GEOMETRIC ALGEBRA:

Parallel Processing for the Mind

Timothy F. Havel (Nuclear Engineering)

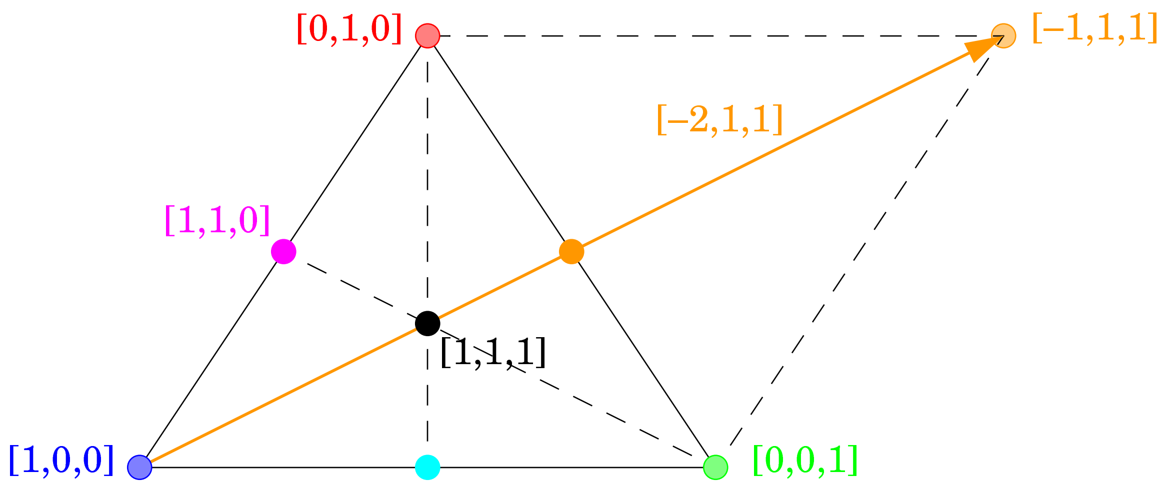
LECTURE #2

In its geometrical applications, multiple algebra will naturally take on one of two principal forms, according as vectors or points are taken as the elementary quantities. These forms of multiple algebra may be named vector analysis and point analysis. The former is included in the latter, since the subtraction of points gives us vectors, and in this way Grassmann's vector analysis is included in his point analysis. On the other hand, if we represent points by vectors drawn from a common origin, and then develop those relations between such vectors representing points, which are independent of the position of the origin, we may obtain a large part, possibly all, of an algebra of points. The vector analysis, thus enlarged, is hardly to be distinguished from a point analysis, but the treatment of the subject in this way has something of a makeshift character, as opposed to the unity and simplicity of the subject when developed directly from the idea of something situated at a point.

J. W. Gibbs, *On Multiple Algebra*, Science Mag. **25:37-66**, 1886.

BARYCENTRIC CALCULUS

August Ferdinand Möbius (1790-1868)



The **barycentric sum** of $n+1$ points $\{\boldsymbol{p}_k \in \mathcal{E}_n\}_{k=0}^{n}$ in an n-D Euclidean space $\mathcal{E}_n$ is denoted by $W\boldsymbol{q} \equiv w_0 \boldsymbol{p}_0 + \cdots + w_n \boldsymbol{p}_n$, where $W = \sum_k w_k$ is the total **weight**. Such sums can also be viewed as a *vector space* $\mathcal{R}_{n+1}$ of dimension $n+1$, wherein the points $\boldsymbol{p}_k$ correspond to a basis (as shown above), namely

$$\{\boldsymbol{p}_k \leftrightarrow \mathbf{p}_k = [\ldots, 0, 1, 0, \ldots] \in \mathcal{R}_{n+1}\}_{k=0}^{n}$$

(so $W\boldsymbol{q} \leftrightarrow [w_0, \ldots, w_n]$). The P.-D. inner product vs. this basis $\mathbf{x} \bullet \mathbf{y} = (n+1)\sum_k x_k y_k$ induces the coordinate change $W\mathbf{q} \rightarrow$

$$W[\mathbf{q} \bullet \mathbf{c}; (\mathbf{q} \wedge \mathbf{c})\mathbf{c}^{-1}], \quad \text{where } \boldsymbol{c} \equiv (\boldsymbol{p}_0 + \ldots + \boldsymbol{p}_n)/(n+1) \leftrightarrow \mathbf{c}$$

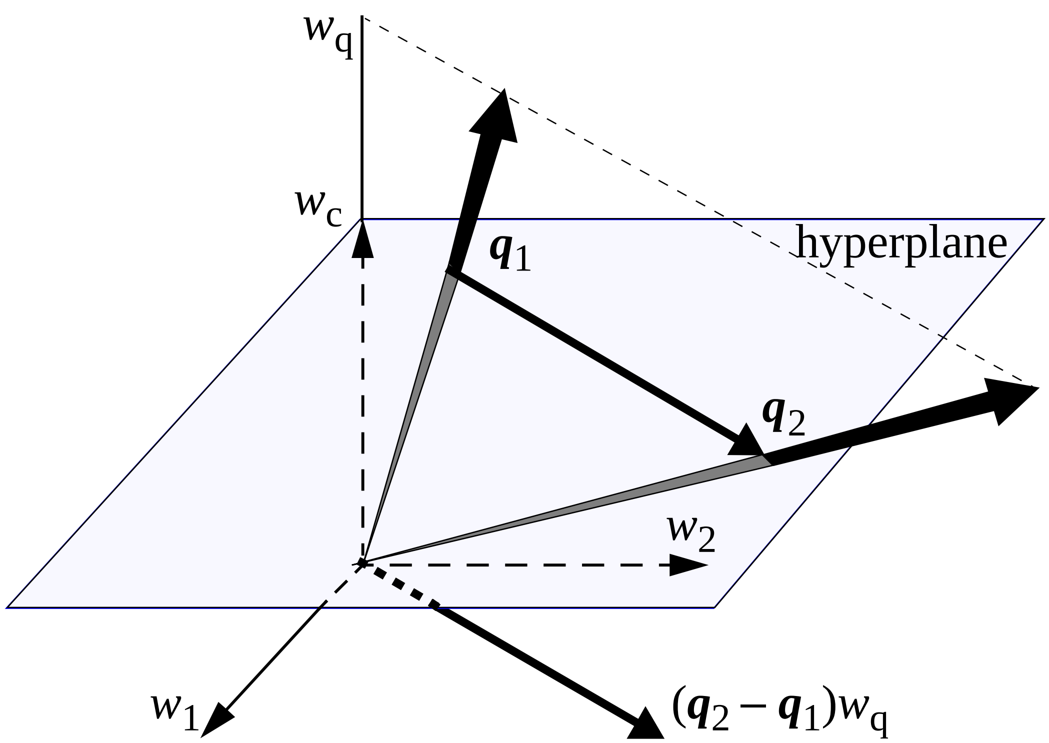
is the **centroid** of the $\boldsymbol{p}_k$; the coordinates of $\boldsymbol{q}$ vs. this new basis $[\sum_k w_k; \sum_k w_k(\boldsymbol{p}_k - \boldsymbol{c})]$ are called its **affine coordinates**.

Points at Infinity

If *every* (unit weight!) point of $\mathcal{E}_n$ can be *uniquely* expressed as a barycentric sum $\sum_k w_k \boldsymbol{p}_k$ of a system of points $\{\boldsymbol{p}_k\}_{k=0}^{n}$, this system is called a **point basis** for $\mathcal{E}_n$.

Points of zero weight are the limit of a sequence of points which moves off to infinity in a fixed direction as the sum of the weights goes to zero; therefore they are called **points at infinity**, and identified with a *direction*. In general, they also have a magnitude, but this depends on how the limit is taken.

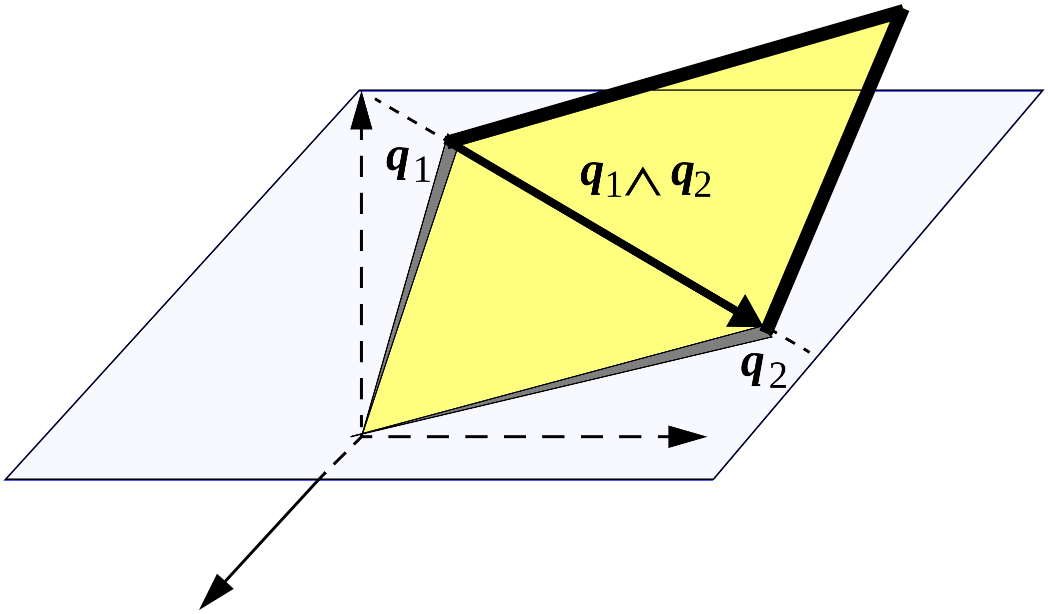
If we choose our basis (or metric!) so that $\{\boldsymbol{p}_1 - \boldsymbol{c}, \ldots, \boldsymbol{p}_n - \boldsymbol{c}\}$ are (ortho)normal, the weights vs. the basis $\{\boldsymbol{c}, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ are affine coordinates, and (unit weight) points can be viewed as vectors to an *affine hyperplane* in $\mathcal{R}_{n+1}$, as shown:



Line-Bound Vectors

Thus the points at infinity $\boldsymbol{q}_2 - \boldsymbol{q}_1$ (and their magnitudes) can be viewed as **vectors** *parallel* to the affine hyperplane.

This interpretation in $\mathcal{R}_{n+1}$ shows that the outer product of a two points is an *oriented segment* of the line between them, i.e.



Since the magnitude of the bivector is twice the length of the segment times its height above the origin, any other pair of points separated by the same distance along the line generate the *same* line bound vector; this can also be proven as follows:

$$(\boldsymbol{q}_1 + \alpha(\boldsymbol{q}_2 - \boldsymbol{q}_1)) \wedge (\boldsymbol{q}_2 + \alpha(\boldsymbol{q}_2 - \boldsymbol{q}_1)) =$$

$$\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 + \alpha(\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 - \boldsymbol{q}_1 \wedge \boldsymbol{q}_2) + \alpha^2(\boldsymbol{q}_2 - \boldsymbol{q}_1) \wedge (\boldsymbol{q}_2 - \boldsymbol{q}_1) = \boldsymbol{q}_1 \wedge \boldsymbol{q}_2$$

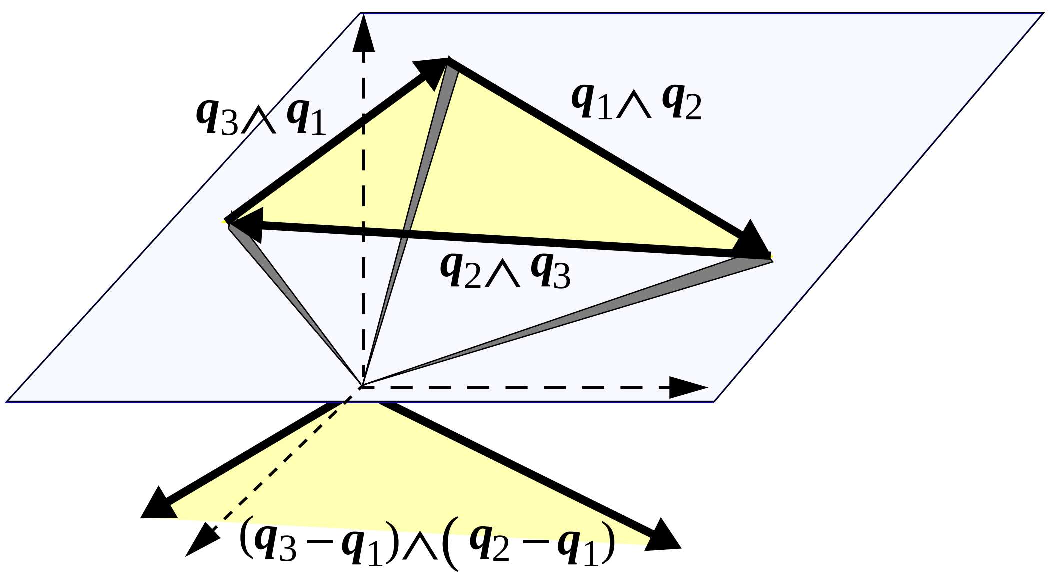
Note that a line-bound vector is geometrically distinct from a **free vector** representing a point at infinity!

Free Areal Magnitudes

The outer product of two free vectors is called a **free areal magnitude**. We can write this as

$$(\boldsymbol{q}_2 - \boldsymbol{q}_1) \wedge (\boldsymbol{q}_3 - \boldsymbol{q}_1) = \boldsymbol{q}_2 \wedge \boldsymbol{q}_3 - \boldsymbol{q}_1 \wedge \boldsymbol{q}_3 + \boldsymbol{q}_1 \wedge \boldsymbol{q}_2 .$$

This shows that the ordered sum of the line-bound vectors around a triangle yields a free areal magnitude, i.e.



This is just a discrete version of *Stokes' theorem* ... with a geometric interpretation. To go from here to the continuous version, just approximate the curve by a polygon, triangulate it, apply the discrete version to each triangle, and take the limit as the number of sides goes to infinity.

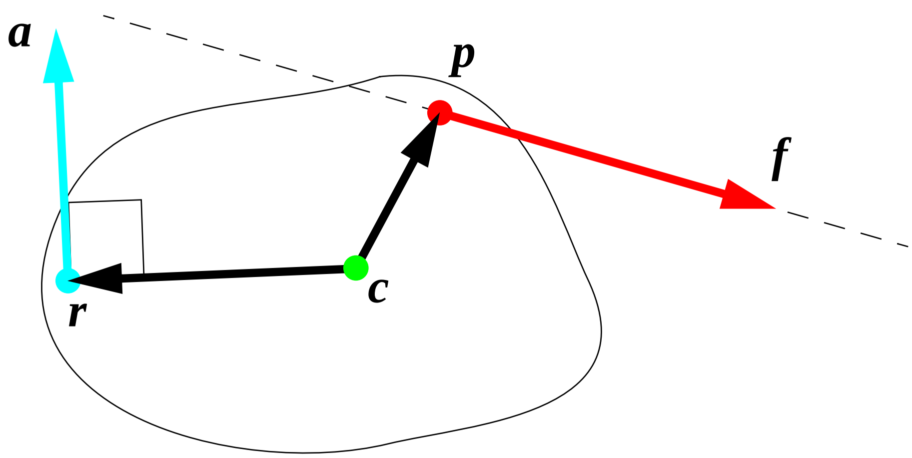
The outer product of three points is a **plane-bound area** ... and so on into as many dimensions as you like!

Forces and Torques in One

We regard a force as a free vector $\boldsymbol{f}$; taking the outer product with a point $\boldsymbol{p}$ in a rigid body yields a line-bound vector $\boldsymbol{f} \wedge \boldsymbol{p}$ which contains *all the information* needed to determine how the force affects the body. To see this, observe that if the body is pivoted about the point $\boldsymbol{c}$, then the acceleration at each point $\boldsymbol{r}$ is given (up to a constant factor) by

$$\boldsymbol{a} = (\boldsymbol{f} \wedge (\boldsymbol{p} - \boldsymbol{c})) \bullet (\boldsymbol{r} - \boldsymbol{c}),$$

as shown in the drawing below:



This illustrates a general rule that we shall see many examples of: *The generators of motion are bivectors*.

A second force $\boldsymbol{f}$ applied to another point $\boldsymbol{q}$ produces the same response at any point $\boldsymbol{r}$ only if $\boldsymbol{f} \wedge (\boldsymbol{p} - \boldsymbol{c}) = \boldsymbol{g} \wedge (\boldsymbol{q} - \boldsymbol{c})$, or

$$\boldsymbol{f} \wedge \boldsymbol{p} - \boldsymbol{g} \wedge \boldsymbol{q} = (\boldsymbol{f} - \boldsymbol{g}) \wedge \boldsymbol{c}.$$

This in turn can be true for all $\boldsymbol{c}$ only if $\boldsymbol{f} = \boldsymbol{g}$ and hence $\boldsymbol{f} \wedge \boldsymbol{p} = \boldsymbol{g} \wedge \boldsymbol{q}$, which proves our claim above.

The Theory of Screws by Sir Robert Ball

A complementary interpretation of line-bound vectors is as an *infinitesimal motion*. In the plane, *rotation* about a point $\boldsymbol{c}$ with angular velocity $\dot{\theta}$ is represented by a weighted point $\dot{\theta}\boldsymbol{c}$, and all information on the instantaneous motion of $\boldsymbol{p}$ is in

$$\dot{\theta}\boldsymbol{c} \wedge \boldsymbol{p} = \dot{\theta}(\boldsymbol{c} - \boldsymbol{p}) \wedge \boldsymbol{p} = \boldsymbol{p} \wedge (\dot{\theta}(\boldsymbol{p} - \boldsymbol{c})) = \boldsymbol{p} \wedge \boldsymbol{v}_{\perp},$$

where $\boldsymbol{v} = \dot{\boldsymbol{p}}$ is the linear velocity of $\boldsymbol{p}$. To prove this, note that the derivative of the squared distance to any fixed point $\boldsymbol{q}$ is

$$\partial_t \|\boldsymbol{p} - \boldsymbol{q}\|^2 = 2(\boldsymbol{p} - \boldsymbol{q}) \bullet \boldsymbol{v} = 2\iota((\boldsymbol{p} - \boldsymbol{q}) \wedge (\iota\boldsymbol{v})).$$

Now if $\boldsymbol{E} \equiv \sigma_1\sigma_2$ is the unit free area, then $\boldsymbol{v}_{\perp} = \boldsymbol{vE} = -\boldsymbol{Ev}$. Also, since $\iota = \sigma_0\sigma_1\sigma_2$, $\iota\boldsymbol{E} = -\sigma_0$, so $\iota\boldsymbol{v} = -\sigma_0\boldsymbol{v}_{\perp}$, and:

$$\partial_t \|\boldsymbol{p} - \boldsymbol{q}\|^2 = 2\iota((\boldsymbol{q} - \boldsymbol{p}) \wedge \sigma_0 \wedge \boldsymbol{v}_{\perp}) = 2\iota((\boldsymbol{q} - \boldsymbol{p}) \wedge \boldsymbol{p} \wedge \boldsymbol{v}_{\perp})$$

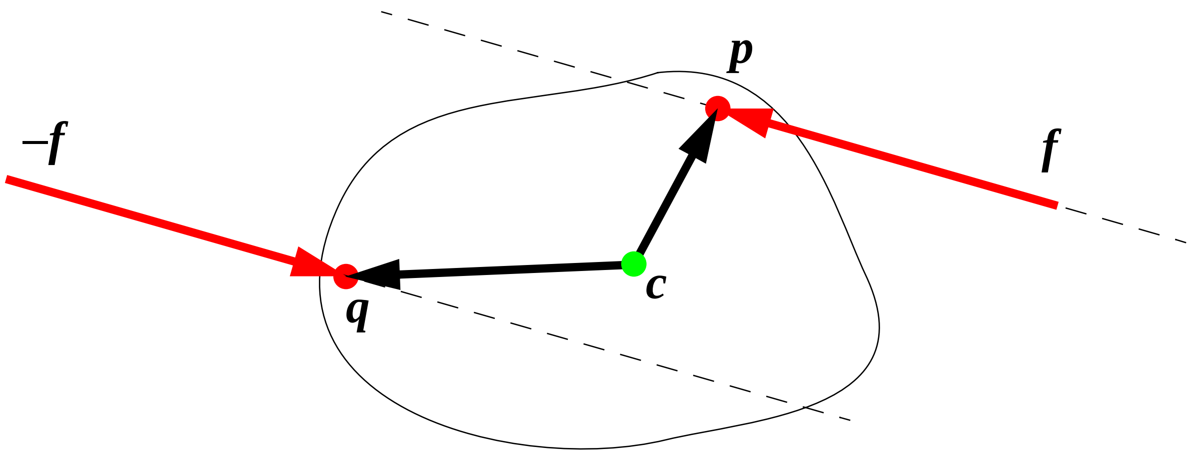
$$= 2\iota(\boldsymbol{q} \wedge \boldsymbol{p} \wedge \boldsymbol{v}_{\perp}) = 2\iota\dot{\theta}(\boldsymbol{q} \wedge \boldsymbol{p} \wedge (\boldsymbol{p} - \boldsymbol{c})) = 2\iota((\dot{\theta}\boldsymbol{c} \wedge \boldsymbol{p}) \wedge \boldsymbol{q})$$

A *translation* is represented by the free vector $\boldsymbol{t}_{\perp} = \boldsymbol{Et}$, i.e. as a rotation about a point-at-infinity.

In *3-D space*, an instantaneous rotation about an axis $\boldsymbol{a}$ thru a point $\boldsymbol{c}$ is represented by a line-bound vector, i.e. by a **rotor** $\dot{\theta}(\boldsymbol{c} \wedge \boldsymbol{a})$, and the resulting motion of a point $\boldsymbol{p}$ by $\dot{\theta}(\boldsymbol{c} \wedge \boldsymbol{a}) \wedge \boldsymbol{p}$. Instantaneous translations are represented by a **translator** $\boldsymbol{t}_{\perp} \wedge \boldsymbol{a}$, while the sum of a rotor & translator is a general **screw**.

Known Only by Their Effects

The analog of a translator for forces is a sum of two forces, whose line-bound vectors that are equal in magnitude, opposite in direction, and on different lines:



The sum of such a pair of forces $\boldsymbol{f} \wedge \boldsymbol{p} - \boldsymbol{f} \wedge \boldsymbol{q} = \boldsymbol{f} \wedge (\boldsymbol{p} - \boldsymbol{q})$ is called a **couple**, and is the outer product of two free vectors.

This brings us to one of the deepest *mysteries of geometry*: The reality of **nonfactorizable** elements in the algebra. For example, a general sum of forces cannot itself be written as the outer product of any two points or free vectors $\boldsymbol{x} \wedge \boldsymbol{y}$. This follows since the l.h.s. below is 0 but the r.h.s. vanishes only if the points / vectors are linearly dependent:

$$\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{x} \wedge \boldsymbol{y} = (\boldsymbol{f} \wedge \boldsymbol{p} + \boldsymbol{g} \wedge \boldsymbol{q}) \wedge (\boldsymbol{f} \wedge \boldsymbol{p} + \boldsymbol{g} \wedge \boldsymbol{q}) = 2\boldsymbol{f} \wedge \boldsymbol{p} \wedge \boldsymbol{g} \wedge \boldsymbol{q}$$

Since $\boldsymbol{g} = (\boldsymbol{f} + \boldsymbol{g}) - \boldsymbol{f}$, any such **wrench** can always be written as the sum of a couple and a finite force; similarly, any screw can be written as the sum of a rotor and a translator.

THE REGRESSIVE PRODUCT

Grassmann actually defined many kinds of geometrical multiplication, including ultimately the geometric product itself.

Of particular interest was the **regressive** (outer) product, which may be defined via duality as

$$\boldsymbol{X} \vee \boldsymbol{Y} \equiv ((\boldsymbol{X}\iota^{-1}) \wedge (\boldsymbol{Y}\iota^{-1}))\iota^{-1} .$$

While the usual (**progressive**) outer product is a blade in the *direct sum* of the nonintersecting subspaces of its factors, the regressive product is a blade in the *intersection* of the spanning subspaces of its factors. In $\mathcal{E}_2$, for example,

$$(\boldsymbol{p} \wedge \boldsymbol{q}) \vee (\boldsymbol{r} \wedge \boldsymbol{s}) = -((\boldsymbol{p} \wedge \boldsymbol{q})\iota) \wedge ((\boldsymbol{r} \wedge \boldsymbol{s})\iota)\iota = ((\boldsymbol{p} \wedge \boldsymbol{q})\iota) \bullet (\boldsymbol{r} \wedge \boldsymbol{s})$$

$$= \boldsymbol{r} \bullet ((\boldsymbol{p} \wedge \boldsymbol{q})\iota)\boldsymbol{s} - \boldsymbol{s} \bullet ((\boldsymbol{p} \wedge \boldsymbol{q})\iota)\boldsymbol{r} = \iota((\boldsymbol{r} \wedge \boldsymbol{p} \wedge \boldsymbol{q})\boldsymbol{s} - (\boldsymbol{s} \wedge \boldsymbol{p} \wedge \boldsymbol{q})\boldsymbol{r})$$

More generally, the progressive & regressive products are related by the "shuffle" formula, $(\boldsymbol{a}_1 \wedge \ldots \wedge \boldsymbol{a}_k) \vee (\boldsymbol{b}_1 \wedge \ldots \wedge \boldsymbol{b}_l)$

$$= \sum_{\text{shuffles } \pi} (-1)^{\pi} \left[\boldsymbol{a}_{\pi(1)}, \ldots, \boldsymbol{a}_{\pi(n-l)}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_l \right] \ldots$$

$$\ldots \boldsymbol{a}_{\pi(n-l+1)} \wedge \ldots \wedge \boldsymbol{a}_{\pi(k)} ,$$

where a **shuffle** is a permutation of $\{1, \ldots, k\}$ that preserves the order of the first k and last $n-k$ elements, $(-1)^{\pi}$ is the parity of that permutation, and the square brackets indicates the dual of the outer product of the n enclosed factors ($k \geq l$).

The Metric Connection

Now let us bring a metric in, by defining a quadratic form in the barycentric coordinates of the points $\boldsymbol{q} \leftrightarrow [q_0\ q_1\ \dots\ q_n]$ vs. a basis $[\boldsymbol{p}_0, \boldsymbol{p}_1, \dots, \boldsymbol{p}_n]$; the corresponding symmetric bilinear form may be written using matrices as

$$D(\boldsymbol{r}, \boldsymbol{s}) \equiv [r_0\ \ r_1\ \ \dots\ \ r_n] \begin{bmatrix} 0 & -d_{01}^2/2 & \dots & -d_{0n}^2/2 \\ -d_{01}^2/2 & 0 & \dots & -d_{1n}^2/2 \\ \dots & \dots & \dots & \dots \\ -d_{0n}^2/2 & -d_{1n}^2/2 & \dots & 0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \dots \\ s_n \end{bmatrix},$$

where d_{ij}^2 $(i, j = 0, \dots, n)$ are the squared distances among the basis points $[\boldsymbol{p}_0, \boldsymbol{p}_1, \dots, \boldsymbol{p}_n]$. Note that on the difference of a pair of basis points, e.g. $\boldsymbol{p}_0 - \boldsymbol{p}_1 \leftrightarrow [1\ {-1}\ 0\ \dots\ 0]$, this form evaluates to $D(\boldsymbol{p}_0 - \boldsymbol{p}_1, \boldsymbol{p}_0 - \boldsymbol{p}_1) = d_{01}^2$; more generally, it gives the *length* of any free vector directly. On any pair of basis points is the form is clearly $D(\boldsymbol{p}_i, \boldsymbol{p}_j) = -d_{ij}^2/2$, and a general **inner product** of pairs of unit weight *points* is $D(\boldsymbol{r}, \boldsymbol{s}) = -\|\boldsymbol{r} - \boldsymbol{s}\|^2/2$. To be convinced of this, take the vertices of a right pyramid as a basis, i.e. $d_{0i}^2 = 1$ & $d_{ij}^2 = 2$ for all $0 \le i, j \le n$ $(i < j)$. The barycentric coordinates w.r.t. points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_n$ *are* Cartesian coordinates vs. the frame with origin $\boldsymbol{p}_0$ & orthonormal axes $\boldsymbol{p}_i - \boldsymbol{p}_0$, and you can show that $D(\boldsymbol{u}, \boldsymbol{v}) = \sum_i u_i v_i$ for all free vectors $\boldsymbol{u}, \boldsymbol{v}$.

I don't think we're in a Euclidean space anymore, Toto...

The astonishing fact is that this quadratic form is *indefinite*, i.e. can be negative, and hence it does *not* correspond to a Euclidean metric outside of the subspace of free vectors. In fact its signature is $(n, -1)$, which for $n = 3$ is the Minkowski metric of relativity. What a coincidence!

At the basis centroid $\boldsymbol{o} \leftrightarrow [1\ 1\ \ldots\ 1]/(n+1)$, the form is

$$D(\boldsymbol{o}, \boldsymbol{o}) = \frac{-1}{(n+1)^2} \sum_{i<j} d_{ij}^2 ,$$

which by a theorem of Lagrange is the negative squared radius of gyration of the basis (vs. unit weights). Thus if we define

$$\tilde{D}(\boldsymbol{r}, \boldsymbol{s}) \equiv D(\boldsymbol{r}, \boldsymbol{s}) + (n+1)(r_0 s_0 c_0^2 + \ldots + r_n s_n c_n^2),$$

where $c_k \equiv \|\boldsymbol{p}_k - \boldsymbol{o}\|$ is the distance of $\boldsymbol{p}_k$ to the centroid, then $\tilde{D}(\boldsymbol{o}, \boldsymbol{o}) = 0$, whereas on the points $\tilde{D}(\boldsymbol{p}_k, \boldsymbol{p}_k) = (n+1)c_k^2$.

I have a conjecture that for free vectors, $\tilde{D}$ is the *inertial tensor*. If so, a simplex of weighted points would describe the dynamical properties of a rigid body more simply than a total mass, center of mass and inertial tensor, and would enable the above static theory of motions and forces to be extended to a dynamical theory (in 3-D, both involve 10 parameters).