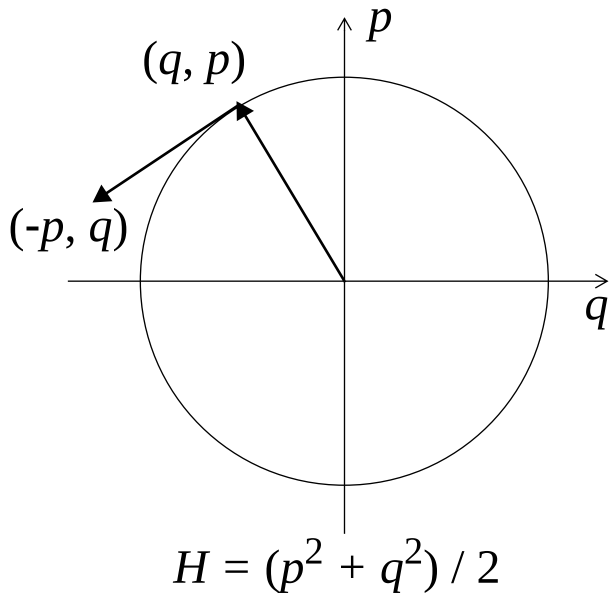
GEOMETRIC ALGEBRA:
Imaginary Numbers Are Real

Timothy F. Havel (Nuclear Engineering)

LECTURE 3

The Return of Hamilton



We have seen that "complex numbers" arise in geometric algebras over the reals in many ways, e.g. as the even sub-algebra of the Euclidean plane. Thus an alternative to abstract complex geometry is to work in an *even* dimensional real space. This leads to more geometric way to think about Hamiltonian mechanics in phase (position-momentum) space. Given a phase space vector $\boldsymbol{r} \equiv \Sigma_k(p_k\boldsymbol{u}_k + q_k\boldsymbol{v}_k)$ versus an orthonormal basis $\{\boldsymbol{u}_k, \boldsymbol{v}_k\}$ and a Hamiltonian H, Hamilton's equations of motion $\dot{q}_k = \partial H/\partial p_k$, $\dot{p}_k = -\partial H/\partial q_k$ imply that

$$\dot{\boldsymbol{r}} = \sum_k (\dot{p}_k \boldsymbol{u}_k + \dot{q}_k \boldsymbol{v}_k) = \sum_k \left(-\frac{\partial H}{\partial q_k} \boldsymbol{u}_k + \frac{\partial H}{\partial p_k} \boldsymbol{v}_k \right)$$
$$= \sum_k \left(\frac{\partial H}{\partial q_k} \boldsymbol{v}_k + \frac{\partial H}{\partial p_k} \boldsymbol{u}_k \right) \bullet (\boldsymbol{u}_k \wedge \boldsymbol{v}_k) \equiv \nabla H \bullet \boldsymbol{J}$$

where $\boldsymbol{J} \equiv \sum_k (\boldsymbol{u}_k \wedge \boldsymbol{v}_k)$ is the so-called **doubling bivector**. The time derivative of any other phase space function G is given by its **Poisson bracket** with H, i.e. $\dot{G} = \nabla G \bullet \dot{\boldsymbol{r}} =$

$$\nabla G \bullet (\nabla H \bullet \boldsymbol{J}) = \sum_k \left(\frac{\partial G}{\partial q_k} \boldsymbol{v}_k + \frac{\partial G}{\partial p_k} \boldsymbol{u}_k \right) \bullet \sum_k \left(\frac{\partial H}{\partial p_k} \boldsymbol{v}_k - \frac{\partial H}{\partial q_k} \boldsymbol{u}_k \right)$$
$$= \sum_k \left(\frac{\partial G}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial G}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \equiv \{G, H\}$$

Like *any* bivector, $\boldsymbol{J}$ defines a *symplectic* quadratic form via

$$\omega(\boldsymbol{x}, \boldsymbol{y}) \equiv \boldsymbol{x} \bullet (\boldsymbol{y} \bullet \boldsymbol{J}) = (\boldsymbol{x} \wedge \boldsymbol{y}) \bullet \boldsymbol{J} = \boldsymbol{J} \bullet (\boldsymbol{x} \wedge \boldsymbol{y}) = -\boldsymbol{J} \bullet (\boldsymbol{y} \wedge \boldsymbol{x})$$

which in turn defines a “complex structure” on phase space via

$$\boldsymbol{u}_k \bullet \boldsymbol{J} = \boldsymbol{v}_k = -\boldsymbol{J} \bullet \boldsymbol{u}_k \qquad \boldsymbol{v}_k \bullet \boldsymbol{J} = -\boldsymbol{u}_k = -\boldsymbol{J} \bullet \boldsymbol{v}_k ,$$

and hence $(\boldsymbol{u}_k \bullet \boldsymbol{J}) \bullet \boldsymbol{J} = -\boldsymbol{u}_k \qquad (\boldsymbol{v}_k \bullet \boldsymbol{J}) \bullet \boldsymbol{J} = -\boldsymbol{v}_k$.

We may thus regard any phase space vector as a “complex” vector $\boldsymbol{r} = \boldsymbol{q} + \boldsymbol{p} \bullet \boldsymbol{J}$ (with $\boldsymbol{p}, \boldsymbol{q} \in \langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle$). The corresponding Hermitian inner product has real & imaginary parts:

$$Re(\langle \boldsymbol{r} | \boldsymbol{r}' \rangle) = \boldsymbol{r} \bullet \boldsymbol{r}' = \boldsymbol{p} \bullet \boldsymbol{p}' + \boldsymbol{q} \bullet \boldsymbol{q}'$$
$$Im(\langle \boldsymbol{r} | \boldsymbol{r}' \rangle) = (\boldsymbol{r} \wedge \boldsymbol{r}') \bullet \boldsymbol{J} = \boldsymbol{q}' \bullet \boldsymbol{p} - \boldsymbol{q} \bullet \boldsymbol{p}'$$

Symplectic or *canonical transformations* are those which preserve the symplectic form, $\boldsymbol{S} \boldsymbol{J} \tilde{\boldsymbol{S}} = \boldsymbol{J}$, and so include all unitary transformations w.r.t. this complex structure.

GEOMETRIC CALCULUS

Starting with the "real" complex derivative:

$$\nabla f(z) = \partial_x f(x,y) - \iota\,\partial_y f(x,y) \qquad (z = x + \iota y).$$

The conjugate operator annihilates analytic functions, since

$$\begin{aligned} 0 = \bar{\nabla} f(z) &= \partial_x f(x,y) + \iota\,\partial_y f(x,y) \\ &= \partial_x (u(x,y) + \iota v(x,y)) + \iota\,\partial_y (u(x,y) + \iota v(x,y)) \\ &= (\partial_x u(x,y) - \partial_y v(x,y)) + \iota\,(\partial_x v(x,y) + \partial_y u(x,y)) \end{aligned}$$

are the **Cauchy-Riemann** equations. On analytic functions,

$$\begin{aligned} \nabla f(z) &= \partial_x (u(x,y) + \iota v(x,y)) - \iota\,\partial_y (u(x,y) + \iota v(x,y)) \\ &= (\partial_x u(x,y) + \partial_y v(x,y)) + \iota\,(\partial_x v(x,y) - \partial_y u(x,y)) \end{aligned}$$

is *twice* the usual complex derivative, since on complex powers

$$\nabla z^n = n z^{n-1}((\partial_x x + \partial_y y) + \iota\,(\partial_x y - \partial_y x)) = 2 n z^{n-1},$$

but ∇ is well-defined even for *nonanalytic* functions, for example $\nabla \bar{z} = (\partial_x x - \partial_y y) + \iota(\partial_x y + \partial_y x) = 0$, which is *nowhere* analytic!

From Gauss' and Stoke's theorems in the plane, we can also derive the first Cauchy integral formula as follows:

$$\begin{aligned} \oint f(z)dz &= \oint (u\,dx - v\,dy) + \iota \oint (u\,dy + v\,dx) = \oint (\bar{\boldsymbol{f}} \bullet d\boldsymbol{z}) + \iota \oint (\bar{\boldsymbol{f}} \times d\boldsymbol{z}) \\ &= \iint (\vec{\nabla} \times \bar{\boldsymbol{f}})\,dx\,dy + \iota \iint (\vec{\nabla} \bullet \bar{\boldsymbol{f}})\,dx\,dy = \iota \iint (\bar{\nabla} f)\,dx\,dy = 0 \end{aligned}$$

GC in Three (& Higher) Dimensions

Geometric algebra also enables one to put standard results in multivariate calculus (on manifolds) into a geometric form. The key is the **vector derivative** of Hestenes & Sobczyk, namely

$$\nabla \boldsymbol{F}(\boldsymbol{x}) = \sigma_1 \frac{\partial \boldsymbol{F}}{\partial x_1} + \cdots + \sigma_n \frac{\partial \boldsymbol{F}}{\partial x_n} = \sigma_1 \partial_1 \boldsymbol{F} + \cdots + \sigma_n \partial_n \boldsymbol{F} ,$$

where $\boldsymbol{F}: V \to G(n)$ and the basis vectors σ_k act on the partials by *geometric multiplication*.

In 2-D with $\boldsymbol{F} \equiv \boldsymbol{f}: V \to V$, we get the "complex" derivative as

$$\nabla \boldsymbol{f} = (\sigma_1 \partial_1 + \sigma_2 \partial_2)(\sigma_1)^2(\sigma_1 u + \sigma_2 v) = (\partial_x - \iota \partial_y)(u + \iota v) .$$

Breaking the 3-D up into symmetric and antisymmetric parts, we obtain the **divergence** and the (dual of the) **curl** as

$$\nabla \boldsymbol{F}(\boldsymbol{x}) = \nabla \bullet \boldsymbol{F}(\boldsymbol{x}) + \nabla \wedge \boldsymbol{F}(\boldsymbol{x}) .$$

The **directional derivative** is given by $(\boldsymbol{a} \bullet \nabla)\boldsymbol{F}(\boldsymbol{x})$, which for vector $\boldsymbol{F} \equiv \boldsymbol{f}: V \to V$ is the usual Jacobian applied to $\boldsymbol{a}$.

A very general form of the fundamental theorem of calculus can also be given, which includes the *divergence*, *Green's* & *Stokes'* theorems as special cases:

$$\int_V d\boldsymbol{x} \nabla \boldsymbol{F} = \oint_{\partial V} d\boldsymbol{s} \boldsymbol{F}$$

This integral formula generalizes to any number of dimensions!

THE SPACE-TIME ALGEBRA

If only Einstein had known!

The geometric algebra $\mathcal{G}(1,3)$ of Minkowski space-time $\mathcal{V}$ is generated by an orthonormal basis $[\gamma_0, \gamma_1, \gamma_2, \gamma_3]$, and is linearly spanned by:

$1 \in \mathcal{R}$ **scalars (1-D)**

$\gamma_k \in \mathcal{V} \qquad (\gamma_k^2 = 2\delta_{0k} - 1)$ **vectors (4-D)**

$\gamma_k\gamma_l = -\gamma_l\gamma_k \in \wedge_2\mathcal{V}$ **bivectors (6-D)**

$\gamma_k\gamma_l\gamma_m \in \wedge_3\mathcal{V}$ **dual vectors (4-D)**

$\iota \equiv \gamma_0\gamma_1\gamma_2\gamma_3 \in \wedge_4\mathcal{V}$ **pseudo-scalars (1-D)**

For $1 \le k, l \le 3, k \ne l$, consider the involutory mapping between the *even subalgebra* $\mathcal{G}^+(1,3)$ and $\mathcal{G}(3)$, given by

$$1 \leftrightarrow 1 \qquad \gamma_k\gamma_0 \leftrightarrow \sigma_k \qquad \gamma_k\gamma_l \leftrightarrow \sigma_l\sigma_k \qquad \iota \leftrightarrow \iota.$$

Since $\sigma_k^2 = (\gamma_k\gamma_0)^2 = -\gamma_k^2\gamma_0^2 = 1$ and

$$(\gamma_k\gamma_0)(\gamma_l\gamma_0) = -\gamma_0\gamma_k\gamma_l\gamma_0 = \gamma_0\gamma_l\gamma_k\gamma_0 = -(\gamma_l\gamma_0)(\gamma_k\gamma_0)$$

this mapping is an algebra isomorphism $\mathcal{G}^+(1,3) \approx \mathcal{G}(3)$. Note that this isomorphism is *frame-dependent*.

In a given frame of reference $[\gamma_0, \gamma_1, \gamma_2, \gamma_3]$, we can *split* any event $\mathbf{x} \in \mathcal{V}$ into

$$\mathbf{x}\gamma_0 = \mathbf{x} \bullet \gamma_0 + \mathbf{x} \wedge \gamma_0 = t + \boldsymbol{x}$$

where $t \equiv \mathbf{x} \bullet \gamma_0$ is the *time* and $\boldsymbol{x} \equiv \mathbf{x} \wedge \gamma_0 \in \langle \gamma_1\gamma_0, \gamma_2\gamma_0, \gamma_3\gamma_0 \rangle$ the *place* of the event in this frame. Henceforth, we will assume a fixed frame, and denote the relative spatial vectors in this frame as $\sigma_1 \equiv \gamma_1\gamma_0$, $\sigma_2 \equiv \gamma_2\gamma_0$, $\sigma_3 \equiv \gamma_3\gamma_0$.

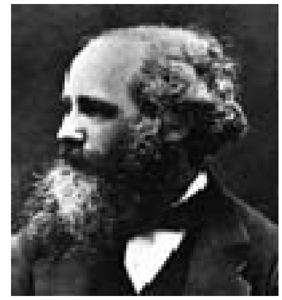
For $1 \le k, l \le 3$, $k \ne l$, any **Lorentz transformation** is given by $\pm \mathbf{L}\mathbf{x}\tilde{\mathbf{L}}$, where $\mathbf{L} \in \mathcal{G}^+(1, 3)$ with $\mathbf{L}\tilde{\mathbf{L}} = 1$. Once again, the corresponding *Lie algebra* can be identified with the *bivector algebra* under its *commutator product*, and

$$\begin{aligned} \exp(\vartheta\sigma_k) &= \cosh(\vartheta) + \sigma_k \sinh(\vartheta) \\ &= (\gamma(1 + \sigma_k\beta))^{1/2} \\ \exp(\theta\iota\sigma_k) &= \cos(\theta) + \iota\sigma_k \sin(\theta) \end{aligned}$$

boost by $\beta = \tanh(2\vartheta)$; **rotate by** 2θ.

The provides a concise means of expressing many physical relations, even ostensibly nonrelativistic ones. For example, if $\mathbf{v} = \mathbf{v}\gamma_0^2 = (\mathbf{v} \bullet \gamma_0 + \mathbf{v} \wedge \gamma_0)\gamma_0 = (dt/d\tau + d\boldsymbol{x}/d\tau)\gamma_0$ is the **space-time velocity** (τ proper time), and we define the **Faraday bivector** as $\mathbf{F} = \boldsymbol{E} + \iota \boldsymbol{B} \in \wedge_2 \mathcal{V}$, a *manifestly covariant* form of the Lorentz force law is $m\dot{\mathbf{v}} = q\mathbf{F} \bullet \mathbf{v}$ (where m is mass & q charge), so that $\mathbf{F}$ is the *generator of motion*.

JAMES CLERK MAXWELL



An even more spectacularly concise way of expressing the geometry behind the physics is found in the S.T.A. version of *Maxwell's equations*. If $\mathbf{J} = (\rho + \boldsymbol{J})\gamma_0$ is the **space-time current density**, these equations are simply

$$\nabla \mathbf{F} = \mathbf{J}$$

where $\boldsymbol{\nabla} \equiv \gamma_0\partial_0 - \gamma_1\partial_1 - \gamma_2\partial_2 - \gamma_3\partial_3 = \gamma_0(\partial_0 - \vec{\nabla})$ is the (contra-variant) *vector derivative* operator.

To show that this is equivalent to the usual four equations, we begin by breaking it into its vector & trivector parts:

$$\nabla \mathbf{F} = \nabla \bullet \mathbf{F} + \nabla \wedge \mathbf{F} = (\rho + \boldsymbol{J})\gamma_0$$

The vector part yields $\rho + \boldsymbol{J} = (\nabla \bullet \mathbf{F}) \bullet \gamma_0 + (\nabla \bullet \mathbf{F}) \wedge \gamma_0$, or

$$(\gamma_0 \wedge \nabla) \bullet \mathbf{F} - \gamma_0 \wedge (\nabla \bullet \mathbf{F}) = \vec{\nabla} \bullet (\boldsymbol{E} + \iota\boldsymbol{B}) - \partial_0(\boldsymbol{E} + \iota\boldsymbol{B}) + \nabla \bullet (\gamma_0 \iota\boldsymbol{B}),$$

where we have expanded the second term as follows:

$$\gamma_0 \wedge (\nabla \bullet \mathbf{F}) = (\nabla \bullet \gamma_0)\mathbf{F} - \nabla \bullet (\gamma_0 \wedge \mathbf{F}) = \partial_0(\boldsymbol{E} + \iota\boldsymbol{B}) - \nabla \bullet (\gamma_0 \iota\boldsymbol{B})$$

The scalar part is $\vec{\nabla} \bullet \boldsymbol{E} = \rho$, the electric field source equation, while the bivector part is:

$$\boldsymbol{J} + \partial_0\boldsymbol{E} = \vec{\nabla} \bullet (\iota\boldsymbol{B}) - \partial_0(\iota\boldsymbol{B}) + (\gamma_0(\partial_0 - \vec{\nabla})) \bullet (\gamma_0 \iota\boldsymbol{B})$$

The first term is zero since $\vec{\nabla}$ and $\iota\boldsymbol{B}$ are orthogonal as space-time bivectors, while the last term is on the right is the sum of

$$(\gamma_0\partial_0)\bullet(\gamma_0\iota\boldsymbol{B}) = \partial_0(\gamma_0\bullet(\gamma_0\wedge\iota\boldsymbol{B})) = \partial_0(\iota\boldsymbol{B})$$

(which cancels with the second term), and $(\vec{\nabla}\gamma_0)\bullet(\gamma_0\wedge\iota\boldsymbol{B})$

$$= (\gamma_0\vec{\nabla})\bullet(\iota(\gamma_0\boldsymbol{B})) = \iota((\vec{\nabla}\gamma_0)\wedge(\gamma_0\boldsymbol{B})) = -\iota(\vec{\nabla}\wedge\boldsymbol{B}).$$

Putting all this together gives the usual magnetic field source equation $\boldsymbol{J}+\partial_0\boldsymbol{E} = \vec{\nabla}\times\boldsymbol{B}$.

We now go back and take the trivector part of ☸, i.e.

$$0 = \nabla\wedge\mathbf{F} = (\gamma_0(\partial_0-\vec{\nabla}))\wedge(\boldsymbol{E}+\iota\boldsymbol{B}) = \text{four terms}.$$

The term $\gamma_0\partial_0\wedge\boldsymbol{E} = \partial_0(\gamma_0\wedge\boldsymbol{E}) = 0$ because the electric field is $\boldsymbol{E} = E_1\gamma_1\gamma_0+E_2\gamma_2\gamma_0+E_3\gamma_3\gamma_0$, while the term

$$(\vec{\nabla}\gamma_0)\wedge(\iota\boldsymbol{B}) = \iota(\gamma_0\vec{\nabla})\bullet\boldsymbol{B} = \frac{\iota}{2}(\gamma_0\vec{\nabla}\boldsymbol{B}+\boldsymbol{B}\gamma_0\vec{\nabla}) = -\gamma_0\iota(\vec{\nabla}\bullet\boldsymbol{B})$$

gives us the equation $\vec{\nabla}\bullet\boldsymbol{B} = 0$ on wedging with γ_0. Finally,

$$(\vec{\nabla}\gamma_0)\wedge\boldsymbol{E} = \gamma_0\iota(\vec{\nabla}\times\boldsymbol{E})$$

while $(\gamma_0\partial_0)\wedge(\iota\boldsymbol{B}) = -\gamma_0\iota(\partial_0\boldsymbol{B})$, so that the last equation $\vec{\nabla}\times\boldsymbol{E} = \partial_0\boldsymbol{B}$ follows on dotting ☸ by $\gamma_0\iota = \gamma_1\gamma_2\gamma_3$.

Note the field's space-time energy-momentum is $\mathbf{P}\equiv$

$$(\tfrac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)+\boldsymbol{E}\times\boldsymbol{B})\gamma_0 = \tfrac{1}{2}(\boldsymbol{E}+\iota\boldsymbol{B})(\boldsymbol{E}-\iota\boldsymbol{B})\gamma_0 = -\tfrac{1}{2}\mathbf{F}(\gamma_0\mathbf{F}\gamma_0)\gamma_0,$$

so the electromagnetic *stress-energy tensor* is just $-(\mathbf{F}\gamma_0\mathbf{F})/2$.

WHAT IS "GEOMETRY"?

A general definition of "geometry" was given by Felix Klein (right) in his famous Erlanger Programm address (1872), as follows: *Geometry is the study of those quantities and relations which are preserved under a group of transformations*.

This definition inspired a generation of research, but is more general than we need. The classical geometries (projective, Euclidean and hyperbolic) involve only **Lie groups** (named after Klein's friend and competitor, Sophus Lie). These are essentially groups whose operations can be described analytically.

Lie groups are usually studied via their linear (matrix) *representations*, which were introduced by Ferdinand Frobenius. The irreducible representations are the building blocks of his theory, & they map everything in their **carrier space** onto everything else.

Thus, according to Klein, the vectors of the carrier space are objects in the geometry of the corresponding group. It was Hermann Weyl's great insight to realize that the "tensor product" is just a means of constructing new group representations, and he therefore defined tensors more generally as the vectors in the carrier space of any group. Rota called this *Weyl's principle*.

Geometric algebras can be constructed via tensor products, and so it may seem that tensors are more general. As far as Lie groups are concerned, however, all the corresponding tensors can be built from the geometric product! For example, to construct the tensor product $(\mathcal{G}(3))^{\otimes N} \approx (\mathcal{G}^{+}(1,3))^{N}$ from the space-time algebra, we assume there exists a *common* frame of reference, i.e. a natural choice of time-like γ_0^m in every particle space, then:

- The even subalgebras $\mathcal{G}^{+}(1,3)$ of different particle spaces *commute*, since for $1 \le i, j \le 3$ and $1 \le k < l \le N$

$$\sigma_k^m \sigma_l^n \leftrightarrow (\gamma_k^m \gamma_0^m)(\gamma_l^n \gamma_0^n) = (\gamma_l^n \gamma_0^n)(\gamma_k^m \gamma_0^m) \leftrightarrow \sigma_l^n \sigma_k^m$$

where the superscripts are particle indices.

You want another example?

Well, here's one for you: *Line-bound vectors*! In order to relate this to ordinary mathematics, we're going to resort to coordinates for a moment here (very sad). Thus consider the affine coordinates of a unit weight point, i.e.

$$\boldsymbol{p} = \sigma_0 + p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3 \leftrightarrow [1\ p_1\ p_2\ p_3] \ .$$

In this coordinate system, the group of all linear transformations which preserve the weights of the points (and hence, in particular, zero weight points at infinity or free vectors, with coordinates $[0\ v_1\ v_2\ v_3]$) is represented by matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & r_{11} & r_{12} & r_{13} \\ t_2 & r_{21} & r_{22} & r_{23} \\ t_3 & r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 1 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \end{bmatrix}$$

The invertible matrices of this form form the group of **affine transformations**, consisting of the translations and nonsingular linear transformations of the 3-D coordinates (note that the sub-group of translations admits no *linear* 3-D representation, since $(\boldsymbol{p} + \boldsymbol{q}) + \boldsymbol{t} \neq (\boldsymbol{p} + \boldsymbol{t}) + (\boldsymbol{q} + \boldsymbol{t})$). If $[r_{ij}]$ is a proper orthogonal matrix, we obtain a subgroup consisting of all *translations* and *rotations*, i.e. the group of **rigid motions** or proper Euclidean group.

Now consider the outer product of $\boldsymbol{p}$ with a second point $\boldsymbol{q}$; the coordinates of this bivector vs. the induced basis $\sigma_i \wedge \sigma_j$ are

$$\boldsymbol{p} \wedge \boldsymbol{q} = (q_1 - p_1)\sigma_0 \wedge \sigma_1 + \cdots + (p_2 q_3 - p_3 q_2)\sigma_2 \wedge \sigma_3,$$

which are just the six 2×2 minors of the 2×4 matrix shown on the right here. These are called the **Plücker coordinates** of the line-bound vector after Julius Plücker (left), who used them to study line geometry at about the same time as Grassmann.

$$\begin{bmatrix} 1 & p_1 & p_2 & p_3 \\ 1 & q_1 & q_2 & q_3 \end{bmatrix}$$

It can be shown that if the points $\boldsymbol{p}$ & $\boldsymbol{q}$ are both subjected to the same affine transformation, then the Plücker coord-inates are transformed by the **second compound** of the matrix, i.e. by the matrix of all its 2×2 minors. The collection of all such 6×6 matrices again forms a group, which maps the free areal magnitudes onto the same, and thereby gives us a *new* representation of the group of affine transformations.

The locus $\{\boldsymbol{r} | \boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r} = 0\}$ of the tensor $\boldsymbol{p} \wedge \boldsymbol{q}$ is just a **line** in an affine space, which likewise uniquely determines the tensor in accord with Weyl's principle. General affine transformations do not preserve the length of a line-bound vector, but they do if $[r_{ij}]$ forms a rotation matrix. It follows that line-bound vectors are *Euclidean* tensors. Other Euclidean tensors are spheres of fixed radius, ellipsoids of fixed semi-axes, etc.

There is a Better Way

Matrices are for the computers ...

The point is that, with geometric algebra, we don't need to use coordinates and matrices at all! As we have seen, the rotation of a 3D vector $\boldsymbol{v} \in \mathcal{R}_3$ is given by $\boldsymbol{R}\boldsymbol{v}\tilde{\boldsymbol{R}} = \boldsymbol{R}\boldsymbol{v}\boldsymbol{R}^{-1}$, where the **spinor** (or quaternion if you will) is given by

$$\boldsymbol{R} = \exp((\iota \boldsymbol{a})/2) = \sum_{k=0}^{\infty} (\iota \boldsymbol{a}/2)^k / k! \qquad (\boldsymbol{a} \in \mathrm{R}_3 \ \&\ \iota = \sigma_1\sigma_2\sigma_3)$$

$$= \sum_{k=0}^{\infty}\left((-1)^k \|\boldsymbol{a}/2\|^{2k}/(2k)! + \frac{\iota \boldsymbol{a}}{\|\boldsymbol{a}\|}\sum_{k=0}^{\infty}(-1)^k \frac{\|\boldsymbol{a}/2\|^{2k+1}}{(2k+1)!}\right)$$

$$= \cos(\|\boldsymbol{a}\|/2) + \iota \boldsymbol{a}/\|\boldsymbol{a}\| \sin(\|\boldsymbol{a}\|/2) \qquad \text{(by the series for cos \& sin)}$$

More generally, given a metric vector space $\mathcal{R}_{p,q}$ of signature p, q and a product of unit vectors $\boldsymbol{R} = \boldsymbol{u}_1\boldsymbol{u}_2\ldots$ within it, the mapping $\boldsymbol{v} \in \mathrm{R}_{p,q} \to \boldsymbol{R}\boldsymbol{v}\boldsymbol{R}^{-1}$ is a composition of reflections and inversions, and hence an **isometry** of $\mathcal{R}_{p,q}$ (meaning that it preserves the inner product). Moreover, by the Cartan-Dieudonné theorem, *every* isometry is of this form.

The multiplicative subgroup of the algebra $\mathcal{G}(p,q)$ generat-ed by the unit vectors is denoted by $\mathbf{Pin}(p,q)$, while the sub-group of the even subalgebra $\mathcal{G}^+(p,q) \cap \mathbf{Pin}(p,q)$ is denoted by $\mathbf{Spin}(p,q)$. These are two-fold covers of the rotation groups $O(p,q)$ and $SO(p,q)$, resp., since $\pm\boldsymbol{R}$ gives the *same* rotation.

How higher-rank tensors transform:

Now consider how a bivector $\boldsymbol{p} \wedge \boldsymbol{q}$ transforms under rotation of its factors by a spinor $\boldsymbol{R} \in \mathbf{Spin}(p, q)$, i.e.

$$(\boldsymbol{R}\boldsymbol{p}\boldsymbol{R}^{-1}) \wedge (\boldsymbol{R}\boldsymbol{q}\boldsymbol{R}^{-1}) = \tfrac{1}{2}((\boldsymbol{R}\boldsymbol{p}\boldsymbol{R}^{-1})(\boldsymbol{R}\boldsymbol{q}\boldsymbol{R}^{-1}) - (\boldsymbol{R}\boldsymbol{q}\boldsymbol{R}^{-1})(\boldsymbol{R}\boldsymbol{p}\boldsymbol{R}^{-1}))$$

$$= \tfrac{1}{2}(\boldsymbol{R}\boldsymbol{p}\boldsymbol{q}\boldsymbol{R}^{-1} - \boldsymbol{R}\boldsymbol{q}\boldsymbol{p}\boldsymbol{R}^{-1}) = \boldsymbol{R}(\boldsymbol{p} \wedge \boldsymbol{q})\boldsymbol{R}^{-1}$$

It follows $\boldsymbol{p} \wedge \boldsymbol{q}$ transforms by the same spinor as $\boldsymbol{p}$ and $\boldsymbol{q}$, i.e. the carrier space has changed, but the representation of the group itself has not! This holds more generally for all tensors that can be constructed via the geometric product.

Side remarks: Since geometric algebra can only treat the isometries of metric vector spaces, it might seem limited to metric tensors, but in fact the general linear group $GL(n)$ is the sub-group of $\mathbf{Pin}(n, n)$, and hence this is *not* a limitation. Similarly, it may seem to be limited to anti-symmetric tensors (i.e. outer products), but we will soon see how general tensor products may be built from geometric products.

Languages for Geometry