GEOMETRIC ALGEBRA:

Imaginary Numbers Are Real

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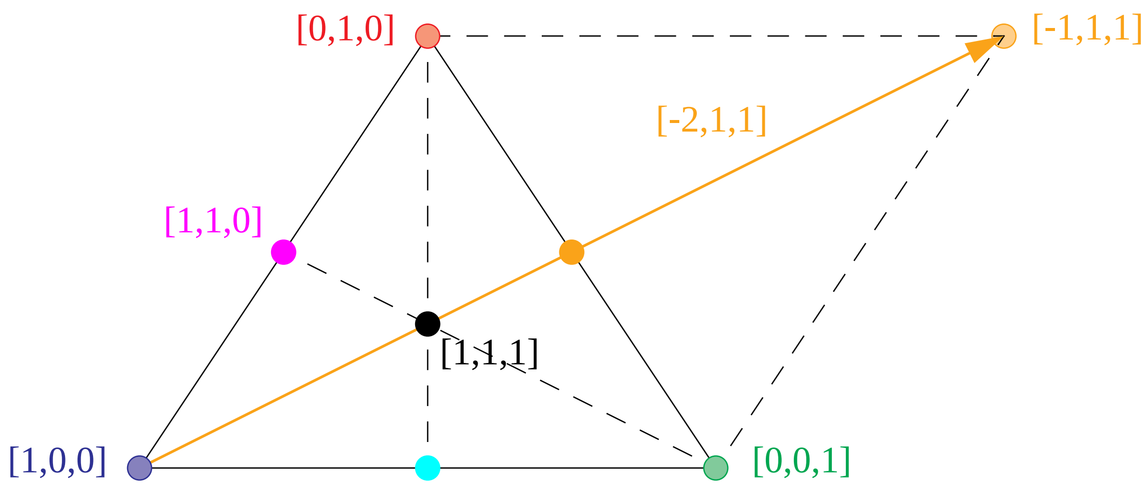
LECTURE #4

In its geometrical applications, multiple algebra will naturally take on one of two principal forms, according as vectors or points are taken as the elementary quantities. These forms of multiple algebra may be named vector analysis and point analysis. The former is included in the latter, since the subtraction of points gives us vectors, and in this way Grassmann's vector analysis is included in his point analysis. On the other hand, if we represent points by vectors drawn from a common origin, and then develop those relations between such vectors representing points, which are independent of the position of the origin, we may obtain a large part, possibly all, of an algebra of points. The vector analysis, thus enlarged, is hardly to be distinguished from a point analysis, but the treatment of the subject in this way has something of a makeshift character, as opposed to the unity and

J. W. Gibbs, *On Multiple Algebra*, Science Mag. **25:37-66**, 1886.

BARYCENTRIC CALCULUS

August Ferdinand Möbius (1790-1868)



The **barycentric sum** of $n + 1$ points $\{\boldsymbol{p}_k \in \mathcal{E}_n\}_{k=0}^{n}$ in an n-D Euclidean space $\mathcal{E}_n$ is denoted by $W\boldsymbol{q} \equiv w_0\boldsymbol{p}_0 + \cdots + w_n\boldsymbol{p}_n$, where $W = \sum_k w_k$ is the total **weight**. Such sums can also be viewed as a *vector space* R_{n+1} of dimension $n + 1$, wherein the points $\boldsymbol{p}_k$ correspond to a basis (as shown above), namely

$$\{\boldsymbol{p}_k \leftrightarrow \mathbf{p}_k = [\ldots, 0, 1, 0, \ldots] \in R_{n+1}\}_{k=0}^{n}$$

(so $W\boldsymbol{q} \leftrightarrow [w_0, \ldots, w_n]$). The P.-D. inner product vs. this basis $\mathbf{x} \bullet \mathbf{y} = (n+1)\sum_k x_k y_k$ induces the coordinate change $W\mathbf{q} \rightarrow$

$$W[\mathbf{q} \bullet \mathbf{c}; (\mathbf{q} \wedge \mathbf{c})\mathbf{c}^{-1}], \quad \text{where } \boldsymbol{c} \equiv (\boldsymbol{p}_0 + \ldots + \boldsymbol{p}_n)/(n+1) \leftrightarrow \mathbf{c}$$

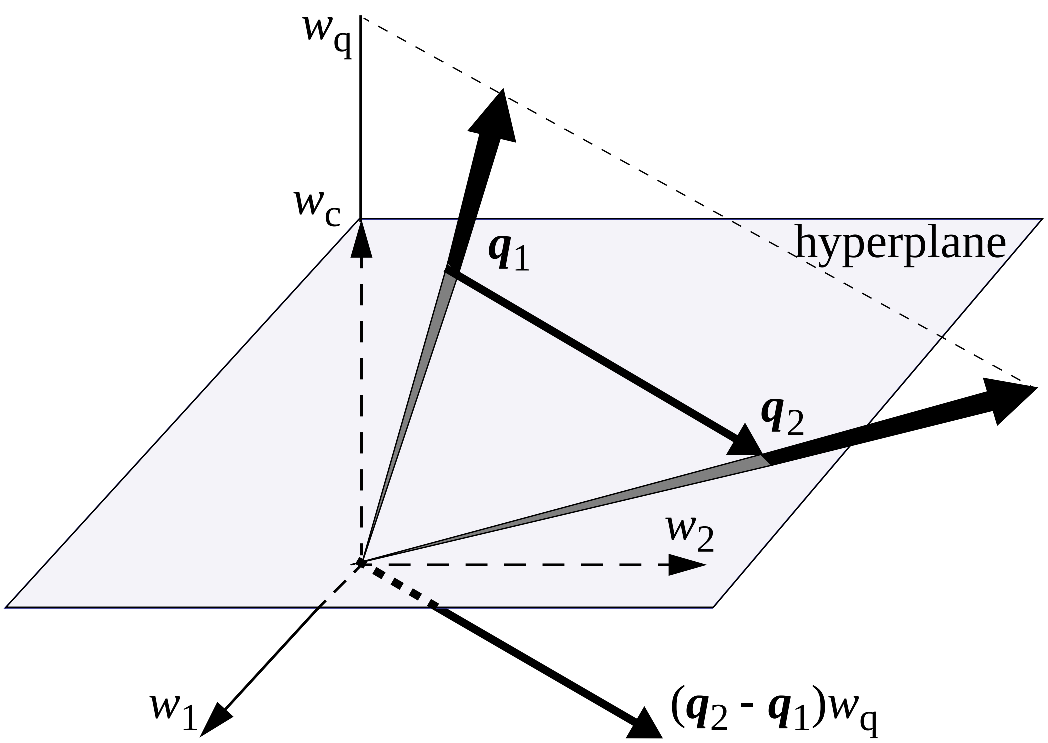
is the **centroid** of the $\boldsymbol{p}_k$; the coordinates of $\boldsymbol{q}$ vs. this new basis $[\sum_k w_k; \sum_k w_k(\boldsymbol{p}_k - \boldsymbol{c})]$ are called its **affine coordinates**.

Points at Infinity

If *every* (unit weight!) point of $\mathcal{E}_n$ can be *uniquely* expressed as a barycentric sum $\sum_k w_k \boldsymbol{p}_k$ of a system of points $\{\boldsymbol{p}_k\}_{k=0}^{n}$, this system is called a **point basis** for $\mathcal{E}_n$.

Points of *zero* weight are the limit of a sequence of points which moves off to infinity in a fixed direction as the sum of the weights goes to zero; therefore they are called **points at infinity**, and identified with a *direction*. In general, they also have a magnitude, but this depends on how the limit is taken.

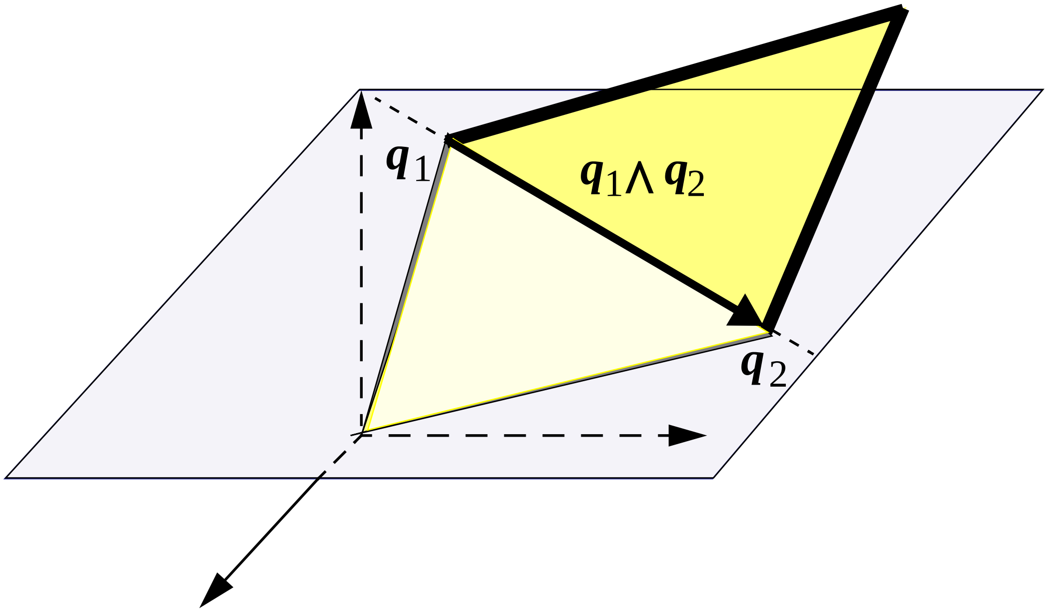
If we choose our basis (or metric!) so that $\{\boldsymbol{p}_1 - \boldsymbol{c}, \ldots, \boldsymbol{p}_n - \boldsymbol{c}\}$ are (ortho)normal, the weights vs. the basis $\{\boldsymbol{c}, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ are affine coordinates, and (unit weight) points can be viewed as vectors to an *affine hyperplane* in $\mathcal{R}_{n+1}$, as shown:



Line-Bound Vectors

Thus the points at infinity $\boldsymbol{q}_2 - \boldsymbol{q}_1$ (and their magnitudes) can be viewed as **vectors** *parallel* to the affine hyperplane.

This interpretation in $\mathcal{R}_{n+1}$ shows that the outer product of a two points is an *oriented segment* of the line between them, i.e.



Since the magnitude of the bivector is twice the length of the segment times its height above the origin, any other pair of points separated by the same distance along the line generate the *same* line bound vector; this can also be proven as follows:

$$(\boldsymbol{q}_1 + \alpha(\boldsymbol{q}_2 - \boldsymbol{q}_1)) \wedge (\boldsymbol{q}_2 + \alpha(\boldsymbol{q}_2 - \boldsymbol{q}_1)) =$$

$$\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 + \alpha(\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 - \boldsymbol{q}_1 \wedge \boldsymbol{q}_2) + \alpha^2(\boldsymbol{q}_2 - \boldsymbol{q}_1) \wedge (\boldsymbol{q}_2 - \boldsymbol{q}_1) = \boldsymbol{q}_1 \wedge \boldsymbol{q}_2$$

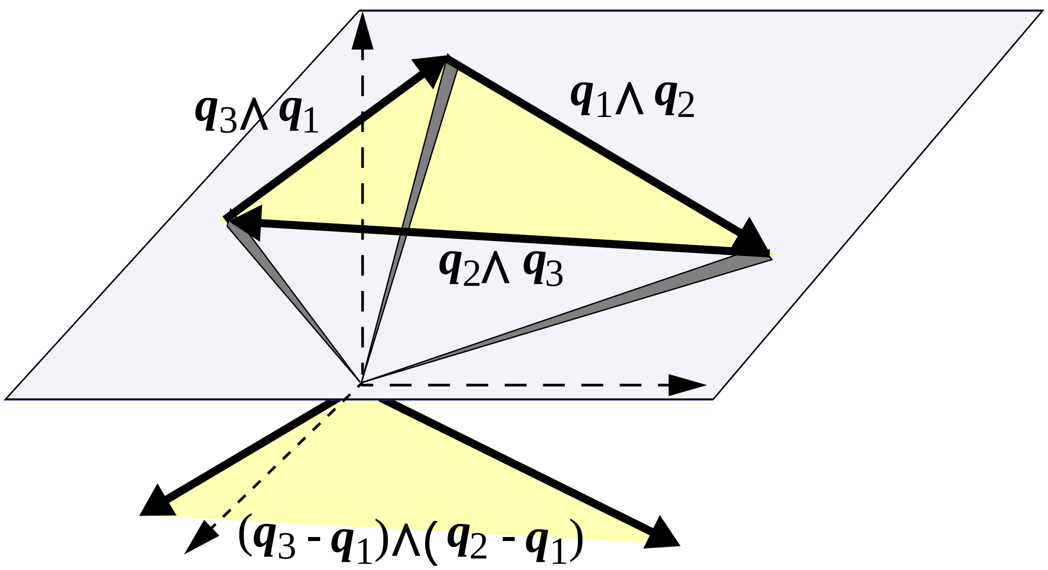
Note that a line-bound vector is geometrically distinct from a **free vector** representing a point at infinity!

Free Areal Magnitudes

The outer product of two free vectors is called a **free areal magnitude**. We can write this as

$$(\boldsymbol{q}_2 - \boldsymbol{q}_1) \wedge (\boldsymbol{q}_3 - \boldsymbol{q}_1) = \boldsymbol{q}_2 \wedge \boldsymbol{q}_3 - \boldsymbol{q}_1 \wedge \boldsymbol{q}_3 + \boldsymbol{q}_1 \wedge \boldsymbol{q}_2.$$

This shows that the ordered sum of the line-bound vectors around a triangle yields a free areal magnitude, i.e.



This is just a discrete version of *Stokes' theorem* ... with a geometric interpretation. To go from here to the continuous version, just approximate the curve by a polygon, triangulate it, apply the discrete version to each triangle, and take the limit as the number of sides goes to infinity.

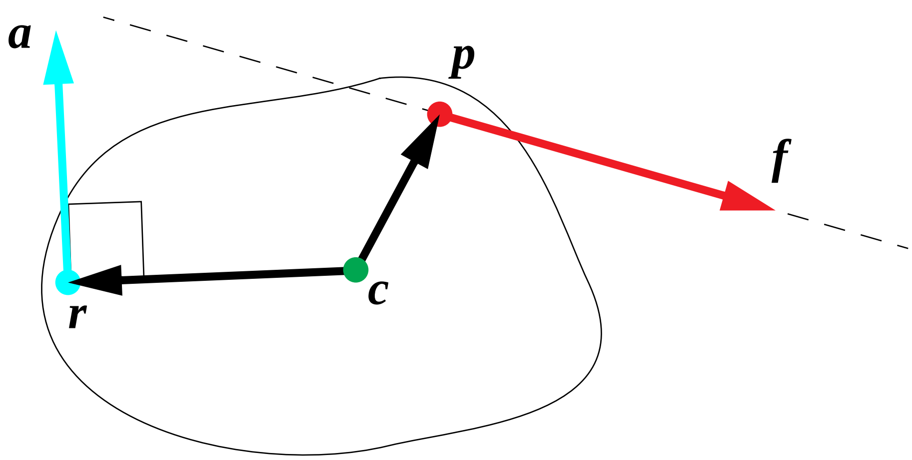
The outer product of three points is a **plane-bound area** ... and so on into as many dimensions as you like!

Forces and Torques in One

We regard a force as a free vector $\boldsymbol{f}$; taking the outer product with a point $\boldsymbol{p}$ in a rigid body yields a line-bound vector $\boldsymbol{f} \wedge \boldsymbol{p}$ which contains *all the information* needed to determine how the force affects the body. To see this, observe that if the body is pivoted about the point $\boldsymbol{c}$, then the acceleration at each point $\boldsymbol{r}$ is given (up to a constant factor) by

$$\boldsymbol{a} = (\boldsymbol{f} \wedge (\boldsymbol{p} - \boldsymbol{c})) \bullet (\boldsymbol{r} - \boldsymbol{c}),$$

as shown in the drawing below:



This illustrates a general rule that we shall see many examples of: *The generators of motion are bivectors*.

A second force $\boldsymbol{f}$ applied to another point $\boldsymbol{q}$ produces the same response at any point $\boldsymbol{r}$ only if $\boldsymbol{f} \wedge (\boldsymbol{p} - \boldsymbol{c}) = \boldsymbol{g} \wedge (\boldsymbol{q} - \boldsymbol{c})$, or

$$\boldsymbol{f} \wedge \boldsymbol{p} - \boldsymbol{g} \wedge \boldsymbol{q} = (\boldsymbol{f} - \boldsymbol{g}) \wedge \boldsymbol{c}.$$

This in turn can be true for all $\boldsymbol{c}$ only if $\boldsymbol{f} = \boldsymbol{g}$ and hence $\boldsymbol{f} \wedge \boldsymbol{p} = \boldsymbol{g} \wedge \boldsymbol{q}$, which proves our claim above.

The Theory of Screws by Sir Robert Ball

A complementary interpretation of line-bound vectors is as an *infinitesimal motion*. In the plane, *rotation* about a point $\boldsymbol{c}$ with angular velocity $\dot{\theta}$ is represented by a weighted point $\dot{\theta}\boldsymbol{c}$, and all information on the instantaneous motion of $\boldsymbol{p}$ is in

$$\dot{\theta}\boldsymbol{c} \wedge \boldsymbol{p} = \dot{\theta}(\boldsymbol{c} - \boldsymbol{p}) \wedge \boldsymbol{p} = \boldsymbol{p} \wedge (\dot{\theta}(\boldsymbol{p} - \boldsymbol{c})) = \boldsymbol{p} \wedge \boldsymbol{v}_{\perp},$$

where $\boldsymbol{v} = \dot{\boldsymbol{p}}$ is the linear velocity of $\boldsymbol{p}$. To prove this, note that the derivative of the squared distance to any fixed point $\boldsymbol{q}$ is

$$\partial_t \|\boldsymbol{p} - \boldsymbol{q}\|^2 = 2(\boldsymbol{p} - \boldsymbol{q}) \bullet \boldsymbol{v} = 2\iota((\boldsymbol{p} - \boldsymbol{q}) \wedge (\iota \boldsymbol{v})).$$

Now if $\boldsymbol{E} \equiv \sigma_1\sigma_2$ is the unit free area, then $\boldsymbol{v}_{\perp} = \boldsymbol{v}\boldsymbol{E} = -\boldsymbol{E}\boldsymbol{v}$. Also, since $\iota = \sigma_0\sigma_1\sigma_2$, $\iota\boldsymbol{E} = -\sigma_0$, so $\iota\boldsymbol{v} = -\sigma_0\boldsymbol{v}_{\perp}$, and:

$$\partial_t \|\boldsymbol{p} - \boldsymbol{q}\|^2 = 2\iota((\boldsymbol{q} - \boldsymbol{p}) \wedge \sigma_0 \wedge \boldsymbol{v}_{\perp}) = 2\iota((\boldsymbol{q} - \boldsymbol{p}) \wedge \boldsymbol{p} \wedge \boldsymbol{v}_{\perp})$$

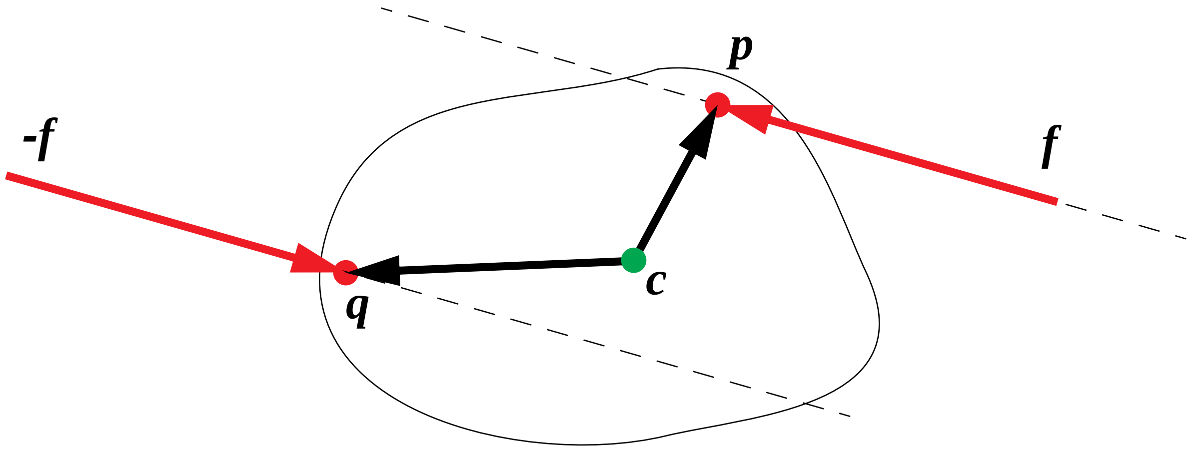
$$= 2\iota(\boldsymbol{q} \wedge \boldsymbol{p} \wedge \boldsymbol{v}_{\perp}) = 2\iota\dot{\theta}(\boldsymbol{q} \wedge \boldsymbol{p} \wedge (\boldsymbol{p} - \boldsymbol{c})) = 2\iota((\dot{\theta}\boldsymbol{c} \wedge \boldsymbol{p}) \wedge \boldsymbol{q})$$

A *translation* is represented by the free vector $\boldsymbol{t}_{\perp} = \boldsymbol{E}\boldsymbol{t}$, i.e. as a rotation about a point-at-infinity.

In *3-D space*, an instantaneous rotation about an axis $\boldsymbol{a}$ thru a point $\boldsymbol{c}$ is represented by a line-bound vector, i.e. by a **rotor** $\dot{\theta}(\boldsymbol{c} \wedge \boldsymbol{a})$, and the resulting motion of a point $\boldsymbol{p}$ by $\dot{\theta}(\boldsymbol{c} \wedge \boldsymbol{a}) \wedge \boldsymbol{p}$. Instantaneous translations are represented by a **translator** $\boldsymbol{t}_{\perp} \wedge \boldsymbol{a}$, while the sum of a rotor & translator is a general **screw**.

Known Only by Their Effects

The analog of a translator for forces is a sum of two forces, whose line-bound vectors that are equal in magnitude, opposite in direction, and on different lines:



The sum of such a pair of forces $\boldsymbol{f} \wedge \boldsymbol{p} - \boldsymbol{f} \wedge \boldsymbol{q} = \boldsymbol{f} \wedge (\boldsymbol{p} - \boldsymbol{q})$ is called a **couple**, and is the outer product of two free vectors.

This brings us to one of the deepest *mysteries of geometry*: The reality of **nonfactorizable** elements in the algebra. For example, a general sum of forces cannot itself be written as the outer product of any two points or free vectors $\boldsymbol{x} \wedge \boldsymbol{y}$. This follows since the l.h.s. below is 0 but the r.h.s. vanishes only if the points / vectors are linearly dependent:

$$\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{x} \wedge \boldsymbol{y} = (\boldsymbol{f} \wedge \boldsymbol{p} + \boldsymbol{g} \wedge \boldsymbol{q}) \wedge (\boldsymbol{f} \wedge \boldsymbol{p} + \boldsymbol{g} \wedge \boldsymbol{q}) = 2\boldsymbol{f} \wedge \boldsymbol{p} \wedge \boldsymbol{g} \wedge \boldsymbol{q}$$

Since $\boldsymbol{g} = (\boldsymbol{f} + \boldsymbol{g}) - \boldsymbol{f}$, any such **wrench** can always be written as the sum of a couple and a finite force; similarly, any screw can be written as the sum of a rotor and a translator.

THE REGRESSIVE PRODUCT

Grassmann actually defined many kinds of geometrical multiplication, including ultimately the geometric product itself.

Of particular interest was the **regressive** (outer) product, which may be defined via duality as

$$\boldsymbol{X} \vee \boldsymbol{Y} \equiv ((\boldsymbol{X}\iota^{-1}) \wedge (\boldsymbol{Y}\iota^{-1}))\iota^{-1} .$$

While the usual (**progressive**) outer product is a blade in the *direct sum* of the nonintersecting subspaces of its factors, the regressive product is a blade in the *intersection* of the spanning subspaces of its factors. In $\mathcal{E}_2$, for example,

$$(\boldsymbol{p} \wedge \boldsymbol{q}) \vee (\boldsymbol{r} \wedge \boldsymbol{s}) = -((\boldsymbol{p} \wedge \boldsymbol{q})\iota) \wedge ((\boldsymbol{r} \wedge \boldsymbol{s})\iota)\iota = ((\boldsymbol{p} \wedge \boldsymbol{q})\iota) \bullet (\boldsymbol{r} \wedge \boldsymbol{s})$$

$$= \boldsymbol{r} \bullet ((\boldsymbol{p} \wedge \boldsymbol{q})\iota)\boldsymbol{s} - \boldsymbol{s} \bullet ((\boldsymbol{p} \wedge \boldsymbol{q})\iota)\boldsymbol{r} = \iota((\boldsymbol{r} \wedge \boldsymbol{p} \wedge \boldsymbol{q})\boldsymbol{s} - (\boldsymbol{s} \wedge \boldsymbol{p} \wedge \boldsymbol{q})\boldsymbol{r})$$

More generally, the progressive & regressive products are related by the "shuffle" formula, $(\boldsymbol{a}_1 \wedge \ldots \wedge \boldsymbol{a}_k) \vee (\boldsymbol{b}_1 \wedge \ldots \wedge \boldsymbol{b}_l)$

$$= \sum_{\text{shuffles } \pi} (-1)^{\pi}[\boldsymbol{a}_{\pi(1)}, \ldots, \boldsymbol{a}_{\pi(n-l)}, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_l] \ldots$$

$$\ldots \boldsymbol{a}_{\pi(n-l+1)} \wedge \ldots \wedge \boldsymbol{a}_{\pi(k)} \quad ,$$

where a **shuffle** is a permutation of $\{1, \ldots, k\}$ that preserves the order of the first k and last $n-k$ elements, $(-1)^{\pi}$ is the parity of that permutation, and the square brackets indicates the dual of the outer product of the n enclosed factors ($k \geq l$).

The Metric Connection

Now let us bring a metric in, by defining a quadratic form in the barycentric coordinates of the points $\boldsymbol{q} \leftrightarrow [q_0\ q_1\ \dots\ q_n]$ vs. a basis $[\boldsymbol{p}_0, \boldsymbol{p}_1, \dots, \boldsymbol{p}_n]$; the corresponding symmetric bilinear form may be written using matrices as

$$D(\boldsymbol{r}, \boldsymbol{s}) \equiv [r_0\ \ r_1\ \ \dots\ \ r_n] \begin{bmatrix} 0 & -d_{01}^2/2 & \dots & -d_{0n}^2/2 \\ -d_{01}^2/2 & 0 & \dots & -d_{1n}^2/2 \\ \dots & \dots & \dots & \dots \\ -d_{0n}^2/2 & -d_{1n}^2/2 & \dots & 0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \dots \\ s_n \end{bmatrix},$$

where d_{ij}^2 $(i, j = 0, \dots, n)$ are the squared distances among the basis points $[\boldsymbol{p}_0, \boldsymbol{p}_1, \dots, \boldsymbol{p}_n]$. Note that on the difference of a pair of basis points, e.g. $\boldsymbol{p}_0 - \boldsymbol{p}_1 \leftrightarrow [1\ -1\ 0\ \dots\ 0]$, this form evaluates to $D(\boldsymbol{p}_0 - \boldsymbol{p}_1, \boldsymbol{p}_0 - \boldsymbol{p}_1) = d_{01}^2$; more generally, it gives the *length* of any free vector directly. On any pair of basis points is the form is clearly $D(\boldsymbol{p}_i, \boldsymbol{p}_j) = -d_{ij}^2/2$, and a general **inner product** of pairs of unit weight *points* is $D(\boldsymbol{r}, \boldsymbol{s}) = -\|\boldsymbol{r} - \boldsymbol{s}\|^2/2$. To be convinced of this, take the vertices of a right pyramid as a basis, i.e. $d_{0i}^2 = 1$ & $d_{ij}^2 = 2$ for all $0 \le i, j \le n$ $(i < j)$. The barycentric coordinates w.r.t. points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_n$ *are* Cartesian coordinates vs. the frame with origin $\boldsymbol{p}_0$ & orthonormal axes $\boldsymbol{p}_i - \boldsymbol{p}_0$, and you can show that $D(\boldsymbol{u}, \boldsymbol{v}) = \sum_i u_i v_i$ for all free vectors $\boldsymbol{u}, \boldsymbol{v}$.

Distance Geometry

As a Mathematical Theory

- Studies mathematical spaces through their *metrics*.
- Includes the classical *hyperbolic* and *elliptic* as well as general *Riemannian* geometries.
- The *Euclidean* case corresponds to the **invariant theory** of the group of rigid motions.

The Fundamental Theorems

I) Any rational polynomial in the Cartesian coordinates of a set of points, *invariant* under the rigid motions, can be rewritten as a polynomial in:

1) The squared distances $D(\mathrm{a}, \mathrm{b})$ between pairs of points a & b.

2) The oriented volumes $V(\mathrm{a}, \mathrm{b}, \ldots, \mathrm{c})$ spanned by the simplices of points $\langle \mathrm{a}, \mathrm{b}, \ldots, \mathrm{c} \rangle$.

Note these **invariants** are *not* algebraically independent over the real numbers.

II) The algebraic relations among the invariants (called syzygies) can be rewritten as a system of *(in)*equalities in the N-point Cayley-Menger determinants:

$0 \le D(\mathrm{a}, \mathrm{b}, \ldots, \mathrm{c})$ **(vanishing if $N > \dim + 1$)**

$$= 2\left(\frac{-1}{2}\right)^N \det \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & D(\mathrm{a}, \mathrm{b}) & \ldots & D(\mathrm{a}, \mathrm{c}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & D(\mathrm{a}, \mathrm{c}) & D(\mathrm{b}, \mathrm{c}) & \ldots & 0 \end{bmatrix}$$ **; in addition,**

if $N = \dim + 1$, the oriented volumes are related to the *non*symmetric determinants by

$$V(\mathrm{a}, \mathrm{b}, \ldots, \mathrm{c})V(\mathrm{p}, \mathrm{q}, \ldots, \mathrm{r}) = D(\mathrm{a}, \mathrm{b}, \ldots, \mathrm{c}; \mathrm{p}, \mathrm{q}, \ldots, \mathrm{r}).$$

- In the case of three points, for example,

$$V^2(\mathrm{a}, \mathrm{b}, \mathrm{c}) = D(\mathrm{a}, \mathrm{b}, \mathrm{c}) =$$

$$\tfrac{1}{4}(d(\mathrm{a}, \mathrm{b}) + d(\mathrm{a}, \mathrm{c}) + d(\mathrm{b}, \mathrm{c}))(d(\mathrm{a}, \mathrm{b}) + d(\mathrm{a}, \mathrm{c}) - d(\mathrm{b}, \mathrm{c}))$$

$$(d(\mathrm{a}, \mathrm{b}) - d(\mathrm{a}, \mathrm{c}) + d(\mathrm{b}, \mathrm{c}))(-d(\mathrm{a}, \mathrm{b}) + d(\mathrm{a}, \mathrm{c}) + d(\mathrm{b}, \mathrm{c}))$$

where $d \equiv \sqrt{D}$. This is just **Heron's formula** for the area of a triangle in terms of its sides.

- Thus the condition $D(\mathrm{a}, \mathrm{b}, \mathrm{c}) \ge 0$ is equivalent to *all three* triangle inequalities among the points.

The nonnegativity of Cayley-Menger determinants can therefore be viewed as a *nonlinear* generalization of the triangle inequality.

The Quadratic Form of a Cayley-Menger Matrix

(or perhaps "Menger Meets Artin")

- *Any* symmetric matrix is the matrix of *some* quadratic form, so *any* symmetric determinant is a **Gramian** of vectors in *some* metric vector space.
- For a Cayley-Menger determinant, $D(\mathrm{a}, \mathrm{b}, \ldots, \mathrm{c})$

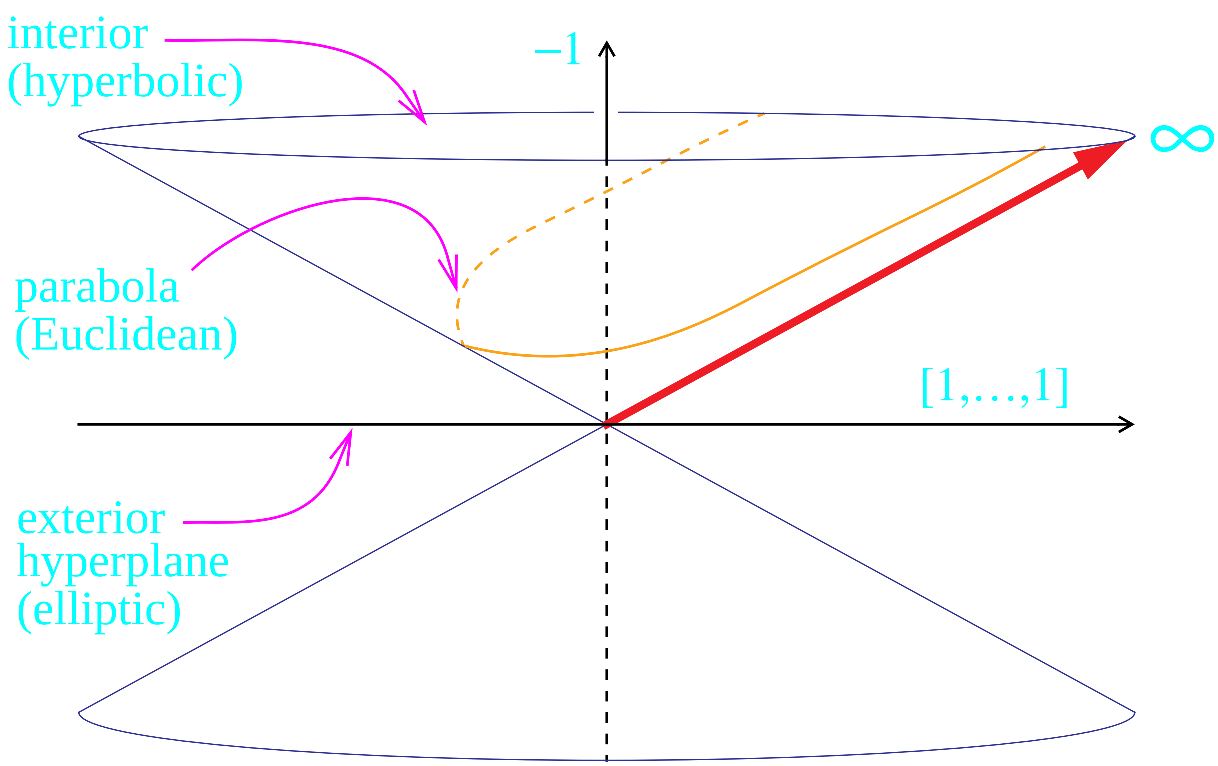
$$= -\det\begin{bmatrix} 0 & -1 & -1 & \ldots & -1 \\ -1 & 0 & -D(\mathrm{a},\mathrm{b})/2 & \ldots & -D(\mathrm{a},\mathrm{c})/2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & -D(\mathrm{a},\mathrm{c})/2 & -D(\mathrm{b},\mathrm{c})/2 & \ldots & 0 \end{bmatrix} \geq 0,$$

  the zero's down the diagonal shows these are *null* vectors, so the metric must be *indefinite*.
- The border of -1's shows that one vector has the same inner product with *all* others, which thus lie in an *affine hyperplane* M intersecting the *null cone* N in a "parabola".
- For nD Euclidean distances, the indicated sign conditions on the principle minors shows the vectors live in an $(n+2)$D *Minkowski space* $\mathcal{V} = \mathcal{R}^{n+1,1}$ (i.e. $[-1, 1, \ldots, 1]$).

- The null cone is $\mathcal{N} \equiv \{\mathbf{v} \in \mathcal{V} \mid \mathbf{v} \bullet \mathbf{v} = 0\}$ and the hyperplane is $\mathcal{M} = \{\mathbf{v} \in \mathcal{V} \mid \mathbf{v} \bullet \mathbf{e}_\infty = -1\}$, where the *point at infinity* is $\mathbf{e}_\infty \equiv [1, 1, \mathbf{0}]$. These are depicted below.

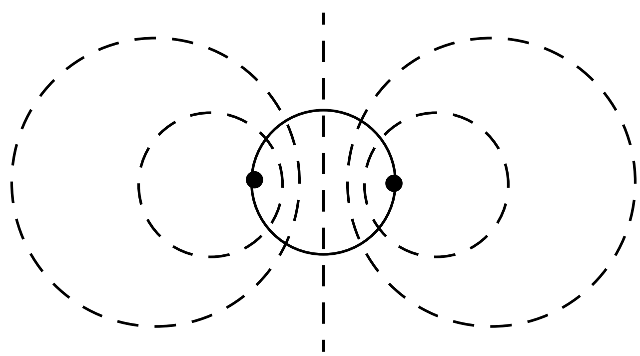
- Relative to this metric, the *inner product of two points* is minus half the squared distance between them, as follows:

$$\frac{-\|\boldsymbol{p}-\boldsymbol{q}\|^2}{2} = \begin{bmatrix} \frac{(\|\boldsymbol{p}\|^2+1)}{2} & \frac{(\|\boldsymbol{p}\|^2-1)}{2} & \boldsymbol{p} \end{bmatrix} \begin{bmatrix} -1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} (\|\boldsymbol{q}\|^2+1)/2 \\ (\|\boldsymbol{q}\|^2-1)/2 \\ \boldsymbol{q} \end{bmatrix}$$



Möbius Sphere Geometry

- More generally, any vector $\mathbf{p} = [(P+1)/2, (P-1)/2, \boldsymbol{p}]$ on $\mathcal{M}$ with $\|\mathbf{p}\|^2 = \|\boldsymbol{p}\|^2 - P > 0$ determines a *sphere* of radius $\|\mathbf{p}\|$ & center $\boldsymbol{p}$. The points of the sphere are exactly $\mathbf{p}^\perp \cap \mathcal{M} \cap \mathcal{N}$, i.e. $\{\mathbf{s} \in \mathrm{V} \mid \mathbf{s} \bullet \mathbf{e}_\infty = -1, \mathbf{s} \bullet \mathbf{s} = 0, \mathbf{s} \bullet \mathbf{p} = 0\}$.
- In addition, vectors on $\mathcal{M}$ orthogonal to a fixed sphere's vector $\mathbf{p}$ define a linearly dependent family of spheres which intersect that of $\mathbf{p}$ orthogonally.



- Vectors $\mathbf{u} \in \mathcal{V}$ other than $\mathbf{e}_\infty$ with $\mathbf{u} \bullet \mathbf{e}_\infty = 0$ determine a Euclidean *plane*; these can be scaled so that $\|\mathbf{u}\| = 1$ and $\mathbf{u} \bullet \mathbf{p}$ is the $\perp$ distance to the center of $\mathbf{p}$. They are the limit of a sequence of spheres whose centers go to $\mathbf{e}_\infty$ (as above).
- The isometry group of $\mathcal{V}$ is isomorphic to the **conformal group**, which maps spheres & planes to the same (or $\mathbf{e}_\infty$); the stabilizer of $\mathbf{e}_\infty$ in this group is the **Euclidean group**.
- Vectors $\mathbf{p}$ with $\mathbf{p}^2 < 0$ are projectively points in Klein's model of *hyperbolic geometry*; those on the hyperplane with a given norm form a pencil of **horospheres** (*a la* Wachter).

Doing It in the Algebra

(via Hestenes' *conformal split*)

Let $\mathbf{e}_\circ$, $\mathbf{e}_\infty$ be null vectors in Minkowski space $\mathcal{R}^{n+1,1}$ with $\mathbf{e}_\circ \bullet \mathbf{e}_\infty = -1$, and let $\mathbf{E}$ be the bivector $\mathbf{e}_\infty \wedge \mathbf{e}_\circ$; then the *conformal split* of any other vector $\mathbf{x}$ is

$$\mathbf{xE} = \mathbf{x} \bullet \mathbf{E} + \mathbf{x} \wedge \mathbf{E} = (\mathbf{x} \bullet \mathbf{e}_\infty)\mathbf{e}_\circ - (\mathbf{x} \bullet \mathbf{e}_\circ)\mathbf{e}_\infty + \mathbf{x} \wedge \mathbf{e}_\infty \wedge \mathbf{e}_\circ ,$$

and hence $\mathbf{x} = \mathbf{xE}^2 =$

$$-(\mathbf{x} \bullet \mathbf{e}_\infty)\mathbf{e}_\circ - (\mathbf{x} \bullet \mathbf{e}_\circ)\mathbf{e}_\infty + (\mathbf{x} \wedge \mathbf{E}) \bullet \mathbf{E} \quad \equiv \quad \mathbf{x}_\circ + \mathbf{x}_\infty + \mathbf{x}_\perp \ .$$

It follows that $\mathbf{e}_\infty \wedge \mathbf{x} = -(\mathbf{x} \bullet \mathbf{e}_\infty)\mathbf{E} + \mathbf{e}_\infty \mathbf{x}_\perp$, so that

$$(\mathbf{e}_\infty \wedge \mathbf{x}) \bullet \mathbf{e}_\circ = -(\mathbf{x} \bullet \mathbf{e}_\infty)\mathbf{e}_\circ + \mathbf{x}_\perp = -(\mathbf{x} \bullet \mathbf{e}_\infty)\left(\mathbf{e}_\circ + \frac{\mathbf{x}_\perp}{-(\mathbf{x} \bullet \mathbf{e}_\infty)}\right)$$

projects $\mathbf{x}$ into a scale factor $-(\mathbf{x} \bullet \mathbf{e}_\infty)$ & a homogeneous part $\mathbf{e}_\circ - \mathbf{x}_\perp / \mathbf{x} \bullet \mathbf{e}_\infty$ on an affine hyperplane thru the "origin" $\mathbf{e}_\circ$.

In the special case of a point $\mathbf{p} \in \mathcal{M} \cap \mathcal{N}$, the scale factor is $-(\mathbf{p} \bullet \mathbf{e}_\infty) = 1$, so $\mathbf{p}_\circ = \mathbf{e}_\circ$, and letting $\boldsymbol{p} \equiv \mathbf{p}_\perp$, we find that

$$0 = \mathbf{p}^2 = 2(\mathbf{p}_\infty \bullet \mathbf{p}_\circ) + \boldsymbol{p}^2 = -2(\mathbf{p} \bullet \mathbf{e}_\circ) + \boldsymbol{p}^2 ,$$

i.e. $\mathbf{p}_\infty = (\boldsymbol{p}^2/2)\mathbf{e}_\infty$, and hence $\mathbf{p} = \mathbf{e}_\circ + (\boldsymbol{p}^2/2)\mathbf{e}_\infty + \boldsymbol{p}$.

Thus the product of any two such “homogenous points” is:

$$\begin{aligned}\mathbf{pq} &= (\mathbf{e}_\circ + (p^2/2)\mathbf{e}_\infty + \boldsymbol{p})(\mathbf{e}_\circ + (q^2/2)\mathbf{e}_\infty + \boldsymbol{q}) \\ &= \boldsymbol{p}\bullet\boldsymbol{q} + (\boldsymbol{p}^2(\mathbf{e}_\infty\mathbf{e}_\circ) + \boldsymbol{q}^2(\mathbf{e}_\circ\mathbf{e}_\infty))/2 + \boldsymbol{p}\wedge\boldsymbol{q} \\ &\qquad + \boldsymbol{p}(\mathbf{e}_\circ + (\boldsymbol{q}^2/2)\mathbf{e}_\infty) + (\mathbf{e}_\circ + (\boldsymbol{p}^2/2)\mathbf{e}_\infty)\boldsymbol{q} \\ &= -\|\boldsymbol{p}-\boldsymbol{q}\|^2/2 + ((\boldsymbol{p}^2-\boldsymbol{q}^2)/2)\mathbf{E} + \boldsymbol{p}\wedge\boldsymbol{q} \\ &\qquad + (\boldsymbol{p}-\boldsymbol{q})\mathbf{e}_\circ + ((\boldsymbol{p}\boldsymbol{q}^2-\boldsymbol{p}^2\boldsymbol{q})/2)\mathbf{e}_\infty\end{aligned}$$

This shows that the scalar part of this *inner product of points* is just $-1/2$ the **Euclidean distance** between the points,

$$\mathbf{p}\bullet\mathbf{q} = -\|\boldsymbol{p}-\boldsymbol{q}\|^2/2.$$

Outer-multiplying the bivector part by $\mathbf{e}_\infty$, on the other hand,

$$\mathbf{e}_\infty\wedge\mathbf{p}\wedge\mathbf{q} = (\boldsymbol{p}\wedge\boldsymbol{q})\mathbf{e}_\infty + (\boldsymbol{q}-\boldsymbol{p})\mathbf{E},$$

gives us the **moment** and **direction** of the line-bound vector generated by the points. Squaring this then gives the scalar $-\|\mathbf{e}_\infty\wedge\mathbf{p}\wedge\mathbf{q}\|^2 = (\mathbf{q}\wedge\mathbf{p}\wedge\mathbf{e}_\infty)\bullet(\mathbf{e}_\infty\wedge\mathbf{p}\wedge\mathbf{q}) =$

$$\det\begin{bmatrix}\mathbf{e}_\infty\bullet\mathbf{e}_\infty & \mathbf{e}_\infty\bullet\mathbf{p} & \mathbf{e}_\infty\bullet\mathbf{q}\\ \mathbf{e}_\infty\bullet\mathbf{p} & \mathbf{p}\bullet\mathbf{p} & \mathbf{p}\bullet\mathbf{q}\\ \mathbf{e}_\infty\bullet\mathbf{q} & \mathbf{p}\bullet\mathbf{q} & \mathbf{q}\bullet\mathbf{q}\end{bmatrix} = \det\begin{bmatrix}0 & -1 & -1\\ -1 & 0 & -\|\boldsymbol{p}-\boldsymbol{q}\|^2/2\\ -1 & -\|\boldsymbol{p}-\boldsymbol{q}\|^2/2 & 0\end{bmatrix}$$

i.e. the *Cayley-Menger determinant* $-D(\mathrm{p},\mathrm{q}) = -\|\boldsymbol{p}-\boldsymbol{q}\|^2$!

Circles in the Plane or Hyperplanes?

Consider now the case $n = 2$ (the Euclidean plane). Then the outer product of any three independent homogeneous points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ may be written as the dual of a vector, i.e. $\mathbf{s}\iota = -\iota\mathbf{s}$, where the unit pseudo-scalar of $\mathcal{R}^{3,1}$ satisfies $\iota^2 = -1$. Let us also normalize $\mathbf{s}$ such that $\mathbf{s} \bullet \mathbf{e}_\infty = -1$ as usual, so that

$$\mathbf{s} \equiv \frac{-(\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r})\iota}{\mathbf{e}_\infty \bullet ((\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r})\iota)} = \frac{-(\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r})\iota}{(\mathbf{e}_\infty \wedge \mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r})\iota} = \frac{-(\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r})\iota}{\|\mathbf{e}_\infty \wedge \mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}\|} .$$

Also, observe that the intersection of the plane $\langle \mathbf{s}, \mathbf{e}_\infty \rangle$ with $\mathcal{M} \cap \mathcal{N}$ is $\mathbf{o} \equiv \mathbf{s} + (\mathbf{s}^2/2)\mathbf{e}_\infty$, since $\mathbf{e}_\infty \bullet \mathbf{o} = -1$ and $\mathbf{o}^2 = 0$.

Thus the locus of the equation $\mathbf{s} \bullet \mathbf{x} = 0$ (with $\mathbf{x}$ a homogeneous point) can be described in the following equivalent ways,

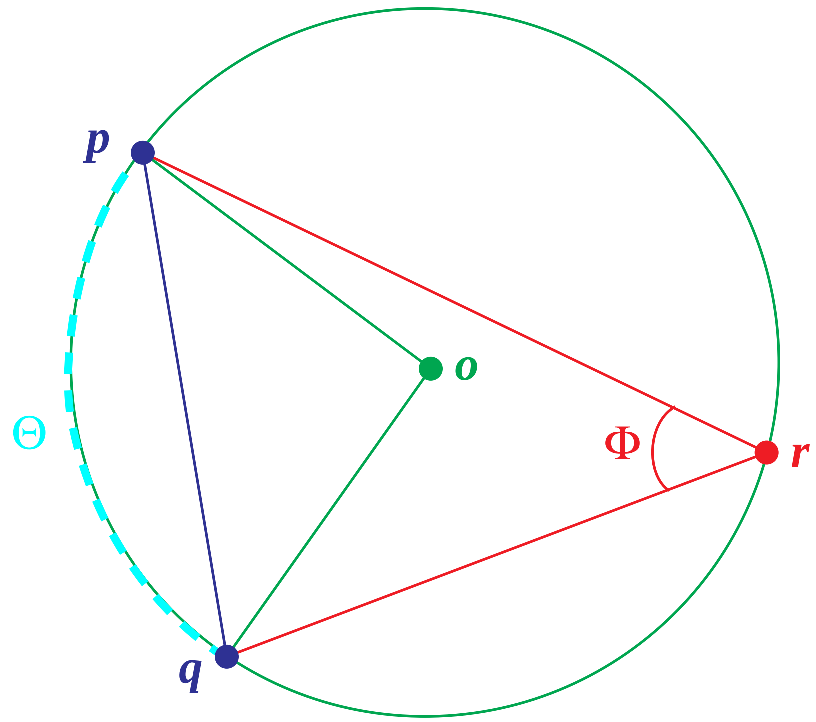
$$0 = \frac{\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r} \wedge \mathbf{x}}{\|\mathbf{e}_\infty \wedge \mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}\|} = \frac{-((\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r})\iota) \bullet \mathbf{x}}{\|\mathbf{e}_\infty \wedge \mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}\|}$$

$$\equiv \mathbf{s} \bullet \mathbf{x} = \mathbf{s}^2/2 + \mathbf{o} \bullet \mathbf{x} = (\mathbf{s}^2 - \|\boldsymbol{x} - \boldsymbol{o}\|^2)/2 ,$$

thereby proving in (almost) one line the following:

Theorem: If $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathcal{M} \cap \mathcal{N}$ are homogeneous points with $\mathbf{e}_\infty \wedge \mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r} \neq 0$ (so that the corresponding points $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ of $\mathcal{R}^2$ are affinely independent), a homogeneous point $\mathbf{x}$ is linearly dependent on $\mathbf{p}, \mathbf{q}, \mathbf{r}$ iff $\boldsymbol{x}$ lies on the circle with center $\boldsymbol{o}$ & squared radius $\mathbf{s}^2$, where $\mathbf{s} \in \mathcal{M}$ is the dual of $\mathbf{p} \wedge \mathbf{q} \wedge \mathbf{r}$.

An Example in 2D = 4D



The above figure shows three points on a unit circle, for which:

$$\|\boldsymbol{p}-\boldsymbol{q}\|^2 = (\mathbf{p}-\mathbf{q})^2 = ((\mathbf{p}-\mathbf{o})-(\mathbf{q}-\mathbf{o}))^2$$
$$= ((\mathbf{p}-\mathbf{o})-\boldsymbol{e}^{\iota\mathbf{E}\Theta}(\mathbf{p}-\mathbf{o}))^2 = (1-\boldsymbol{e}^{\iota\mathbf{E}\Theta})(1-\boldsymbol{e}^{-\iota\mathbf{E}\Theta})$$
$$= 2(1-\cos(\Theta)) = 4(\sin(\Theta/2))^2$$

It follows that $1 = \mathbf{s}^2 =$

$$\frac{(\iota(\mathbf{p}\wedge\mathbf{q}\wedge\mathbf{r}))^2}{\|\mathbf{e}_\infty\wedge\mathbf{p}\wedge\mathbf{q}\wedge\mathbf{r}\|^2} = \frac{(1/4)(\mathbf{p}-\mathbf{q})^2(\mathbf{p}-\mathbf{r})^2(\mathbf{q}-\mathbf{r})^2}{(\mathbf{p}-\mathbf{r})^2(\mathbf{q}-\mathbf{r})^2-((\mathbf{p}-\mathbf{r})\bullet(\mathbf{q}-\mathbf{r}))^2},$$

& thus $\sin^2(\Theta/2) = \|\boldsymbol{p}-\boldsymbol{q}\|^2 = 1-\cos^2(\Phi)$, i.e. $\Phi = \Theta/2$!

A Hot Night in Liouville!

As in $\mathcal{R}^n$, we can write an arbitrary isometry of $\mathcal{R}^{n+1,1}$ as a product of reflections. Since $\mathbf{N}$ is automatically invariant under the isometries, these act on homogeneous points as

$$-\mathbf{v}\mathbf{p}\mathbf{v}^{-1} = -\mathbf{v}(\mathbf{e}_\circ + (p^2/2)\mathbf{e}_\infty + p)\mathbf{v}^{-1}$$

$$= \vartheta \cdot (\mathbf{e}_\circ + ((\Theta(p))^2/2)\mathbf{e}_\infty + \Theta(p)) \; ,$$

where $\Theta\colon \mathcal{R}^n \to \mathcal{R}^n$ and $\vartheta \equiv (\mathbf{v}\mathbf{p}\mathbf{v}^{-1}) \bullet \mathbf{e}_\infty \neq 0$. However, we can also write it as

$$-\mathbf{v}\mathbf{p}\mathbf{v}^{-1} = (\mathbf{p} \wedge \mathbf{v} - \mathbf{p} \bullet \mathbf{v})\mathbf{v}^{-1} = \mathbf{p} - 2(\mathbf{p} \bullet \mathbf{v})\mathbf{v}^{-1} \qquad ✦$$

In the case that $\mathbf{v} \equiv \mathbf{s} \equiv \mathbf{o} - (\mathbf{s}^2/2)\mathbf{e}_\infty \equiv \mathbf{e}_\circ - \mathbf{e}_\infty/2$ represents a unit sphere centered on the origin $\mathbf{e}_\circ$, we have $\mathbf{v}^{-1} = \mathbf{v}$ and

$$-2(\mathbf{p} \bullet \mathbf{v}) = -2(\mathbf{e}_\circ + (p^2/2)\mathbf{e}_\infty + p) \bullet (\mathbf{e}_\circ - \mathbf{e}_\infty/2) = p^2 - 1,$$

so that by ✦ above

$$-\mathbf{v}\mathbf{p}\mathbf{v}^{-1} = (\mathbf{e}_\circ + (p^2/2)\mathbf{e}_\infty + p) + (p^2 - 1)(\mathbf{e}_\circ - \mathbf{e}_\infty/2)$$

$$= p^2\mathbf{e}_\circ + \mathbf{e}_\infty/2 + p = p^2(\mathbf{e}_\circ + (p^{-2}/2)\mathbf{e}_\infty + p^{-1})$$

i.e. $\Theta(p) = p^{-1}$ and $\vartheta = p^2$. This reflection is thus an **inversion** in the unit sphere at $\mathbf{e}_\circ$ ... *a conformal transformation*!

Now consider the reflection w.r.t. a $\mathbf{v} \in \mathcal{R}^{n+1,1}$ with $\mathbf{v} \bullet \mathbf{e}_\infty = 0$, so that $\mathbf{v} = \mathbf{e}_\infty \delta + \boldsymbol{v}$ represents an affine hyperplane with **normal** $\boldsymbol{v}$ and **distance** to the origin $-\mathbf{v} \bullet \mathbf{e}_\circ = \delta$; then by our previous equation ✦, we get

$$-\mathbf{vpv} = \mathbf{p} - 2(\mathbf{p} \bullet \mathbf{v})\mathbf{v} = \mathbf{p} - 2(\boldsymbol{p} \bullet \boldsymbol{v} - \delta)(\mathbf{e}_\infty \delta + \boldsymbol{v})$$
$$= (\mathbf{e}_\circ + ((\Theta(\boldsymbol{p}))^2/2)\mathbf{e}_\infty + \Theta(\boldsymbol{p}))$$

where $\Theta(\boldsymbol{p}) = \boldsymbol{p} - 2(\boldsymbol{p} \bullet \boldsymbol{v})\boldsymbol{v} + 2\delta\boldsymbol{v} = -\boldsymbol{v}(\boldsymbol{p} - \delta\boldsymbol{v})\boldsymbol{v} + \delta\boldsymbol{v}$ is the **reflection** of $\boldsymbol{p}$ w.r.t. this hyperplane. In particular, the composition of two reflections in parallel hyperplanes is

$$\mathbf{vv}' = \boldsymbol{v}^2 + (\delta' - \delta)\boldsymbol{v} \wedge \mathbf{e}_\infty \equiv 1 + (\boldsymbol{t}/2)\mathbf{e}_\infty ,$$

where $\boldsymbol{t} \equiv 2(\delta' - \delta)\boldsymbol{v}$ is thus the vector of **translation** by twice the distance between the hyperplanes. Hence $\mathbf{t} = 1 + (\boldsymbol{t}/2)\mathbf{e}_\infty$ represents the translation as a *Lorentz transformation*!

Finally, consider the action of the exponential

$$\exp(\eta\mathbf{E}) = \cosh(\eta) + \mathbf{E}\sinh(\eta) = (1 + \mathbf{E})e^{\eta}/2 + (1 - \mathbf{E})e^{-\eta}/2$$

on homogenous points, i.e.

$$\exp(\eta\mathbf{E})\mathbf{p}\exp(-\eta\mathbf{E}) = \exp(2\eta\mathbf{E})(\mathbf{e}_\circ + (\boldsymbol{p}^2/2)\mathbf{e}_\infty) + \boldsymbol{p}$$
$$= e^{2\eta}(\mathbf{e}_\circ + ((e^{-2\eta}\boldsymbol{p})^2/2)\mathbf{e}_\infty + e^{-2\eta}\boldsymbol{p}) \ .$$

This is thus a **dilation** about $\mathbf{e}_\circ$ by $e^{-2\eta}$, which with the above generates the whole conformal group by *Liouville's theorem*.

Invariant Theory Revisited

In Tribute to Gian-Carlo Rota

- Projective geometry can be done at four distinct levels: **(1)** *synthetic*, **(2)** *Grassmann algebra*, **(3)** *invariant theoretic*, and **(4)** *algebraic geometry* (usually over the complex numbers).
- Analogous levels can also be identified in the **Cayley-Klein** (projective metric) geometries (over the real numbers).
- The extension to metric affine geometry is nontrivial, since *transvections are not isometries*. We now see that levels **(1)** & **(3)** are Möbius & distance geometry, resp.; the *missing link* is: