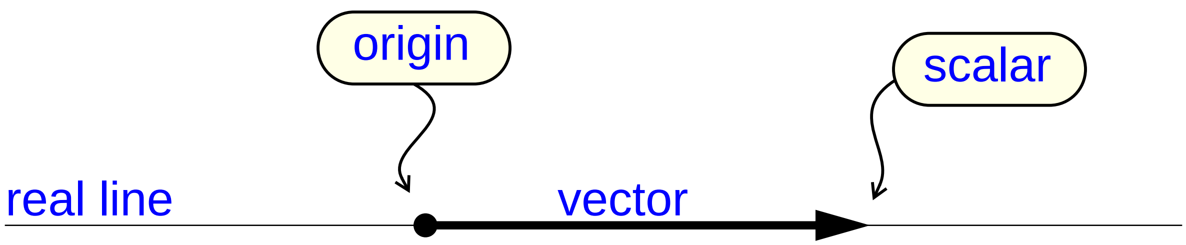
One-Dimensional Space

- Recall that a vector space $\mathcal{V}$ over the real numbers $\mathcal{R}$ is defined by the following operations:

  1) An associative multiplication by scalars, $(\alpha, \boldsymbol{v}) \to \alpha\boldsymbol{v}$ with $(\alpha\beta)\boldsymbol{v} = \alpha(\beta\boldsymbol{v})$; multiplying by zero gives a distinguished element called the **origin**: $\mathbf{0} \equiv 0\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{V}$.

  2) An associative and commutative addition of vectors, $(\boldsymbol{u}, \boldsymbol{v}) \to \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$, such that $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathcal{V}$.

  3) These operations are distributive, i.e. $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v}$ and $(\alpha + \beta)\boldsymbol{v} = \alpha\boldsymbol{v} + \beta\boldsymbol{v}$ for all $\alpha, \beta \in \mathcal{R}$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$.



- The vector space is **one-dimensional** if in addition for all $\boldsymbol{a}, \boldsymbol{u} \in \mathcal{V}$ with $\boldsymbol{u} \neq \mathbf{0}$ there exists $\alpha \in \mathcal{R}$ such that $\alpha\boldsymbol{u} = \boldsymbol{a}$.
- On choosing an arbitrary **unit** $\mathbf{0} \neq \boldsymbol{u} \in \mathcal{V}$, we obtain a one-to-one mapping between $\mathcal{R}$ and $\mathcal{V}$: $(\alpha \leftrightarrow \boldsymbol{a}) \Leftrightarrow (\alpha\boldsymbol{u} = \boldsymbol{a})$.
- This mapping allows us to define the **length** of $\boldsymbol{a}$ as $|\alpha|$.

Note the scalars are the linear transformations of $\mathcal{V}$, while the vectors are the objects on which the scalars act.

How to Multiply Vectors

It's so simple, so very simple, that only a child can do it!

- Let $\boldsymbol{a}, \boldsymbol{u} \in \mathcal{V}$ with $\boldsymbol{a} = \alpha\boldsymbol{u}$ as above.
- Suppose that we can "solve" this equation for $\alpha = \boldsymbol{a}\boldsymbol{u}^{-1}$, and see where this leads us.
- On setting $\boldsymbol{a} \equiv \boldsymbol{u}^{-1}$, we get $\alpha = \boldsymbol{u}^{-2}$, so $\boldsymbol{u}^2 = \alpha^{-1}$ is a *scalar*.
- Assuming further that $\boldsymbol{u} \equiv \boldsymbol{u}^{-1}$, the product of any two vectors becomes the product of their lengths (up to sign).

A Perhaps Yet Stranger Idea

- Both the real numbers $\mathcal{R}$ and $\mathcal{V}$ are 1-D spaces, and we can regard them as two orthogonal axes in a 2-D space. This 2-D "space" consists of formal *sums of scalars and vectors*.
- A product of two such entities (if associative & distributive) is:

$$(\alpha + \beta\boldsymbol{u})(\gamma + \delta\boldsymbol{u}) = (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma)\boldsymbol{u}$$

- This is our first geometric algebra $\mathcal{G}(1)$, which looks a lot like the complex numbers except that $\boldsymbol{u}^2 = 1$ not -1.
- In fact if we throw reflection in the origin into our transformation, i.e. $-\alpha = \boldsymbol{a}\boldsymbol{u}^{-1}$, we do get $\boldsymbol{u}^2 = -1$ & $\mathcal{G}(0, 1)$
- The only geometrically interesting alternative is $\boldsymbol{u}^2 = 0$.

THE 4 DIMENSIONS OF 2D

And the geometry behind complex numbers:

- Let us now suppose we can something similar in 2-D, i.e. that the square of a vector is its length squared: $\boldsymbol{a}^2 = \|\boldsymbol{a}\|^2$.
- There is one thing we *won't* assume: That the product of vectors is commutative (it was in 1-D, but nevermind...).
- We can still get a commutative product by averaging the results of multiplying both ways around, i.e. $(\boldsymbol{ab} + \boldsymbol{ba})/2$, which we call the **symmetric part** of the product.
- This is interpreted using the law of cosines, as follows:

$$\frac{1}{2}\left(\boldsymbol{ab} + \boldsymbol{ba}\right) = \frac{1}{2}\left(\boldsymbol{a}^2 + \boldsymbol{b}^2 - (\boldsymbol{a} - \boldsymbol{b})^2\right)$$
$$= \frac{1}{2}\left(\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - \|\boldsymbol{a} - \boldsymbol{b}\|^2\right) = \boldsymbol{a} \bullet \boldsymbol{b}$$

  We see have rediscovered the usual vector **inner product**!

- The antisymmetric part $\boldsymbol{a} \wedge \boldsymbol{b} = (\boldsymbol{ab} - \boldsymbol{ba})/2 = \boldsymbol{ab} - \boldsymbol{a} \bullet \boldsymbol{b}$ by way of contrast, is called the **outer product**. This is:
  - **Nilpotent**, i.e. $\boldsymbol{a} \wedge \boldsymbol{a} = 0$.
  - **Alternating**, i.e. $\boldsymbol{a} \wedge \boldsymbol{b} = -(\boldsymbol{b} \wedge \boldsymbol{a})$.
  - Has *nonpositive* square, since by Cauchy-Schwarz:

$$(\boldsymbol{a} \wedge \boldsymbol{b})^2 = -(\boldsymbol{a} \wedge \boldsymbol{b})(\boldsymbol{b} \wedge \boldsymbol{a}) = -(\boldsymbol{ab} - \boldsymbol{a} \bullet \boldsymbol{b})(\boldsymbol{ba} - \boldsymbol{a} \bullet \boldsymbol{b})$$
$$= -(\boldsymbol{abba} - (\boldsymbol{a} \bullet \boldsymbol{b})(\boldsymbol{ab} + \boldsymbol{ba}) + (\boldsymbol{a} \bullet \boldsymbol{b})^2)$$
$$= -(\|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2 - (\boldsymbol{a} \bullet \boldsymbol{b})^2) < 0$$

NB: This last property shows that the outer product of two vectors *cannot* itself be a vector! This new entity is called a **bivector**.

- This shows that the magnitude $\|\boldsymbol{a} \wedge \boldsymbol{b}\|^2 \equiv -(\boldsymbol{a} \wedge \boldsymbol{b})^2$ of a bivector is the area of the parallelogram spanned by $\boldsymbol{a}, \boldsymbol{b}$, justifying its geometric interpretation as a *oriented plane segment* (just as a vector is an oriented line segment).
- The outer product of orthonormal vectors σ_1, σ_2 is in fact a square root of -1, as $\iota \equiv \sigma_1 \wedge \sigma_2 = \sigma_1\sigma_2 - \sigma_1 \bullet \sigma_2 = \sigma_1\sigma_2$ so $\iota^2 = -(\sigma_1\sigma_2)(\sigma_2\sigma_1) = -\sigma_1(\sigma_2\sigma_2)\sigma_1 = -\sigma_1\sigma_1 = -1$.
- It follows that the products of pairs of vectors generate a subalgebra $\mathcal{G}^+(2)$, called the **even subalgebra**, which in 2-D is *isomorphic* to the complex numbers.
- The usual mapping of the plane onto the complex numbers $\mathcal{C}$ is obtained by multiplication by a unit vector $\boldsymbol{u}$, i.e.

$$\boldsymbol{a} \rightarrow \boldsymbol{au} = \boldsymbol{a} \bullet \boldsymbol{u} + \boldsymbol{a} \wedge \boldsymbol{u} = a_{\|} + a_{\perp}\iota.$$

- Thus the unit bivector can also be interpreted as a half-turn in the plane, or as the *generator* of rotations via the polar form: $\exp(\theta\iota) = \cos(\theta) + \iota\sin(\theta)$.

THE ALGEBRA OF 3D SPACE

The outer product of three 3-D vectors:

◆ Now consider the *symmetric* part of the Clifford product of a vector $\boldsymbol{a}$ and a bivector $\boldsymbol{b} \wedge \boldsymbol{c}$, which we denote as

$$\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = \frac{1}{2}\Big(\boldsymbol{a}(\boldsymbol{b} \wedge \boldsymbol{c}) + (\boldsymbol{b} \wedge \boldsymbol{c})\boldsymbol{a}\Big). \qquad ♥$$

Setting $\boldsymbol{b} \wedge \boldsymbol{c} = \boldsymbol{bc} - \boldsymbol{b} \bullet \boldsymbol{c}$ & $\boldsymbol{b} \wedge \boldsymbol{c} = \boldsymbol{b} \bullet \boldsymbol{c} - \boldsymbol{cb}$ on the r.h.s. gives

$$\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) \equiv \frac{1}{2}\Big(\boldsymbol{abc} - \boldsymbol{cba}\Big) \equiv (\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c} \;, \qquad ♣$$

Since ♥ is clearly antisymmetric in $\boldsymbol{b}$ and $\boldsymbol{c}$, swapping them in ♣ shows that it is equal to

$$\boldsymbol{b} \wedge (\boldsymbol{c} \wedge \boldsymbol{a}) \equiv \frac{1}{2}\Big(\boldsymbol{bca} - \boldsymbol{acb}\Big) \equiv (\boldsymbol{b} \wedge \boldsymbol{c}) \wedge \boldsymbol{a} \;. \qquad ♦$$

Similarly, swapping $\boldsymbol{a}$ and $\boldsymbol{b}$ shows ♣ to be the same as

$$\boldsymbol{c} \wedge (\boldsymbol{a} \wedge \boldsymbol{b}) \equiv \frac{1}{2}\Big(\boldsymbol{cab} - \boldsymbol{bac}\Big) \equiv (\boldsymbol{c} \wedge \boldsymbol{a}) \wedge \boldsymbol{b} \;. \qquad ♠$$

It follows that we have found an **outer product** of *three* vectors $\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}$, which is:

- ➤ Multi-linear (because the Clifford product is).
- ➤ Associative (according to the above definitions).
- ➤ Alternating (just take the average of ♣, ♦ and ♠).

Coordinates (ugh!)

- A big advantage of these techniques is that they do *not* require coordinate expansions relative to a basis.

- Nevertheless, dimensionality is most easily established in this way; thus let $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{V}$ be an orthonormal basis, and

$$\boldsymbol{a} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3, \quad \boldsymbol{b} = b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3 .$$

Then their outer product can be expanded to

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_1 b_2 - a_2 b_1)\sigma_1\sigma_2 + (a_1 b_3 - a_3 b_1)\sigma_1\sigma_3 + (a_2 b_3 - a_3 b_2)\sigma_2\sigma_3$$

since $\sigma_1\sigma_2 = \sigma_1 \bullet \sigma_2 + \sigma_1 \wedge \sigma_2 = \sigma_1 \wedge \sigma_2 = -\sigma_2\sigma_1$, & so on. Thus any outer product can be expanded in the basis bivectors $\sigma_1\sigma_2$, $\sigma_1\sigma_3$, $\sigma_2\sigma_3$, and the space of bivectors is again *3-D*.

- For three vectors, a similar but longer calculation shows

$$\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} = \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})\, \sigma_1\sigma_2\sigma_3 ,$$

so that the space of **trivectors** is *1-D*. The unit trivector $\iota = \sigma_1\sigma_2\sigma_3$ is again a square-root of -1, since:

$$\iota^2 = -(\sigma_1\sigma_2\sigma_3)(\sigma_3\sigma_2\sigma_1) = -(\sigma_1\sigma_2)\sigma_3^2(\sigma_2\sigma_1) = \ldots = -1$$

Because they *commute* with everything but *change sign* under inversion, trivectors are also called **pseudo-scalars**.

- The outer products of four or more vectors is always 0, and hence the dimension of the whole algebra is $1 + 3 + 3 + 1 = 8$.

GEOMETRIC INTERPRETATION

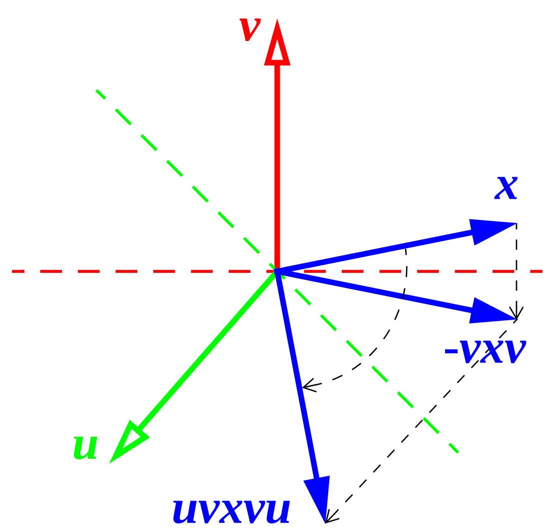
The Point of It All

- Any element of $\mathcal{G}(3)$ is simultaneously an *additive operator*, a (right, left, two-sided, inner, outer) *multiplicative operator*, and also an *operand* in the carrier space of the corresponding (semi)group representations.
- A positive scalar, for example, is both a magnitude as well as a dilatation about the origin.
- A vector can be viewed as a lineal magnitude, a transla-tion, or a *reflection-dilatation*, since for $\boldsymbol{u}, \boldsymbol{x} \in \mathcal{V}$ with $\boldsymbol{u}^2 = 1$,

$$-\boldsymbol{uxu} = -\boldsymbol{u}(\boldsymbol{x}_\perp + \boldsymbol{x}_\parallel)\boldsymbol{u} = \boldsymbol{u}^2\boldsymbol{x}_\perp - \boldsymbol{u}^2\boldsymbol{x}_\parallel = \boldsymbol{x}_\perp - \boldsymbol{x}_\parallel$$

(where $\boldsymbol{x}_\parallel = (\boldsymbol{x} \bullet \boldsymbol{u})\boldsymbol{u}$ and $\boldsymbol{x}_\perp = \boldsymbol{x} - (\boldsymbol{x} \bullet \boldsymbol{u})\boldsymbol{u} = (\boldsymbol{x} \wedge \boldsymbol{u})\boldsymbol{u}$).

- Thus a product of unit vectors $\boldsymbol{R} \equiv \boldsymbol{uv}$ represents the composition of their reflections, which is a **rotation** $\boldsymbol{Rx\tilde{R}}$ by *twice* the lesser angle between the normal planes (where $\boldsymbol{\tilde{R}} \equiv \boldsymbol{vu}$ is the **reverse** of $\boldsymbol{R}$). Left multiplication by a vector maps other vectors into a rotation-dilatation in their mutual plane.



◆ The **even subalgebra** $\mathcal{G}^{+}(3)$ is isomorphic to Hamilton's quaternions Q, since the basis elements $I \equiv \sigma_2\sigma_3$, $J \equiv \sigma_3\sigma_1$, $K \equiv \sigma_1\sigma_2$ *anticommute*, *square to* -1, and *satisfy the relations*

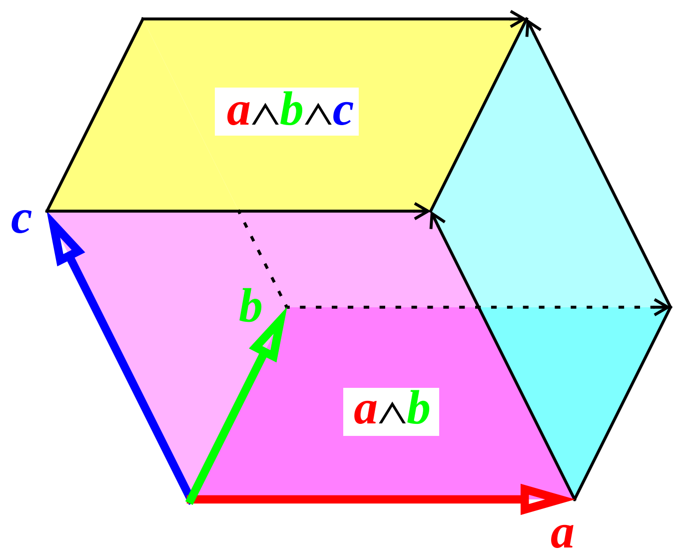
$$IJ = -K, \quad JK = -I, \quad IK = -J \quad \& \quad IJK = 1$$

(the signs are in accord with a right-handed basis $\sigma_1, \sigma_2, \sigma_3$).

◆ Next, consider the unit trivector ι: the two-sided operation is trivial ($\iota x \tilde{\iota} = \iota\tilde{\iota} x = x$), but right and left-multiplication are the *orthogonal complement* operation, e.g.

$$\iota\sigma_1 = -\sigma_3\sigma_2\sigma_1^2 = \sigma_2\sigma_3; \text{ similarly, } \iota\sigma_2 = \sigma_3\sigma_1,\ \iota\sigma_3 = \sigma_1\sigma_2.$$

◆ A vector $\boldsymbol{a}$ operates by outer multiplication on some other vector $\boldsymbol{b}$ by mapping it to the bivector $\boldsymbol{a} \wedge \boldsymbol{b}$, which has dimensions of area and so is best visualized as the *oriented parallelogram* swept out by $\boldsymbol{b}$ as it is translated by $\boldsymbol{a}$.



◆ Similarly, the trivector $\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}$ is the *oriented volume* element swept out by $\boldsymbol{a} \wedge \boldsymbol{b}$ as it is translated by $\boldsymbol{c}$; the associativity of the outer product means that the same volume is obtained on sweeping $\boldsymbol{b} \wedge \boldsymbol{c}$ by $\boldsymbol{a}$.

Relations to Other Mathematical Notions

- The foregoing geometric interpretations show that Gibbs' **cross product** is related to the outer product as follows:

$$\boldsymbol{a} \wedge \boldsymbol{b} = \iota(\boldsymbol{a} \times \boldsymbol{b}) \quad \Leftrightarrow \quad \boldsymbol{a} \times \boldsymbol{b} = -\iota(\boldsymbol{a} \wedge \boldsymbol{b})$$

(NB: the cross product changes sign on inversion in the origin, but the outer product is fully basis independent).

- It also follows that the **triple product** is the same as

$$\boldsymbol{a} \bullet (-\iota(\boldsymbol{b} \wedge \boldsymbol{c})) \;=\; \boldsymbol{a} \bullet (\boldsymbol{b} \times \boldsymbol{c}) \;=\; -\iota(\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c})$$

Writing $\iota\boldsymbol{d} \equiv \boldsymbol{b} \wedge \boldsymbol{c}$ allows this to be rewritten as

$$\iota(\boldsymbol{a} \bullet \boldsymbol{d}) \;=\; \boldsymbol{a} \wedge (\iota\boldsymbol{d}).$$

The **inner product** of a *vector* and a *bivector* is defined so that the reciprocal relation is also true, i.e.

$$\iota(\boldsymbol{a} \wedge \boldsymbol{d}) \;=\; \iota(\boldsymbol{a}\boldsymbol{d} - \boldsymbol{d}\boldsymbol{a})/2 \;=\; (\boldsymbol{a}(\iota\boldsymbol{d}) - (\iota\boldsymbol{d})\boldsymbol{a})/2 \;=\; \boldsymbol{a} \bullet (\iota\boldsymbol{d}).$$

- Thus one can regard the space of bivectors as the **dual space** $\mathcal{V}^*$, and multiplication by ι as the *isomorphism* defined by the given metric with $\mathcal{V}$ (which Gibbs identified with $\mathcal{V}^*$).

- Finally, consider the **commutator product** of bivectors:

$$[\iota\boldsymbol{a}, \iota\boldsymbol{b}] \;\equiv\; (\boldsymbol{b}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{b})/2 \;=\; -\iota(\boldsymbol{a} \times \boldsymbol{b}).$$

This shows that the *Lie algebra* $\mathrm{so}(3)$ is (isomorphic to) the commutator algebra of bivectors. The exponential map

$$\exp(-\iota\boldsymbol{a}/2) \;=\; \cos(\|\boldsymbol{a}/2\|) \;-\; \iota\sin(\|\boldsymbol{a}/2\|)\,\boldsymbol{a}/\|\boldsymbol{a}\|$$

is the *quaternion* for a rotation about $\boldsymbol{a}$ by the angle $\|\boldsymbol{a}\|$.

General Definitions

Sylvester's Law of Inertia:

Definition: A **metric vector space** $(\mathcal{V}, Q)$ is a real v.s. with a quadratic form $Q: \mathcal{V} \to \mathcal{R}$, usually written as $\|\mathbf{v}\|^2 \equiv Q(\mathbf{v})$.

Theorem: Any quadratic form can, by a suitable choice of coordinates, be written in the canonical form:

$$Q(v_1\sigma_1 + \cdots + v_n\sigma_n) = v_1^2 + \cdots + v_p^2 - v_{p+1}^2 - \cdots - v_{p+q}^2$$

Definition: (p, q) is the **signature** of the form, which is **nondegenerate** if $p + q = n$.

Geometric algebra of metric vector spaces:

Definition: An associative algebra over $\mathcal{R}$ is the **geometric algebra** $\mathcal{G}(Q)$ of a nondegenerate metric vector space $(\mathcal{V}, Q)$ if it contains $\mathcal{V}$ and $\mathcal{R}$ as distinct subspaces such that:

1) $\mathbf{v}^2 = Q(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}$;

2) $\mathcal{V}$ generates $\mathcal{G}(Q)$ as an algebra over $\mathcal{R}$;

3) $\mathcal{G}(Q)$ is not generated by any proper subspace of $\mathcal{V}$.

Theorem: All Clifford algebras are isomorphic to a direct sum of matrix algebras over $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{Q}$.

PARTING SHOTS

The following is an opinion gained by experience; it cannot be "proven" save to oneself: The merger of geometric and algebraic notions provided by **GA** allows one to more directly and efficiently use ones *geometric intuition* to formulate and solve mathematical problems than any other mathematical system (tensors, differential forms, etc.). In essence, this means using the brain's *visual information processing* abilities to solve mathematical problems by **parallel**, rather than **sequential**, reasoning. It is amazing how long it has taken the scientific community to grasp this simple idea! But as Grassmann himself said:

> ***I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, still that time will come when it will be brought forth from the dust of oblivion, and when the ideas now dormant will bring forth fruit ... For truth is eternal and divine, and no phase in the development of truth, however small may be the region encompassed, can pass on without leaving a trace; truth remains, even though the garment in which poor mortals clothe it may fall to dust.***

Hermann Grassmann,
forward to the 2nd *Ausdehnungslehre*, 1862.